The Estimation of Dynamic Discrete Games

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Setup for a dynamic discrete choice game: choices $a_{it} \in A$

- static payoff (AS): $\pi_i(a_t, x_t) + \epsilon_{it}$), ϵ_{it} is **private** shock and distributed (IID) across players/overtime $p(\epsilon_t) = \Pi_{i=1}^N g_i(\epsilon_{it})$
- transition probability (CIX): $p(x_{t+1}, \epsilon_{t+1} | a_t, x_t, \epsilon_t) = f(x_{t+1} | a_t, x_t) p_\epsilon(\epsilon_{t+1})$
- define a set of stationary strategies $\sigma = \{\sigma_i(x, \epsilon_i)\}$ and its associated CCP P^σ such that:

$$P_i^\sigma(a_i | x) = \int I(\sigma_i(x, \epsilon_i) = a_i) g_i(\epsilon_i) d\epsilon_i$$

Firm's optimization problem

- Firmm's optimization problem, given all other firms follow σ:

$$V_i^\sigma(x,\epsilon_i) = max_{a_i}\{\pi^\sigma(a_i,x) + \epsilon_i(a_i) + \beta\sum_{x'}[\int V_i^\sigma(x',\epsilon_i')g_i(\epsilon_i')d\epsilon_i']f_i^\sigma(x'|x,a_i)\}$$

$$f_i^\sigma(x'|x,a_i) = \sum_{a_{-i}}(\prod_{j\neq i} P_j^\sigma(a_{-i}|x))f(x'|x,a_i,a_{-i})$$

- As in the single agent case, easy to define Integrated value function and Choice-specific value function (which depends on Ivf):

$$V_i^\sigma(x) = \int V_i^\sigma(x,\epsilon_i)g_i(d\epsilon_i) = \int max\{v_i^\sigma(a_i,x) + \epsilon_i(a_i)\}g_i(d\epsilon_i)$$

$$v_i^\sigma(a_i,x) \equiv \pi^\sigma(a_i,x) + \beta\sum_{x'} V_i^\sigma(x')f_i^\sigma(x'|x,a_i)$$

Define MPE

- $\sigma*$ is a set of MPE strategies if for any firm i and any (x, ϵ_i)

$$\sigma^*(x, \epsilon_i) = argmax_{a_i}\{v_i^{\sigma^*}(a_i, x) + \epsilon_i(a_i)\}$$

- alternatively, can also define in the associated probability space

$$P^* = \Lambda(P^*) = \{\Lambda_i(a_i|x; P^*_{-i})\}$$

$$\Lambda_i(a_i|x; P^*_{-i}) = \int I(a_i = argmax_{a'}\{v_i^{P^*}(a', x) + \epsilon_i(a')\})g_i(\epsilon_i)d\epsilon_i$$

Given **equilibrium** P^* (Best Response), there is an easier way to evaluate IVF

- We rewrite the IVF as:

$$V_i^{P^*}(x) = \sum_{a_i} P_i^*(a_i|x)[\pi^{P^*}(a_i,x) + e_i^{P^*}(a_i,x)] + \beta \sum_{x'} V_i^{P^*}(x') f_i^{P^*}(x'|x)$$

$$f_i^{P^*}(x'|x) = \sum_{a_i} P_i^*(a_i|x) f_i^{P^*}(x'|x,a_i) \equiv \sum_{a \in A^N} \prod_j P_j^*(a_j|x) f(x'|x,a)$$

- $e_i^{P^*}(a_i,x)$ is in general complex, but under IID extreme value (α)

$$e_i^{P^*}(a_i,x) = Euler'scons. - \alpha ln(P_i^*(a_i|x))$$

- Finally, we can then **calculate** a linear equation system

$$(I - \beta F^{P^*})V_i^{P^*} = \sum_{a_i} P^*(a_i)[\pi^{P^*}(a_i) + e_i^{P^*}(a_i)]$$

Then use this mapping to define a pseudo likelihood estimator

- Define $\Psi_i(a_{imt}|x_{mt}; P, \theta)$ is the choice probability derived from IVF associated with P.
- If we assume we can obtain a consistent estimator of P^*, $\hat{P}$. Then we can define a two-step estimator:

$$\hat{\theta} \equiv argmax_{\theta} \sum_{t=1}^{t} \sum_{i=1}^{N} ln\Psi_i(a_{imt}|x_{mt}; \hat{P}, \theta)$$

- Share the same insight as AM (2007)
- Argue that GMM estimator can reduce finite sample bias
- Define it as:

$$\sum_{t=1}^{T}\sum_{i=1}^{N} H_i(x_{mt})[I(a_{imt}=a)-\Psi_i(a|x_{mt};\hat{P},\theta)]=0$$

...

- Another difference is that they argue $f^{P^*}(x'|x)$ can also be estimated directly from data, instead of being constructed using $\sum_{a\in A^N}\prod_j P_j^*(a_j|x)f(x'|x,a)$.