

THE EKELAND VARIATIONAL PRINCIPLE

The Main Result

The following result, which was proved in 1974 by Ivar Ekeland, is the basic tool for establishing the existence of solutions, or approximate solutions, for optimization problems who fail to satisfy the compactness requirement of Weierstrass' Theorem (or Proposition 5).

The Ekeland Variational Principle. *Let X be a complete metric space and $\varphi : X \rightarrow \mathbb{R}$ an upper semicontinuous map that is bounded from above. Then, for any $\varepsilon > 0$, there exists a $y \in X$ such that*

$$\varphi(y) > \varphi(x) - \varepsilon d(x, y) \quad \text{for all } x \in X \setminus \{y\}.$$
¹ (1)

Proof. Since $\sup \varphi(X) < \infty$, there exists a $z^0 \in X$ such that

$$\varphi(z^0) > \sup \varphi(X) - \varepsilon.$$

We construct two sequences (z^0, z^1, \dots) and (S_0, S_1, \dots) inductively as follows: For any $m = 0, 1, \dots$,

- Suppose z^m is known;
- Define

$$S_m := \{x \in X : \varphi(z^m) \leq \varphi(x) - \varepsilon d(x, z^m)\};$$

- Pick any $z^{m+1} \in S_m$ such that

$$\varphi(z^{m+1}) \geq \sup \varphi(S_m) - \frac{1}{m+1}.$$
 (2)

Obviously, $S_m \neq \emptyset$ (because $z^m \in S_m$) for each m . Moreover, thanks to the upper semicontinuity of φ , the map $\varphi(\cdot) - \varepsilon d(\cdot, z^m)$ is upper semicontinuous, and hence, each S_m is closed (Proposition D.4).

Claim 1. $S_{m+1} \subseteq S_m$ for each $m = 0, 1, \dots$

¹In this statement, we can replace the word “upper” with “lower,” provided that the word “above” is replaced with “below” and (1) is replaced with

$$\varphi(y) < \varphi(x) + \varepsilon d(x, y) \quad \text{for all } x \in X \setminus \{y\}.$$

Proof of Claim 1. Take any $m \in \mathbb{Z}_+$, and let $x \in S_{m+1}$. Then, by using the triangle inequality, and the facts that $x \in S_{m+1}$ and $z^{m+1} \in S_m$, we find

$$\begin{aligned} \varphi(x) - \varepsilon d(x, z^m) &\geq \varphi(x) - \varepsilon d(x, z^{m+1}) - \varepsilon d(z^{m+1}, z^m) \\ &\geq \varphi(z^{m+1}) - \varepsilon d(z^{m+1}, z^m) \\ &\geq \varphi(z^m), \end{aligned}$$

that is, $x \in S_m$. \parallel

Claim 2. $\text{diam}(S_m) \leq \frac{2}{\varepsilon m}$ for each $m = 1, 2, \dots$

Proof of Claim 2. Take any $m \in \mathbb{N}$, and let $x \in S_m$. Then, by Claim 1, we have $x \in S_{m-1}$. It follows from this fact and (2) that

$$\begin{aligned} \varepsilon d(x, z^m) &\leq \varphi(x) - \varphi(z^m) \\ &\leq \sup \varphi(S_{m-1}) - \varphi(z^m) \\ &\leq \frac{1}{m}, \end{aligned}$$

and the claim follows. \parallel

We may now apply the Cantor-Fréchet Intersection Theorem to conclude $S_0 \cap S_1 \cap \dots = \{y\}$ for some $y \in X$. It remains to show that y satisfies (1). Suppose this is not the case, that is, for some $x \in X \setminus \{y\}$, we have $\varphi(x) - \varepsilon d(x, y) \geq \varphi(y)$. Then,

$$\varphi(x) - \varepsilon d(x, y) \geq \varphi(y) \geq \varphi(z^m) + \varepsilon d(y, z^m)$$

for all $m = 0, 1, \dots$, so that

$$\begin{aligned} \varphi(x) &\geq \varphi(z^m) + \varepsilon(d(x, y) + d(y, z^m)) \\ &\geq \varphi(z^m) + \varepsilon d(x, z^m) \end{aligned}$$

for all $m = 0, 1, \dots$, that is, $x \in S_0 \cap S_1 \cap \dots$, which means $x = y$, a contradiction. \blacksquare

Remark 1. The proof above gives a bit more than what our statement of the Ekeland Variational Principle asserts. In particular, it shows that, under the hypotheses of this variational principle, for every $\varepsilon > 0$ and $z^0 \in X$ with $\varphi(z^0) > \sup \varphi(X) - \varepsilon$, there exists a $y \in N_{1,X}(z^0)$ such that (1) holds. (*Proof.* Find y as in the proof above, and note that $y \in S_0$ implies

$$\varepsilon d(z^0, y) \leq \varphi(y) - \varphi(z^0) \leq \sup \varphi(X) - \varphi(z^0) < \varepsilon,$$

so $y \in N_{1,X}(z^0)$.) \square

Exercise 1. Prove Proposition 5 by using the Ekeland Variational Principle.

Exercise 2. Let X be a complete metric space and $\varphi : X \rightarrow \mathbb{R}$ an upper semicontinuous map that is bounded from above. Take any $\varepsilon > 0$ and $z^0 \in X$ with $\varphi(z^0) > \sup \varphi(X) - \varepsilon$. Prove that, for any $\lambda \in (0, 1)$, there exists a $y \in X$ such that

- (i) $\varphi(y) \geq \varphi(z^0) + \lambda d(y, z^0)$;
- (ii) $d(y, z^0) < \frac{\varepsilon}{\lambda}$
- (iii) $\varphi(y) > \varphi(x) - \lambda d(x, y)$ for all $x \in X \setminus \{y\}$.

Application: Caristi's Fixed Point Theorem

Somewhat unexpectedly, the Ekeland Variational Principle provides a means for proving metric fixed point theorems. As an illustration, we note how easily this principle yields Caristi's Fixed Point Theorem, which, as we have seen in Section D.5.1, generalizes the Banach Fixed Point Theorem.

Caristi's Fixed Point Theorem. Let Φ be a self-map on a complete metric space X . If

$$d(x, \Phi(x)) \leq \varphi(x) - \varphi(\Phi(x)) \quad \text{for all } x \in X \quad (3)$$

for some lower semicontinuous $\varphi \in \mathbb{R}^X$ that is bounded from below, then Φ has a fixed point in X .

Proof. By the Ekeland Variational Principle, there exists some $y \in X$ such that

$$\varphi(y) < \varphi(x) + d(x, y) \quad \text{for all } x \in X \setminus \{y\}.$$

But (3) says that

$$\varphi(y) \geq \varphi(\Phi(y)) + d(y, \Phi(y)).$$

Therefore $\Phi(y)$ is not a member of $X \setminus \{y\}$, that is, $\Phi(y) = y$. ■

Exercise 3. Let X be a complete metric space and $\varphi \in \mathbb{R}^X$ a upper semicontinuous function that is bounded from above. Suppose that $\Gamma : X \rightarrow 2^X$ with the following property:

$$\text{For every } x \in X \text{ there exists a } y \in \Gamma(x) \text{ such that } \varphi(y) \geq \varphi(x) + d(x, y).$$

Show that $x^* \in \Gamma(x^*)$.

Application: Takahashi's Existence Theorem

The following result, which was proved in 1989 by Wataru Takahashi, is a useful generalization of Weierstrass' Theorem. It gives sufficient conditions for a real function defined on a complete metric space to possess a maximum point.

Takahashi's Existence Theorem. Let X be a complete metric space and $\varphi : X \rightarrow \mathbb{R}$ an upper semicontinuous map that is bounded from above. Suppose that, for every

$x \in X$ with $\varphi(x) < \sup \varphi(X)$, there exists a $z \in X \setminus \{x\}$ such that $\varphi(z) > \varphi(x) + d(x, z)$.² Then, φ achieves its maximum, that is, $\varphi(x^*) = \sup \varphi(X)$ for some $x^* \in X$.

Proof. Let us assume that the assertion is false, that is, $\varphi(x) < \sup \varphi(X)$ for all $x \in X$, and work toward obtaining a contradiction. When combined with the hypothesis of the theorem, this supposition allows us to conclude that, for every $x \in X$, there is a $z_x \in X \setminus \{x\}$ such that

$$\varphi(z_x) > \varphi(x) + d(x, z_x).$$

But by the Ekeland Variational Principle, there exists a $y \in X$ such that

$$\varphi(y) \geq \varphi(x) - d(x, y) \quad \text{for every } x \in X \setminus \{y\}.$$

Since $y \neq z_y$, it follows that

$$\begin{aligned} \varphi(y) &\geq \varphi(z_y) - d(z_y, y) \\ &> \varphi(y) + d(y, z_y) - d(z_y, y) \\ &= \varphi(y), \end{aligned}$$

which is absurd. ■

Exercise 4. Deduce Takahashi's Existence Theorem from Caristi's Fixed Point Theorem.

Application: The Existence of Approximate Stationary Points

Here is another application of the Ekeland Variational Principle, this time to optimization theory. Suppose you want to maximize a bounded differentiable real function on \mathbb{R} , but you realize that the maximum of your function is not attained. Thus “equate the derivative of the function to zero” rule does not help to see around which points on its domain the value of the function is “almost” equal to its supremum value. A natural necessary condition for this, you may think, is the derivative of the function being close to zero. This intuition, which is easily formalized, is quite right indeed, but giving a proof for it is surprisingly difficult. Well, this is so, unless you're endowed with the Ekeland Variational Principle.

Proposition 1. *Let f be a differentiable self-map on \mathbb{R} that is bounded from above. Then, there exists a real sequence (y_m) such that*

$$f(y_m) \rightarrow \sup f(\mathbb{R}) \quad \text{and} \quad f'(y_m) \rightarrow 0.$$

²This condition can be generalized to the following: There exists a $\theta > 0$ such that for every $x \in X$ with $\varphi(x) < \sup \varphi(X)$, there exists a $z \in X \setminus \{x\}$ such that $\varphi(z) > \varphi(x) + \theta d(x, z)$. (The argument that follows would settle this generalization as well.)

Proof. Let $\alpha := \sup f(\mathbb{R})$, which is a real number by hypothesis. By the Ekeland Variational Principle and Remark 1, for every positive integer m , there exists some $y_m \in X$ such that

- (i) $f(y_m) \geq f(t) - \frac{1}{m}|t - y_m|$ for all $t \in \mathbb{R}$; and
- (ii) $|y_m - z_m| < 1$ for some $(z_m) \in \mathbb{R}^\infty$ with $f(z_m) > \alpha - \frac{1}{m}$.

Fix any $m \in \mathbb{N}$, and observe that (i) implies, for any $t > y_m$,

$$\frac{f(t) - f(y_m)}{t - y_m} \leq \frac{1}{m},$$

so letting $t \searrow y_m$, we find $f'_+(y_m) \leq \frac{1}{m}$. A similar method shows that $f'_-(y_m) \geq -\frac{1}{m}$, so it follows from the differentiability of f that $|f'(y_m)| \leq \frac{1}{m}$. Letting $m \rightarrow \infty$, therefore, we find $f'(y_m) \rightarrow 0$. Furthermore, by using (i) and (ii) together,

$$\alpha \geq f(y_m) \geq f(z_m) - \frac{1}{m}|z_m - y_m| > \alpha - \frac{2}{m}.$$

Letting $m \rightarrow \infty$, then, we find $f(y_m) \rightarrow \alpha$. ■

It is not difficult to show that this result generalizes to differentiable functions defined on \mathbb{R}^n ; all we need to do in that case is to apply the arguments with respect to the partial derivatives of the function at hand. In the addendum to Chapter K, in fact, I will prove something much stronger than this.

References

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