Chapter 6

Theory of Utility Representation

This chapter is concerned with the problem of order-embedding a given preordered set in $(\mathbb{R}, \geq)$ or in $(\mathbb{R}^n, \geq)$ (or in another naturally ordered function space). In the lexicon of individual decision theory, this problem is known as the "ordinal utility representation problem." As the bulk of work in this area is in fact done within decision theory, we shall adopt this terminology here as well.

There are mainly two approaches toward the utility representation problem. The first one is order-theoretic, and aims at identifying those complete (or incomplete) preorders that can be order-embeded in $\mathbb{R}$ (or $\mathbb{R}^n$) by means of purely order-theoretic conditions. The second is topological. In that approach the ground set of the preorder is given some topological structure, and the involved preorders are assumed to satisfy suitable continuity conditions. In return, we ask to find continuous (or at least semicontinuous) order embeddings (utility functions). We shall outline the elements of both of these approaches below.

1. Preliminaries

1.1 Order-Preserving Real Functions

The bulk of this chapter is concerned with those order-preserving functions that map a preordered set into the loset $(\mathbb{R}, \geq)$. (As usual, we denote the latter loset simply as $\mathbb{R}$ below.) To conform with the related literature, we shall rename such functions as follows:

Definition. Let $(X, \succsim)$ be a preordered set. We say that a real function f on X is **$\succsim$-increasing** if

$$x \succsim y \quad \text{implies} \quad f(x) \geq f(y)$$

for every $x, y \in X$. This function is said to be **strictly $\succsim$-increasing**, if it is $\succsim$-increasing and

$$x \succ y \quad \text{implies} \quad f(x) > f(y)$$

for every $x, y \in X$.

Given a preordered set $(X, \succsim)$, and a function $f : X \to \mathbb{R}$, the notion of being $\succsim$-increasing is none other than being order-preserving. However, the notion of being strictly $\succsim$-increasing is new. In particular, it is weaker than being an order-embedding. While it is readily checked that if f is an order-embedding, then it must be strictly $\succsim$-increasing, the converse is false. For instance, the real map

$$\mathbf{x} \mapsto x_1 + x_2$$

on $\mathbb{R}^2$ is strictly $\geq$-increasing, but it is far from being an order-embedding. Thus, in general, we have

$$\boxed{\text{order-embedding}} \Longrightarrow \boxed{\text{strictly } \succsim\text{-increasing map}} \Longrightarrow \boxed{\succsim\text{-increasing map}}$$

for any real map defined on a preordered set $(X, \succsim)$. Nonetheless, it is plain that the discrepancy between the former two monotonicity notions disappear when $\succsim$ is a linear order. That is:

$$\boxed{\text{order-embedding}} \Longleftrightarrow \boxed{\text{strictly } \succsim\text{-increasing map}} \Longrightarrow \boxed{\succsim\text{-increasing map}}$$

for any real map defined on a loset $(X, \succsim)$.

1.2 Losets and $\mathbb{Q}$

The following is a simple, but very useful, result of order theory. It was proved in 1895 by Georg Cantor during the course of his work on the development of transfinite numbers. In words, it says that every countable loset can be order-embedded in the set of rational numbers between 0 and 1 (and hence in $\mathbb{Q}$).

Proposition 1.2.1. Let $(X, \succcurlyeq)$ be a loset. If X is countable, then there exists a strictly $\succcurlyeq$-increasing real function from X into $\mathbb{Q} \cap (0, 1)$.

Proof. The claim is trivial when X is finite, so we assume that X is countably infinite. Let us enumerate X and $\mathbb{Q} \cap (0, 1)$ as

$$X = \{x_1, x_2, ...\} \quad \text{and} \quad \mathbb{Q} \cap (0, 1) = \{r_1, r_2, ...\}.$$

We construct the function f from X into $\mathbb{Q}\cap(0,1)$ as follows. First, let $f(x_1) := r_1$. Second, if $x_1 \succcurlyeq x_2$ (or $x_2 \succcurlyeq x_1$), set $f(x_2)$ as the first enumerated element in $\{r_2, ...\}$ such that $r_1 \geq f(x_2)$ ($f(x_2) \geq r_1$, resp.). Proceeding inductively, for any $m = 2, 3, ...$, set

$$\begin{aligned} f(x_m) := \ & \text{the first element of } \{r_1, ...\}\backslash\{f(x_1), ..., f(x_{m-1})\} \\ & \text{which has the same order relation (w.r.t. } \geq \text{) to the numbers } f(x_1), ..., f(x_{m-1}) \\ & \text{as } x_m \text{ has to the elements } x_1, ..., x_{m-1} \text{ (w.r.t. } \succcurlyeq\text{).} \end{aligned}$$

Since there is a rational number strictly between any two distinct rational numbers, this well-defines $f : X \to \mathbb{Q}\cap(0,1)$. Moreover, it follows readily from our construction that, for every x and y in X, we have $f(x) \geq f(y)$ iff $x \succcurlyeq y$.

Let us now pose the following question: Which losets are order-isomorphic to $\mathbb{Q}$? Certainly, not all losets are up to the task. First of all, it is obvious that a finite loset cannot be order-isomorphic to $\mathbb{Q}$ (as there cannot be a bijection between these two sets). A countably infinite loset does not do the job necessarily either. For instance, the losets $(\mathbb{N}, \geq)$ and $(\mathbb{Q}, \geq)$ are not order-isomorphic, as the first one has a minimum element and the latter does not. Neither would the absence of extremum elements solve the problem. Indeed, $(\mathbb{Z}, \geq)$ and $(\mathbb{Q}, \geq)$ are not order-isomorphic. After all, $(\mathbb{Q}, \geq)$ has the property that for any distinct elements p and q in $\mathbb{Q}$, there is an element r in $\mathbb{Q}$ with $p > q > r$. Obviously, $(\mathbb{Z}, \geq)$ does not possess this order-theoretic property, and hence it cannot be order-isomorphic to $(\mathbb{Q}, \geq)$. Put succinctly, $(\mathbb{Q}, \geq)$ is *order-dense* while $(\mathbb{Z}, \geq)$ is not.

Definition. Let $(X, \succsim)$ be a preordered set and $Z \subseteq X$. If, for any x and y in X such that $x \succ y$, there exists an element z of Z such that $x \succ z \succ y$, we say that Z is **$\succsim$-dense in X**. If $Z = X$ here, we simply say that X is **$\succsim$-dense**.

> **Note.** Order-denseness is an order-theoretic property in the sense that a preordered set $(X, \succsim)$ satisfies it iff every preordered set that is order-isomorphic to $(X, \succsim)$ also satisfies it.

It turns out that the absence of extremum elements and order-denseness jointly characterize those countable losets that are order-isomorphic to $(\mathbb{Q}, \geq)$. This is, again, a result due to Georg Cantor.

Proposition 1.2.2. Let $(X, \succcurlyeq)$ be a loset with no $\succcurlyeq$-maximum and no $\succcurlyeq$-minimum elements. If X is countable and $\succcurlyeq$-dense, then $(X, \succcurlyeq)$ is order-isomorphic to $(\mathbb{Q}, \geq)$.

As the proof of this result parallels that of Proposition 1.2.1, we leave it as an exercise here.

1.3 The Utility Representation Problem

In Section 3.2 of Chapter 1, we have mentioned that a "preference relation" on a set X of choice alternatives is defined as a preorder on X in economics, and it is assumed to contain all the information that concerns how an agent compares any two alternatives according to her tastes.[1] Unfortunately, this preorder is often not the most convenient way of summarizing this information. Indeed, maximizing a binary relation (while a well-defined matter) is a much less friendlier exercise than maximizing a real function. Thus, it would be quite useful if we knew how and when one can find a real function that attaches to an alternative x a (strictly) higher value than an alternative y iff x is ranked (strictly) above y by a given preference relation. In economics, such a function is called the *utility function* of the individual who possesses this preference relation. A fundamental question in the theory of individual choice is therefore the following: What sort of preference relations can be described by means of a utility function?

We begin by formally defining what it means to "describe a preference relation by means of a utility function."

Definition. Let $\succsim$ be a preference relation on a nonempty set X. We say that a function $u : X \to \mathbb{R}$ **represents** $\succsim$ provided that

$$x \succsim y \quad \text{if and only if} \quad u(x) \geq u(y)$$

for every x and y in X. If such a function exists, we say that $\succsim$ admits a **utility representation**, and refer to u as a **utility function** for $\succsim$.

Note. In order-theoretic terms, the phrase "u is a utility function for the preference relation $\succsim$," means simply that u is an order-embedding from the preordered set $(X, \succsim)$ into the loset $(\mathbb{R}, \geq)$. Consequently, having a utility function is an order-theoretic property (that is, it is preserved under any order-isomorphism).

In the good part of this chapter, we shall be concerned with identifying those preference relations that admit a utility representation. Let us begin by noting that a utility function for a preference relation is by no means unique. Rather, it is unique up to

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[1] There are more sophisticated notions of preference relations that stem from the theory of boundedly rational choice. We shall not deal with such preferences in this text, however.

strictly increasing transformations. Indeed, if $\succsim$ is a preference relation on a nonempty set X, and $u : X \to \mathbb{R}$ is a utility function for $\succsim$, then, obviously, $f \circ u$ is also a utility function for $\succsim$, where f is any strictly increasing real function on $u(X)$. Less obvious is the fact that the converse of this is also true:

Proposition 1.3.1. Let $(X, \succsim)$ be a preordered set, and u and v two real functions on X. Then, u and v are both utility functions for $\succsim$ if, and only if, there exists a strictly increasing real function f on $u(X)$ such that $v = f \circ u$.

We leave the proof of this result as an exercise.

Remark. The "if" part of Proposition 1.3.1 shows that we can always represent a preference relation $\succsim$ by a *bounded* utility function, provided that this preference relation admits a utility representation. Indeed, if u is a utility function for $\succsim$, then so is the map

$$\arctan u,$$

because $x \mapsto \arctan x$ is a strictly increasing map from $\mathbb{R}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2})$. In fact, we can control the length of the range of an ordinal utility function anyway we wish. After all, for any real numbers with $a < b$, the map

$$\left(\frac{b-a}{\pi}\right) \arctan u + \frac{a+b}{2}$$

is a utility function for $\succsim$ whose range is contained in the interval (a, b).

Let us now turn to the problem of existence of a utility representation. In fact, thanks to Proposition 1.2.1, we already have in our disposal an important result in this regard. The following is deduced easily from Proposition 1.2.1 by the method of "passing to the quotient."

Proposition 1.3.2. There is a utility function $u : X \to (0, 1)$ for any complete preference relation $\succsim$ on a nonempty countable set X.

Proof. Clearly, the symmetric part $\sim$ of $\succsim$ is an equivalence relation. Let $[x]_\sim := \{z \in X : x \sim z\}$ for any $x \in X$, and define the quotient set

$$X/_\sim := \{[x]_\sim : x \in X\}.$$

As we have noted in Proposition 2.1 of Chapter 1, $X/_{\sim}$ is a partition of X.

Now, define the linear order $\succcurlyeq$ on $X/_{\sim}$ as $[x]_{\sim} \succcurlyeq [y]_{\sim}$ iff $x \succsim y$. By Proposition 1.2.1, there exists a strictly $\succcurlyeq$-increasing function f from $X/_{\sim}$ into $(0,1)$. But then $u : X \to (0,1)$, defined by $u(x) := f([x]_{\sim})$, is a utility function for $\succsim$.

Unfortunately, the countability requirement cannot be dismissed in Proposition 1.3.2. We now present some illustrations of this fact.[2]

Example 1.3.1. Consider the linear order $\succcurlyeq_{\text{lex}}$ on $\mathbb{R}^2$ defined as $x \succcurlyeq_{\text{lex}} y$ iff either $x_1 > y_1$, or $x_1 = y_1$ and $x_2 \geq y_2$. This linear order, which is called the **lexicographic order**, cannot be represented by a utility function. For, suppose $u : \mathbb{R}^2 \to \mathbb{R}$ represents $\succcurlyeq_{\text{lex}}$. Then, for any real number a, we have

$$u(a, a+1) > u(a, a)$$

so that $I(a) := (u(a,a), u(a,a+1))$ is a nondegenerate interval. Moreover, $I(a) \cap I(b) = \emptyset$ for any distinct numbers a and b, because

$$u(b,b) > u(a,a+1) \text{ whenever } b > a,$$

and

$$u(b,b+1) < u(a,a) \text{ whenever } b < a.$$

Therefore, the map $a \mapsto I(a)$ is an injection from $\mathbb{R}$ into $\{I(a) : a \in \mathbb{R}\}$. But since $\{I(a) : a \in \mathbb{R}\}$ is countable, this entails that $\mathbb{R}$ is countable, a contradiction.

Example 1.3.2. Let $\succcurlyeq$ be a well-ordering on $\mathbb{R}$ – recall Section 2.5 of Chapter 3. For any real number x, define

$$\Lambda(x) := \{\omega \in \mathbb{R} : x \succ \omega\}$$

and let

$$S := \{x \in \mathbb{R} : \Lambda(x) \text{ is uncountable}\}.$$

Then, set $X := \mathbb{R}$ if $S = \emptyset$, and

$$X := \Lambda(\text{the } \succcurlyeq\text{-minimum element of } S),$$

if S is nonempty. We wish to show that the restriction of $\succcurlyeq$ to X, which we shall denote again by $\succcurlyeq$ for convenience, does not admit a utility representation.

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[2] See Beardon *et al.* (2002) for a systematic analysis of complete preferences that do not admit utility representations.

Let us first make some observations on X. Obviously, this set is uncountable. On the other hand, by definition of X, $\Lambda(x)$ is countable for every x in X. Furthermore, as $\succcurlyeq$ is transitive, we have $\Lambda(x) \subseteq X$ for every $x \in X$. Conversely, if y is a real number that does not belong to $\bigcup\{\Lambda(x) : x \in X\}$, then $y \succ X\backslash\{y\}$, that is, $X\backslash\{y\} \subseteq \Lambda(y)$, which implies that $\Lambda(y)$ is uncountable, thereby showing that y cannot belong to X. Taking stock: X is an uncountable set such that $\{\Lambda(x) : x \in X\}$ is countable for each $x \in X$, and

$$X = \bigcup\{\Lambda(x) : x \in X\}.$$

Now, to derive a contradiction, let us assume that there is a utility function $u : X \to \mathbb{R}$ that represents $\succcurlyeq$. Without loss of generality, we may assume that u is bounded. Let $\alpha := \sup u(\mathbb{R})$. Then, for every rational number r with $\alpha > r$, there exists a real number x such that $\alpha > u(x) > r$, and hence

$$\{\omega \in \mathbb{R} : r > u(\omega)\} \subseteq \{\omega \in \mathbb{R} : u(x) > u(\omega)\} = \Lambda(x).$$

(The last equality follows from the hypothesis that u represents $\succcurlyeq$.) Thus, $\{\omega \in \mathbb{R} : r > u(\omega)\}$ is a countable set for every rational number r with $\alpha > r$. And yet, for every real number x,

$$\Lambda(x) = \{\omega \in \mathbb{R} : u(x) > u(\omega)\} \subseteq \bigcup_{\alpha > r \in \mathbb{Q}} \{\omega \in \mathbb{R} : r > u(\omega)\},$$

(because $\mathbb{Q}$ is $\geq$-dense in $\mathbb{R}$), and hence

$$X = \bigcup\{\Lambda(x) : x \in X\} \subseteq \bigcup_{\alpha > r \in \mathbb{Q}} \{\omega \in \mathbb{R} : r > u(\omega)\}.$$

It follows that X is contained in the union of a countable collection of countable sets, and hence it is countable, a contradiction.

Just in case these examples may strike you as too abstract, we give two further illustrations of preference relations that cannot be represented by a utility function, which arise in economic analysis. The following example is due to Dubra and Echenique (2003), and concerns the representation of preferences over information structures.

Example 1.3.3. Let $\mathcal{P}$ denote the collection of all partitions of $[0, 1]$. (In the context of information theory, any one member of $\mathcal{P}$ is thought of as modeling the information that an agent will have about the values of a given $[0, 1]$-valued random variable, when the value of this variable is realized.) By Szpilrajn's Theorem (or better, by Corollary 3.1.1 of Chapter 3), there exists a complete preference relation $\succsim$ on $\mathcal{P}$ such that $A \subset B$

implies $B \succ A$. (This preference relation ranks finer partitions (more information) higher than coarser ones (less information).) You are asked to prove below that this preference relation cannot be represented by a utility function.

The following example is due to Basu and Mitra (2003), and it concerns the representation of a (social) preference relation over intergenerational income streams in a way that is both egalitarian and efficient.

Example 1.3.4. Let $X := \{0,1\}^\infty$, the set of all sequences whose terms are either 0 or 1. Let $\succsim$ be a complete preference relation on X such that

- $(x_m) \succ (y_m)$ for any distinct $(x_m), (y_m) \in X$ with $x_m \geq y_m$ for each m; and
- $(x_m) \sim (y_m)$ for any $(x_m), (y_m) \in X$ such that there exist two positive integers i and j with $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for every positive integer k other than i and j.

(Think of any one member of X as an intergenerational wealth stream of a society (in which there are two possible levels of income, low and high), and $\succsim$ as a social welfare ordering. Then, this preference relation favors *efficiency* in the sense that it ranks a stream above another if every generation receives at least as much wealth in the former stream as they do in the latter, with at least one generation receiving strictly more. It is also *equitable* in the sense that it treats any two distinct generations equally in terms of social welfare.) It is not difficult to show that such a complete preference relation $\succsim$ exists by using Szpilrajn's Theorem. You are asked to prove below that no such preference relation admits a utility function.

Exercises

1.1. Give an example of a countable and order-dense loset which is not order-isomorphic to $(\mathbb{Q}, \geq)$.

1.2. Prove Proposition 1.2.2.

1.3. Prove Proposition 1.3.1.

1.4. Let X be a nonempty set and $\succsim$ a complete preference relation on X that admits a utility representation. Let S be a nonempty subset of X, and suppose that $u : S \to \mathbb{R}$ is a utility function for $\succsim \cap (S \times S)$. Does there necessarily exist a utility function $U : X \to \mathbb{R}$ for $\succsim$ such that $U|_S = u$?

1.5. Prove the assertion made in Example 1.3.3.

1.6. Prove the assertion made in Example 1.3.4.

2. Representation through Order-Separability

While its countability requirement cannot be relaxed entirely, we can replace this condition by various other properties in Proposition 1.3.2. In particular, it turns out that the existence of a countable order-dense subset is all one needs for guaranteeing the existence of a utility representation.

Proposition 2.1. Let X be a nonempty set, and $\succsim$ a complete preference relation on X. If X contains a countable $\succsim$-dense set, then $\succsim$ can be represented by a utility function.

Proof. If $\succ = \emptyset$, then it is enough to take u as any constant function, so we may assume $\succ \neq \emptyset$ to concentrate on the nontrivial case. Assume that there is a countable $\succsim$-dense set Z in X. By Proposition 1.3.2, there exists a function $v : Z \to (0,1)$ that represents $\succsim \cap (Z \times Z)$.

Now take any $x \in X$ such that $x \succ \omega$ for at least one ω in X, and define

$$\alpha_x := \sup\{v(\omega) : x \succ \omega \in Z\}.$$

As Z is $\succsim$-dense in X, and the range of v is contained in the interval $(0,1)$, α_x is a well-defined real number in $[0,1]$ for each $x \in X$ such that $x \succ \omega$ for at least one ω in X. Define next the function $u : X \to [0,1]$ as follows:

$$u(x) := \begin{cases} 1, & \text{if } \omega \succ x \text{ for no } \omega \in X \\ 0, & \text{if } x \succ \omega \text{ for no } \omega \in X \\ \alpha_x, & \text{otherwise.} \end{cases}$$

$\succ \neq \emptyset$ implies that u is well-defined. The rest of the proof consists of verifying that u actually represents $\succsim$ on X.

Take any $x, y \in X$ with $x \succ y$. We wish to show that $u(x) > u(y)$. First, observe that there exist $z_1, z_2 \in Z$ such that $x \succ z_1 \succ z_2 \succ y$ (because Z is $\succsim$-dense in X). Since $x \succ z_1 \in Z$ and v represents $\succsim \cap (Z \times Z)$, we have

$$u(x) = \alpha_x \geq v(z_1) > v(z_2).$$

On the other hand, if $y \succ \omega$ for no $\omega \in X$, we obviously have $v(z_2) \geq 0 = u(y)$, and if $y \succ \omega$ for some $\omega \in X$, then $v(z_2) > v(\omega)$ for every $\omega \in Z$ with $y \succ \omega$, and hence, again, $v(z_2) \geq \alpha_y = u(y)$. In all contingencies, then, $v(z_2) \geq u(y)$. Combining these observations yields $u(x) > u(y)$. As it is similarly checked that $x \sim y$ implies $u(x) = u(y)$, we may conclude that u is a utility function for $\succsim$.[3]

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[3] See the proof of Proposition 2.2 for an alternative argument that would also yield this result.

It is important to note that Proposition 2.1 does not generalize Proposition 1.3.2. (For instance, the latter result applies to $(\mathbb{N}, \geq)$, while $\mathbb{N}$ does not contain a $\geq$-dense set.) However, it is possible to weaken the order-separability requirement we used in Proposition 2.1 to a condition that is both necessary and sufficient for the representability of $\succsim$. This is a result due to Garett Birkhoff.

Birkhoff's Representation Theorem. Let X be a nonempty set, and $\succsim$ a complete preference relation on X. Then, $\succsim$ admits a utility representation if, and only if, there exists a countable subset Z of X such that for every x and y in $X\backslash Z$ with $x \succ y$, we have

$$x \succ z \succ y \quad \text{for some } z \in Z.$$

Proof. In what follows, we shall first assume that $\succsim$ is antisymmetric, and then relax this hypothesis. The "if" part of our assertion can be proved by an adaptation of the argument we gave for Proposition 2.1; we leave this part as an exercise. To see its "only if" part, assume that $u : X \to \mathbb{R}$ is a utility function for $\succsim$. Let $\mathcal{I}$ stand for the collection of all closed and bounded intervals with distinct rational endpoints. For each $I \in \mathcal{I}$, we use the Axiom of Choice to pick an element ω_I of X with $u(\omega_I) \in I$, provided that there exists at least one such element in X. Define

$$S := \{\omega_I : I \in \mathcal{I}\},$$

which is a countable set (because so is $\mathcal{I}$). Finally, we let T stand for the collection of all ordered pairs (x, y) such that

- $x, y \in X\backslash S$;
- $x \succ y$; and
- $x \succ \omega \succ y$ for no $\omega \in S$.

The key observation is the following:

Claim. If $(x, y) \in T$, then $x \succ \omega \succ y$ for no $\omega \in X$.

Proof. Suppose $x \succ \omega \succ y$ holds for some $\omega \in X$. We wish to show that (x, y) does not belong to T. First, note that we have $u(x) > u(\omega) > u(y)$, so we can choose two rational numbers a and b such that

$$u(x) > b > u(\omega) > a > u(y).$$

Then, by definition, $\omega_{[a,b]}$ belongs to the interval $[a,b]$, and hence,

$$u(x) > u(\omega_{[a,b]}) > u(y).$$

This means that $x \succ \omega_{[a,b]} \succ y$, while, of course, $\omega_{[a,b]}$ belongs to S. By definition of T, therefore, (x,y) does not belong to T.

This claim entails that the collection $\{(u(y),u(x)) : (x,y) \in T\}$ is countable, because there cannot be uncountably many disjoint non-degenerate intervals in $\mathbb{R}$. But, given the hypothesis that $\succsim$ is antisymmetric, $(x,y) \mapsto (u(y),u(x))$ is a bijection from T onto this collection. We may thus conclude that T is countable as well. As we already know that S is countable, therefore,

$$Z := S \cup \{x \in X : \text{either } (\omega,x) \in T \text{ or } (x,\omega) \in T \text{ for some } \omega \in X\}$$

is a countable subset of X.

To complete the argument, take any x and y in $X \backslash Z$ with $x \succ y$. As x does not belong to Z, neither (ω,x) nor (x,ω) belongs to T for any $\omega \in X$. Setting ω as y here, then, we find that (x,y) is not in T. Since neither x nor y is in Z, both are in $X \backslash S$ while $x \succ y$. By definition of T, therefore, $x \succ \omega \succ y$ for some $\omega \in S$. As $S \subseteq Z$, we are done.

We have proved Birkhoff's Representation Theorem in the case where $\succsim$ is a linear order on X. The general case is settled by the method of "passing to the quotient" (as in Proposition 1.3.2).

There are many other ways of characterizing complete preference relations that admit utility representations. The following is, for instance, occasionally useful

Proposition 2.2. Let X be a nonempty set, and $\succsim$ a complete preference relation on X. Then, $\succsim$ admits a utility representation if, and only if, there exists a countable subset Z of X such that for every x and y in X with $x \succ y$, we have

$$x \succsim z \succsim y \quad \text{for some } z \in Z. \tag{1}$$

Proof. The "only if" part of the assertion follows readily from Birkhoff's Representation Theorem. To prove its "if" part, assume that there is a countable subset Z of X such that (1) holds for every x and y in X with $x \succ y$. let us enumerate Z as $\{z_1, z_2, ...\}$, and define

$$M(x) := \{i \in \mathbb{N} : x \succ z_i\}.$$

Notice that the transitivity of $\succsim$ implies $x \succsim y$ iff $\Lambda(x) \supseteq \Lambda(y)$, and hence,

$$x \succsim y \text{ implies } M(x) \supseteq M(y),$$

for every x and y in X. Furthermore, if $x \succ y$, then (1) entails that there is a z in $M(x)\backslash M(y)$, that is,

$$x \succ y \text{ implies } M(x) \supset M(y),$$

for every x and y in X. Consequently, the map $u : X \to [0, 1]$, defined by

$$u(x) := \sum_{i \in M(x)} 2^{-i},$$

is strictly $\succsim$-increasing, and we are done.

The following example illustrates the power of this fact.

Example 2.1. Let $X := \{0, 1\}^{\infty}$, and take any complete preference relation $\succsim$ on X such that the two (efficiency and fairness) properties mentioned in Example 1.3.4 are satisfied. We wish to show that $\succsim$ does not admit a utility representation. Let Z be a subset of X such that for every x and y in X with $(x_m) \succ (y_m)$, we have

$$(x_m) \succsim (z_m) \succsim (y_m) \quad \text{for some } (z_m) \in Z.$$

By Proposition 2.2, it is enough to show that Z is not countable to establish our assertion.

Let $\{r_1, r_2, ...\}$ be an enumeration of $\mathbb{Q}$. For each real number a and positive integer m, we define

$$y_m(a) := \begin{cases} 1, & \text{if } a > r_i \\ 0, & \text{otherwise,} \end{cases}$$

and

$$x_m(a) := \begin{cases} 1, & \text{if } m = \min\{i : x_i(a) = 0\} \\ y_m(a), & \text{otherwise.} \end{cases}$$

Notice that, for any real number a and positive integer m, the first (efficiency) property of $\succsim$ implies $(x_m(a)) \succ (y_m(a))$, and hence there is a $z_m(a) \in Z$ such that

$$(x_m(a)) \succsim (z_m(a)) \succsim (y_m(a)).$$

On the other hand, for any real numbers a and b with $b > a$, the set $\{i : y_m(a) = 1\}$ is a subset of $\{i : y_m(b) = 1\}$, and there are infinitely many integers i that belong to the latter set but not to the former. By the two properties of $\succsim$, therefore, we must have $(y_m(b)) \succ (x_m(a))$ whenever $b > a$. (Why?) It follows that

$$(z_m(b)) \succsim (y_m(b)) \succ (x_m(a)) \succsim (z_m(a))$$

for any real numbers a and b with $b > a$. This means that $t \mapsto (z_m(t))$ is an injection from $\mathbb{R}$ into Z. Thus: Z is not countable.

Exercises

2.1. Prove the "if" part of Birkhoff's Representation Theorem in the case where $\succsim$ is a linear order on X.

2.2. Use the method of "passing to the quotient" to deduce Birkhoff's Representation Theorem from its special case in which we assume that $\succsim$ is a linear order on X.

2.3. Use Birkhoff's Representation Theorem to prove that the lexicographic order on $\mathbb{R}^2$ does not admit a utility representation.

2.4. Use Proposition 2.2 to prove the assertion made in Example 1.3.3.

2.5. (Jaffray) Let X be a nonempty set, and $\succsim$ a complete preference relation on X. Prove: $\succsim$ admits a utility representation iff there exists a countable subset Z of X such that for every x and y in X with $x \succ y$, we have

$$x \succsim z \succ w \succsim y \qquad \text{for some } z, w \in Z.$$

3. Representation through Semicontinuity

For a complicated prerodered set $(X, \succsim)$, it may be quite difficult to check if there is a countable $\succsim$-dense set in X, so neither Proposition 2.1 nor Birkhoff's Representation Theorem may be of immediate use in determining whether $\succsim$ admits a utility representation or not. In most applications, however, X has some additional topological structure, and $\succsim$ satisfies a suitable continuity condition (that links the order structure of $\succsim$ to the topological structure of X). In such situations, and when the topology of X is sufficiently well-behaved, we may approach to the utility representation problem by means of a different approach.

As a major example of what one can achieve by this approach we shall prove below that every semicontinuous complete preference relation – recall Section 4.1 of Chapter 3 – admits a utility representation, provided that it is defined on a separable metric space, or more generally, on a topological space with countable basis. (See Section 1.2 of Appendix.) This was proved first by Trout Rader in 1963.

Proposition 3.1. Let X be a topological space with a countable basis and $\succsim$ a complete preference relation on X. If $\succsim$ is upper (or lower) semicontinuous, then it can be represented by a utility function.

Proof. Let us assume first that $\succsim$ is upper semicontinuous. We begin by noting that, as X is a topological space with a countable basis, there exists a countable collection $\mathcal{O}$ of open subsets of X such that $U = \bigcup\{O \in \mathcal{O} : O \subseteq U\}$ for every open set U in X. (See Section 1.2 of Appendix.) This fact allows us to adapt the argument we gave in the proof of Proposition 2.2 to the present setting.

Enumerate $\mathcal{O}$ as $\{O_1, O_2, ...\}$, and define

$$\Lambda(x) := \{\omega \in X : x \succ \omega\}$$

and

$$M(x) := \{i \in \mathbb{N} : O_i \subseteq \Lambda(x)\}$$

for each x in X. Then, by transitivity of $\succsim$, we have $x \succsim y$ iff $\Lambda(x) \supseteq \Lambda(y)$, and hence,

$$x \succsim y \ \text{ implies } \ M(x) \supseteq M(y),$$

for every x and y in X. Furthermore, because $\succsim$ is a complete and upper semicontinuous preference relation, $\Lambda(x)$ is an open subset of X, and hence,

$$\Lambda(x) = \bigcup\{O_i : i \in M(x)\} \quad \text{for every } x \in X.$$

It follows that

$$x \succ y \ \text{ implies } \ M(x) \supset M(y)$$

for every x and y in X. Consequently, the map $u : X \to [0, 1]$, defined by

$$u(x) := \sum_{i \in M(x)} 2^{-i},$$

is strictly $\succsim$-increasing, and we are done.

Next, assume that $\succsim$ is lower semicontinuous. Then, applying what we have just established to the preordered set $(X, \precsim)$, we find an upper semicontinuous utility function u for $\precsim$. But then $-u$ is a lower semicontinuous utility function for $\succsim$, and our proof is complete.

One can in fact show that an upper semicontinuous and complete preference relation $\succsim$ on a topological space X with a countable basis can be represented by an upper semicontinuous utility function. This is particularly easy when $\succsim$ is antisymmetric, for then $\limsup u$, where u is as found in Proposition 3.1, can be shown to be such a utility function for $\succsim$. To settle the general case, however, one needs a nontrivial modification of the method of "passing to the quotient."

Rader's Representation Theorem. Let X be a topological space with a countable basis and $\succsim$ a complete preference relation on X. If $\succsim$ is upper (lower) semicontinuous, then it can be represented by an upper (lower) semicontinuous utility function.

Proof. In view of the duality argument given in the last paragraph of Proposition 3.1, it is enough to establish the assertion in the case where $\succsim$ is upper semicontinuous. In that case, we begin by applying Proposition 3.1 to obtain a utility function u for $\succsim$. Without loss of generality, we assume that u is bounded. Next, we define the quotient set $X/_{\sim}$ as in the proof of Proposition 1.3.2. We topologize this set by means of the quotient topology, that is, we declare a subset O of $X/_{\sim}$ to be open iff $\bigcup\{[x]_{\sim} : [x]_{\sim} \in O\}$ is an open subset of X. (See Section 1.9 of Appendix.)

Define the map $U : X/_{\sim} \to \mathbb{R}$ by

$$U([x]_{\sim}) := u(x).$$

Since u represents $\succsim$, it assigns the same value to all members of any given $[x]_{\sim}$. Therefore, U is well-defined. Let V denote the limsup of U (relative to the quotient topology), that is, define $V : X/_{\sim} \to \mathbb{R}$ by

$$V([x]_{\sim}) := \inf_{O \in \mathcal{O}(x)} \sup\{u(\omega) : [\omega]_{\sim} \in O\},$$

where $\mathcal{O}(x)$ is the collection of all open subsets of $X/_{\sim}$ that contain $[x]_{\sim}$. As the limsup of any bounded real function on a topological space is upper semicontinuous, V is an upper semicontinuous function on $X/_{\sim}$. Thus, the map $v : X \to \mathbb{R}$, defined by $v(x) := V([x]_{\sim})$, is upper semicontinuous.[4] For future reference, we also note that

$$v(x) \geq u(x) \quad \text{for every } x \in X, \tag{2}$$

which is an obvious consequence of the definitions of V and v.

We shall complete our proof by showing that v represents $\succsim$. To this end, take any elements x and y of X. If $x \sim y$, then, clearly,

$$v(x) = V([x]_{\sim}) = V([y]_{\sim}) = v(y).$$

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[4] After all, this map equals $V \circ \rho_{\sim}$, where $\rho_{\sim} : X \to X/_{\sim}$ is the quotient map (which is rendered continuous by the quotient topology).

Now suppose $x \succ y$. We wish to show that $v(x) > v(y)$ by distinguishing between two cases.

Case 1: $x \succ z \succ y$ for some $z \in X$. In this case, the set $O := \{[\omega]_\sim : z \succ \omega\}$ includes $[y]_\sim$, and we have

$$u(z) \geq \sup\{u(\omega) : [\omega]_\sim \in O\}.$$

Furthermore, by upper semicontinuity and completeness of $\succsim$, the set $\{\omega : z \succ \omega\}$ is open in X, while

$$\bigcup\{[\omega]_\sim : [\omega]_\sim \in O\} = \{\omega \in X : z \succ \omega\}.$$

Thus, by definition of the quotient topology, O is an open subset of $X/_\sim$, that is, $O \in \mathcal{O}(y)$. It then follows from the definition of v that $u(z) \geq V([y]_\sim) = v(y)$. By (2), then,

$$v(x) \geq u(x) > u(z) \geq v(y)$$

since u represents $\succsim$.

Case 2: $x \succ z \succ y$ for no $z \in X$. In this case,

$$u(y) = \max\{u(\omega) : [\omega]_\sim \in O\}$$

where $O := \{[\omega]_\sim : x \succ \omega\}$. As, again, O is an open subset of $X/_\sim$ that includes $[y]_\sim$, that is, $O \in \mathcal{O}(y)$, it follows from the definition of v that $u(y) \geq v(y)$. By (2), then, $u(y) = v(y)$, and hence,

$$v(x) \geq u(x) > u(y) = v(y)$$

since u represents $\succsim$.

Curiously, we cannot furnish a continuous utility function from the assumptions of Rader's Theorem. That is, as you are asked to show below, there need not exist a continuous utility function for an upper semicontinuous preference relation on a Euclidean space.

Exercises

3.1. Give an example of an upper semicontinuous preference relation on $\mathbb{R}^2$ that does not admit a continuous utility representation.

3.2. (*Wold's Theorem*) Let $\succsim$ be a continuous preference relation on $\mathbb{R}^n_+$. Assume that $\succsim$ is **monotonic** in the sense that

$$\mathbf{x} > \mathbf{y} \quad \text{implies} \quad \mathbf{x} \succ \mathbf{y}$$

for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$. Prove that the map

$$\mathbf{x} \mapsto \max\{a \geq 0 : \mathbf{x} \succsim a\mathbf{1}\},$$

where **1** is the n-vector of 1s, is a continuous utility function for $\succsim$.

3.3. Prove or disprove: Every monotonic and upper semicontinuous complete preference relation $\succsim$ on $\mathbb{R}^2_+$ can be represented by a continuous utility function.

3.4. Let X be a nonempty set, and $\succsim$ a complete preference relation on X. Then, the collection of all sets of the form $\{\omega : x \succ \omega\}$, $\{\omega : \omega \succ x\}$ and $\{\omega : x \succ \omega \succ y\}$, where x and y vary over X, constitutes a basis for a topology. Prove that this topology is second countable iff $\succsim$ admits a utility representation.

4. The Open Gap Lemma

One of the most fundamental theorems of utility theory tells us that asking for the continuity of $\succsim$ in Rader's Representation Theorem would guarantee the existence of a continuous utility function for this preference relation. This may not appear surprising at first. After all, Rader's Representation Theorem ensures that there are at least two utility functions for $\succsim$, one upper semicontinuous, and the other lower semicontinuous. Deducing from this that there is, in fact, one continuous utility function for $\succsim$ is, however, quite difficult. To be able to prove this, we need to look into the order structure of $\mathbb{R}$ somewhat closely.

Definition. Let S be a subset of $\mathbb{R}$. A **gap** of S is a $\supseteq$-maximal nondegenerate interval I that is disjoint from S and that has lower and upper bounds in S. If, in addition, I is an open interval, then it is said to be an **open gap** of S.

The following is a useful improvement of Proposition 1.2.1. In words, it says that the function f in that result can be chosen to have the property that every rational number r in $\mathbb{Q} \cap (0,1)$ with $\inf f(X) < r < \sup f(X)$, belongs either to the range of f or to an open gap of the range of f. This result was proved by Jean-Yves Jaffray in 1975.

Jaffray's Lemma. Let $(X, \succcurlyeq)$ be a loset. If X is countable, then there exists a strictly $\succcurlyeq$-increasing function $f : X \to \mathbb{Q} \cap (0,1)$ such that, for every rational number r in the open interval $(\inf f(X), \sup f(X))$, we have either $r \in f(X)$ or

$$f(x) > r > f(y) \quad \text{and} \quad (f(y), f(x)) \cap f(X) = \emptyset$$

for some $x, y \in X$.

Proof. The claim is trivial when X is finite, so we assume that X is countably infinite. Let us enumerate X and $\mathbb{Q} \cap (0,1)$ as

$$X = \{x_1, x_2, ...\} \quad \text{and} \quad \mathbb{Q} \cap (0,1) = \{r_1, r_2, ...\},$$

and construct the function f from X into $\mathbb{Q}\cap(0,1)$ exactly as in the proof of Proposition 1.1. Note that, by its construction, f has the following property: For every $x,y\in X$ with $x\succ y$,

$$f\left(\begin{array}{c}\text{the first enumerated element}\\ \text{of }\{\omega\in X: x\succ\omega\succ y\}\end{array}\right)=\begin{array}{c}\text{the first enumerated element}\\ \text{of }(f(y),f(x))\cap\mathbb{Q}.\end{array}\tag{3}$$

Now take any rational number r in $(\inf f(X),\sup f(X))$. Define

$$J:=\min\{j\in\mathbb{N}:\{r_1,...,r_j\}\text{ contains }r\text{ and elements }p,q\in f(X)\text{ with }p>r>q\}.$$

Next, define

$$p_*:=\max\{t\in\{r_1,...,r_J\}\cap f(X):r>t\}$$

and

$$p^*:=\min\{t\in\{q_1,...,q_J\}\cap f(X):t>r\}.$$

The key observation here is:

Claim. If $(p_*,p^*)\cap f(X)\neq\emptyset$, then r is either the first enumerated element of $(p_*,p^*)\cap f(X)$, or it is enumerated before the first enumerated element of $(p_*,p^*)\cap f(X)$.

Proof. Suppose $(p_*,p^*)\cap f(X)\neq\emptyset$, and the first enumerated element of $(p_*,p^*)\cap f(X)$, say q, is enumerated before r. (In particular, $q\neq p$.) But then, by definition of J, the numbers p_*,p^* and q all belong to $\{r_1,...,r_J\}\cap f(X)$. And yet, we have either $p^*>q>r>p_*$ or $p^*>r>q>p_*$, while these inequalities cannot hold due to the choice of numbers p_* and p^*.

Now, suppose r does not belong to an open gap of $f(X)$. We wish to show that r belongs to $f(X)$. Since $r\in(p_*,p^*)$, the interval (p_*,p^*) cannot be a gap of $f(X)$, that is, $(p_*,p^*)\cap f(X)\neq\emptyset$. Of course, we have $p_*=f(y)$ and $p^*=f(x)$ for some $y,x\in X$ with $x\succ y$. Since f is strictly $\succsim$-increasing, and $(f(y),f(x))\cap f(X)\neq\emptyset$, we must have

$$\{\omega\in X:x\succ\omega\succ y\}\neq\emptyset.$$

Let a be the first enumerated element of this set. Then, by (3),

$$f(a)=\text{ the first enumerated element in }(f(y),f(x))\cap\mathbb{Q}.$$

In particular, either $r=f(a)$ or $f(a)$ is enumerated before r. But, since

$$f(a)\in(f(y),f(x))\cap f(X)=(p_*,p^*)\cap f(X),$$

the Claim above shows that the latter case is impossible. Thus, $r=f(a)$, that is, r belongs to the range of f.

We now come to one of the most crucial building blocks of ordinal utility theory, the so-called "open gap lemma." This result was first stated by Gerard Debreu in 1954, but the proof that Debreu gave had a flaw. A complete, but rather involved. proof first appeared in Debreu (1964). Since then a number of relatively simple proofs are obtained. For instance, Bowen (1968) proved this result by means of a measure-theoretic method, and Beardon (1992) by means of a topological method. We follow here the order-theoretic approach of Jaffray (1975).

The Open Gap Lemma. Let X be a nonempty subset of $[0,1]$. Then, there exists a strictly increasing function $\varphi : X \to [0,1]$ such that every gap of $\varphi(X)$ is either a singleton or an open interval.

Proof. Let us first establish the following auxiliary fact:

Claim. There exists a countable subset A of X such that, for every $x, y \in X$,

$$x > y \quad \text{implies} \quad x \geq b > a \geq y \text{ for some } a, b \in A. \tag{4}$$

Proof. Since $[0,1]$ is a separable metric space, so is X, that is, there is a countable dense subset S of X. Let T be the set of all endpoints of the gaps of X that belong to X. Clearly, T is a countable subset of X (because X may have only countably many gaps). We define

$$A := S \cup T.$$

Obviously, A is a countable subset of X. Now, take any $x, y \in X$ such that $x > y$. If $(y, x) \cap X = \emptyset$, then (y, x) is a gap of X, and hence $x, y \in T$. Then (4) holds with $a := y$ and $b := x$. If, on the other hand, $(y, x) \cap X \neq \emptyset$, then, since $(y, x) \cap X$ is open in X and S is dense in X, there must exist a number b in S such that $x > b > y$. If $(b, y) \cap X = \emptyset$, then (b, y) is a gap of X, and hence $b, y \in T$. Then (4) holds with $a := y$. Otherwise, by denseness of S, there is an $a \in S$ such that $b > a > y$, and we are done.

Given the set A found in the Claim above, we next use Jaffray's Lemma (with X being A) to find a strictly $\succcurlyeq$-increasing function $f : A \to \mathbb{Q} \cap (0,1)$ such that, for every rational number r in $(\inf f(A), \sup f(A))$, we have either $r \in f(A)$ or

$$f(b) > r > f(a) \;\text{ and }\; (f(a), f(b)) \cap f(A) = \emptyset$$

for some $a, b \in A$. Finally, we define $\varphi : X \to [0,1]$ by

$$\varphi(x) := \sup\{f(a) : x \geq a \in A\}.$$

Clearly, φ is an extension of f, that is, $\varphi|_A = f$. In addition, φ is strictly increasing. Indeed, for any $x, y \in X$ with $x > y$, the Claim above ensures that there exist $a, b \in A$ such that $x \geq b > a \geq y$. Then, since f is strictly increasing,

$$\varphi(x) \geq f(b) > f(a) \geq f(c) \quad \text{for all } c \in A \text{ with } y \geq c$$

so that $\varphi(x) > f(a) \geq \varphi(y)$.

It remains to prove that every gap of $\varphi(X)$ is either a singleton or an open interval. Suppose that I is a non-singleton gap of $\varphi(X)$. Then,

$$\emptyset = I \cap \varphi(X) \supseteq I \cap \varphi(A) = I \cap f(A).$$

Moreover, since I is a nondegenerate interval, it contains a rational number r, and we have

$$\sup f(A) \geq \varphi(x) > r > \varphi(y) \geq \inf f(A),$$

where $\varphi(x)$ and $\varphi(y)$ are, respectively, any upper and lower bounds for I in $\varphi(X)$. (We have $\varphi(x) > r > \varphi(y)$ because $r \in I$ and $I \cap \varphi(X) = \emptyset$.) Therefore, by the choice of f, there must exist numbers a and b in A such that

$$r \in (f(a), f(b)) \text{ and } (f(a), f(b)) \cap f(A) = \emptyset.$$

Since f is strictly increasing, the latter equation implies $(a, b) \cap A = \emptyset$, so, by the Claim above, $(a, b) \cap X = \emptyset$. Since φ is a strictly increasing extension of f, then,

$$(f(a), f(b)) \cap \varphi(X) = (\varphi(a), \varphi(b)) \cap \varphi(X) = \emptyset.$$

It follows that $(f(a), f(b))$ is a gap of $\varphi(X)$. Since $f(a)$ and $f(b)$ belong to $\varphi(X)$, $(f(a), f(b))$ is a $\supseteq$-maximal gap of $\varphi(X)$. But then, since I and $(f(a), f(b))$ intersect (for r belongs to both of these sets), we must have $I = (f(a), f(b))$, so I is an open gap of $\varphi(X)$.

5. The Debreu-Eilenberg Representation Theorems

The Open Gap Lemma facilitates the derivation of the two most famous theorems of ordinal utility theory. In particular, we are now in a position to improve any utility representation theorem to a continuous utility representation theorem in the case of continuous preference relations.

Proposition 5.1. Let X be any topological space and let $v : X \to \mathbb{R}$ represent a preference relation $\succsim$ on X. If $\succsim$ is continuous, then it is representable by a continuous utility function.

Proof. Assume that $\succsim$ is continuous, and apply the Open Gap Lemma to find a strictly increasing $\varphi : v(X) \to \mathbb{R}$ such that every gap of $\varphi(v(X))$ is either a singleton or an open interval. Define $u := \varphi \circ v$ and observe that u represents $\succsim$. We will now prove that u is upper semicontinuous – its lower semicontinuity is established similarly.

Take an arbitrary real number α and let

$$S_\alpha := \{\omega \in X : u(\omega) \geq \alpha\}.$$

Our task is to show that S_α is a closed subset of X. If $\alpha = u(x)$ for some $x \in X$, then $S_\alpha = x^\uparrow$ (because u represents $\succsim$), and we are done by the upper semicontinuity of $\succsim$. We then consider the case where $\alpha \in \mathbb{R}\backslash u(X)$. Clearly, if $\alpha \leq \inf u(X)$, then we have $S_\alpha = X$, and if $\alpha \geq \sup u(X)$, then $S_\alpha = \emptyset$; so our claim is trivial in these cases. Assume then that α belongs to the open interval $(\inf u(X), \sup u(X))$, and let I be a gap of $u(X)$ that contains α. (Clearly, I is the union of all intervals in $\mathbb{R}\backslash u(X)$ that contain α.) By definition of u, either $I = \{\alpha\}$ or $I = (\alpha_*, \alpha^*)$ for some α_* and α^* in $u(X)$ with $\alpha^* > \alpha_*$. In the latter case, we have

$$S_\alpha = \{\omega \in X : u(\omega) \geq \alpha^*\} = y^\uparrow,$$

where $y \in X$ satisfies $u(y) = \alpha^*$. Thus S_α is a closed set, thanks to the upper semicontinuity of $\succsim$. In the former case, on the other hand,

$$S_\alpha = \bigcap \{S_\beta : \alpha > \beta \in u(X)\}$$

Since S_β is closed for each $\beta \in u(X)$, and the intersection of any collection of closed sets is closed, we find again that S_α is a closed set.

The following is one of the most celebrated results of utility theory.

The Debreu Representation Theorem. Let X be a topological space with a countable basis, and $\succsim$ a complete preference relation on X. If $\succsim$ is continuous, then it can be represented by a continuous utility function.

Proof. Apply Propositions 3.1 and 5.1.

In applications, X often takes the form of a separable metric space. As such a space is a topological space with a countable basis, the Debreu Representation Theorem deals with such cases readily:

Corollary 5.2. Every continuous preference relation on a separable metric space admits a continuous utility representation.

Samuel Eilenberg proved in 1941 that any continuous preference relation on a connected and separable topological space admits a continuous utility representation. (This result replaces the countable basis requirement of Debreu's Theorem with connectedness.) The technique we used to prove the Debreu Representation Theorem can also be used to prove this result.

The Eilenberg Representation Theorem. Let X be a connected and separable topological space, and $\succsim$ a complete preference relation on X. If $\succsim$ is continuous, then it can be represented by a continuous utility function.

Proof. In view of Proposition 5.1, all we need to prove here is that $\succsim$ admits a utility representation. Let Y be a countable dense subset of X, and pick any $x, y \in X$ with $x \succ y$. (If there is no such x and y in X, then any constant function would represent $\succsim$.) Define

$$\Lambda(x) := \{\omega \in X : x \succ \omega\}$$

and

$$V(y) := \{\omega \in X : \omega \succ y\},$$

which are disjoint open subsets of X (because $\succsim$ is complete and continuous). Since X is connected, we must have $\Lambda(x) \cap V(y) \neq \emptyset$. But then $\Lambda(x) \cap V(y)$ is a nonempty open subset of X, and hence, since a dense set intersects every nonempty open set, we must have $Y \cap \Lambda(x) \cap V(y) \neq \emptyset$. This proves that Y is $\succsim$-dense in X. Applying Proposition 2.1, therefore, we may conclude that $\succsim$ admits a utility representation.

> **Note.** There are various generalizations of the Eilenberg Representation Theorem. Campión et. al. (2007), for instance, have recently shown that it is enough to take X to be locally connected (and separable) in this result.[5]

As we have mentioned above, the underlying set X of alternatives is usually taken to be a metric space, say, in economic applications. (For instance, when modeling choice over commodity bundles, one takes X to be $\mathbb{R}^n$, and when modeling choice over intertemporal income streams, X is often taken as a suitable normed linear space of

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[5] A topological space X is said to be *locally connected* if, for any point x in X and any open neighborhood U of x, there is a connected set V with $x \in V \subseteq U$.

real sequences.) Within such contexts, Debreu's theorem is more general than that of Eilenberg, because it does not require the connectedness hypothesis.

Application. (*Choice Correspondences*) We briefly revisit the theory of choice correspondences introduced in Section 4.3 of Chapter 3. Let X be a metric space, and C a choice correspondence on $\mathbf{k}(X)$, the collection of all nonempty compact subsets of X. Recall that the Fundamental Theorem of Revealed Preference tells us that C can be thought of as arising from the maximization of a continuous preference relation, that is, there exists a continuous and complete preference relation $\succsim$ on X such that

$$C(S) = \max(S, \succsim) \quad \text{for every } S \in \mathbf{k}(X),$$

provided that C satisfies the rationality properties of WARP and C. In this case, if we know also that X is separable, we may invoke the Debreu Representation Theorem – or more precisely, Corollary 5.2 – to conclude that there exists a continuous (utility) function $u : X \to \mathbb{R}$ such that

$$C(S) = \arg\max u(S) \quad \text{for every } S \in \mathbf{k}(X), \tag{5}$$

where, of course, $\arg\max u(S)$ stands for the set of all x in S that maximizes u on S. This gives us a deeper version of our earlier finding on the representation of choice correspondences:

Fundamental Theorem of Revealed Preference 2. Let X be a separable metric space, and C a function from $\mathbf{k}(X)$ into 2^X. Then, C is a choice correspondence on $\mathbf{k}(X)$ that satisfies WARP and C if, and only if, there exists a continuous real function u on X such that (5) holds.

6. Multi-Utility Representation

Obviously, a preference relation that admits a utility representation must be complete. But this does not mean that there is no useful notion of utility representation for possibly incomplete preference relations. Indeed, we have already seen in Section 4.2 of Chapter 1 that every preference relation can be represented as the intersection of a collection of complete preference relations. If every one of the preference relations in that collection admits a utility representation, we arrive at a useful functional representation for our original relation.

Definition. Let $\succsim$ be a preference relation on a nonempty set X. We say that a nonempty set $\mathcal{U}$ of real functions on X **represents** $\succsim$ provided that

$$x \succsim y \quad \text{if and only if} \quad u(x) \geq u(y) \;\; \text{for each} \;\; u \in \mathcal{U}$$

for every $x, y \in X$. If such a set $\mathcal{U}$ exists, we say that $\succsim$ admits a **multi-utility representation**. If $\mathcal{U}$ is finite (countable), we say that there exists a **finite (countable) multi-utility representation** for $\succsim$. If $\mathcal{U}$ represents $\succsim$, and every u in $\mathcal{U}$ is strictly $\succsim$-increasing, then we say that $\mathcal{U}$ **properly represents** $\succsim$, and that $\succsim$ admits a **proper multi-utility representation**.

The following example is prototypical.

Example 6.1. The standard partial order $\geq$ of $\mathbb{R}^n$ admits a (finite) multi-utility representation. Recall that this partial order is defined as $x \geq y$ iff $x_i \geq y_i$ for each $i = 1, ..., n$. Then, where $u_i : \mathbb{R}^n \to \mathbb{R}$ is defined by $u_i(x) := x_i$ for each $i = 1, ..., n$, it is obvious that $\{u_1, ..., u_n\}$ represents $\geq$. This partial order also admits a proper multi-utility representation. For,

$$\left\{ u_j + \lambda \sum_{i=1}^{n} u_i : j = 1, ..., n \text{ and } \lambda > 0 \right\}$$

properly represents $\geq$.

It is worth noting that the reasoning we gave in the context of the example above readily yields the following general principle:

Proposition 6.1. Every preference relation that admits a finite multi-utility representation admits also a proper (but not necessarily finite) multi-utility representation.

Another elementary observation concerns the universality of the notion of multi-utility representation.

Proposition 6.2. There exists a multi-utility representation for every preference relation.

Proof. Let X be a nonempty set and $\succsim$ a preference relation on X. For any subset S of X, let us write $\mathbf{1}_S$ for the indicator function of S on X; that is,

$$\mathbf{1}_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \in X \backslash S. \end{cases}$$

It is easily checked that $\mathcal{U} := \{\mathbf{1}_{x^\uparrow} : x \in X\}$ represents $\succsim$.

Proposition 6.2 shows that the notion of utility representation is strictly more demanding than that of multi-utility representation for *complete* preference relations.

Example 6.2. It is plain that if a preference relation admits a utility representation, then it admits a multi-utility representation. The converse is false, even for complete preference relations. After all, we have seen in Example 2.1 that the lexicographic order on $\mathbb{R}^2$ cannot be represented by a utility function. Yet, this linear order admits a multi-utility representation by Proposition 6.2.

We conclude by noting that the utility functions found in Proposition 6.2 provide us with an upper semicontinuous multi-utility representation.

Proposition 6.3. Let X be a topological space and $\succsim$ a preference relation on X. If $\succsim$ is upper (or lower) semicontinuous, then it can be represented by a set of upper (lower resp.) semicontinuous utility functions.

Proof. Let $\succsim$ be an upper semicontinuous preference relation on X. Then, $\mathcal{U} := \{\mathbf{1}_{x^\uparrow} : x \in X\}$ represents $\succsim$. Moreover, for each $x \in X$, the set $x^\uparrow$ is closed in X by upper semicontinuity of $\succsim$, and hence $\mathbf{1}_{x^\uparrow}$ is upper semicontinuous. For the lower semicontinuous part of the assertion, it is enough to replace $\mathcal{U}$ with $\{-\mathbf{1}_{x^\downarrow} : x \in X\}$ in this reasoning.

Exercises

6.1. Let X be a topological space and $\succsim$ a near-complete and upper semicontinuous preference relation on X. Prove that there exists a finite set $\mathcal{U}$ of upper semicontinuous real functions on X that represents $\succsim$.

(*Hint.* Examine the proof of Proposition 5.2.4 of Chapter 3.)

6.2. (*Uniqueness of Multi-Utility Representations*) Given any nonempty set X, and a nonempty subset $\mathcal{U}$ of $\mathbb{R}^X$, define the map $\Gamma_{\mathcal{U}} : X \to \mathbb{R}^{\mathcal{U}}$ by

$$\Gamma_{\mathcal{U}}(x)(u) := u(x).$$

Prove: Two nonempty subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^X$ represent the same preference relation on X if, and only if, there exists a map $f : \Gamma_{\mathcal{U}}(X) \to \mathbb{R}^{\mathcal{V}}$ such that (i) $\Gamma_{\mathcal{V}} = f \circ \Gamma_{\mathcal{U}}$; and (ii) for every $\alpha, \beta \in \Gamma_{\mathcal{U}}(X)$, we have $\alpha > \beta$ iff $f(\alpha) > f(\beta)$.

7. Continuous Multi-Utility Representation

We now turn our attention to the problem of finding a set of continuous utility functions for a preference relation defined on a topological space. This problem is significantly more complicated than its semicontinuous counterpart. In most cases it requires one to solve first a monotonic extension problem. Fortunately, there are some results in the topological literature which are tailor made to the needs of this problem. We shall start this section by examining one major such result, and return to the continuous multi-utility representation problem subsequently.

7.1 Nachbin's Theorem

In particular, much is accomplished by the following extension theorem, which was proved by Leopoldo Nachbin in 1950, is of fundamental importance for the theory of ordered topological spaces.

Nachbin's Extension Theorem. Let X be a compact Hausdorff topological space, $\succsim$ a closed-continous preorder on X, and Z a closed subset of X. If $f : Z \to \mathbb{R}$ is a continuous and $\succsim$-increasing function, then there is a continuous and $\succsim$-increasing function $F : X \to \mathbb{R}$ such that $F|_Z = f$.

Proof. TBW.

7.2 Nachbin's Theorem

Recall that our present goal is to identify those preference relations on a given topological space that admit multi-utility representations by means of a set of continuous utility functions. It is easy to verify that any preference relation that admits such a representation must be closed-continuous. We shall now show that, provided that the prize space under consideration is suitably well-behaved, the converse of this observation is also true.

Theorem 7.2.1. Let X be a compact Hausdorff topological space, and $\succsim$ a preference relation on X. If $\succsim$ is closed-continuous, then it can be represented by a set of continuous utility functions.

Proof. Define

$$\Omega := \{(x, y) \in X \times X : x \succsim y \text{ is false}\}.$$

If $\Omega = \emptyset$, then $x \sim y$ for all $x, y \in X$, so any nonempty set of constant functions on X represents $\succsim$. We assume, then, $\Omega \neq \emptyset$. In turn, for any $(x, y) \in \Omega$, we define the function $v_{x,y} : \{x, y\} \to \mathbb{R}$ as

$$v_{x,y}(x) := 0 \text{ and } v_{x,y}(y) := 1,$$

which is a continuous and $\succsim$-increasing function on $\{x, y\}$. By Nachbin's Extension Theorem, for each $(x, y) \in \Omega$ there is a continuous and $\succsim$-increasing function $u_{x,y} : X \to \mathbb{R}$ such that $u_{x,y}(x) := 0$ and $u_{x,y}(y) := 1$. Then, $\mathcal{U} := \{u_{x,y} : (x, y) \in \Omega\}$ represents $\succsim$. Indeed, for any x and y in X, if $y \succ x$, then $u(y) \geq u(x)$ for all $u \in \mathcal{U}$ (since each $u \in \mathcal{U}$ is $\succsim$-increasing) while

$$u_{x,y}(y) = 1 > 0 = u_{x,y}(x).$$

If $y \sim x$, then $u(x) = u(y)$ for all $u \in \mathcal{U}$ (since each $u \in \mathcal{U}$ is $\succsim$-increasing). Finally, if neither $x \succsim y$ nor $y \succsim x$ hold, then

$$u_{x,y}(y) = 1 > 0 = u_{x,y}(x) \quad \text{and} \quad u_{y,x}(x) = 1 > 0 = u_{y,x}(y)$$

because we have $(x, y) \in \Omega$ and $(y, x) \in \Omega$, respectively.

Thanks to the compactness of X, it is possible to sharpen Theorem 7.2.1 as follows:

Corollary 7.2.2. Let X be a compact Hausdorff topological space, and $\succsim$ a preference relation on X. If $\succsim$ is closed-continuous, then it can be represented by a countable set of continuous utility functions.

Proof. As usual, we let $\mathbf{C}(X)$ stand for the set of all continuous real functions on X, and metrize this set by means of the sup-metric. Since X is compact, $\mathbf{C}(X)$ is a separable metric space.[6] Now, by Theorem 7.2.1, there exists a nonempty subset $\mathcal{U}$ of the set $\mathbf{C}(X)$ that represents $\succsim$. As a metric subspace of a separable metric space is separable, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ which is dense in $\mathcal{U}$ (relative to the sup-metric).

We claim that $\mathcal{V}$ represents $\succsim$. To see this, take any elements x and y of X. Obviously, if $x \succsim y$, then $v(x) \geq v(y)$ for every $v \in \mathcal{V}$ (because $\mathcal{V} \subseteq \mathcal{U}$). Conversely, suppose that

$$v(x) \geq v(y) \quad \text{for all } v \in \mathcal{V}. \tag{6}$$

Since $\mathcal{U}$ represents $\succsim$, if $x \succsim y$ were false, we would have $u(y) > u(x)$ for some $u \in \mathcal{U}$. Then, by denseness of $\mathcal{V}$ in $\mathcal{U}$, we could find a sequence (v_m) in $\mathcal{V}$ such that

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[6] This is a fairly straightforward consequence of the Stone-Weierstrass Theorem. (In fact, one can show that, for any Hausdorff topological space X, $\mathbf{C}(X)$ is separable iff X is compact.)

$\|v_m - u\|_\infty \to 0$. This implies that $v_m(y) > v_m(x)$ for a positive integer m large enough, which contradicts (6).

Another shortcoming of Theorem 7.1 is that this theorem does not tell us if we can *properly* represent the preference relation under consideration. However, as we shall now show next, there is an easy remedy for this. First, let us recall the following elementary fact from real analysis.

The Weierstrass M-Test. Let X be a topological space, and (u_m) a sequence of continuous functions on X. Suppose that there exist a real sequence $(M_1, M_2, ...)$ such that

$$\|u_i\|_\infty \leq M_i \quad \text{for each } i = 1, 2, ...$$

and

$$\sum_{i=1}^{\infty} M_i < \infty.$$

Then, $\sum^\infty u_i(x)$ exists for each $x \in X$, and $x \mapsto \sum^\infty u_i(x)$ is a continuous map on X.

Proof. Define $f_m := u_1 + \cdots + u_m$ for each positive integer m. Take any $\varepsilon > 0$. Since $\sum^\infty M_i < \infty$, there exists a positive integer m^* such that

$$\sum_{i=m^*}^{\infty} M_i < \varepsilon.$$

Then,

$$\left| \sum_{i=m^*}^{m} u_i(x) \right| \leq \sum_{i=m^*}^{\infty} |u_i(x)| \leq \sum_{i=m^*}^{\infty} M_i < \varepsilon$$

for every integer $m \geq m^*$ and $x \in X$. It follows that

$$|f_m(x) - f_{m^*}(x)| = \left| \sum_{i=m^*}^{m} u_i(x) \right| < \varepsilon \quad \text{for every } x \in X \text{ and } m \geq m^*.$$

Conclusion: (f_m) is a Cauchy sequence in the metric space $\mathbf{C}_b(X)$ of continuous and bounded functions on X. Since $\mathbf{C}_b(X)$ is a complete metric space – see Example 2.4.5 of Appendix – we may thus conclude that $\lim f_m \in \mathbf{C}_b(X)$.

We are now ready for:

Levin's Theorem. Let X be a compact Hausdorff topological space, and $\succsim$ a preference relation on X. If $\succsim$ is closed-continuous, then there is a continuous and strictly $\succsim$-increasing real function on X.

Proof. Assume that $\succsim$ is closed-continuous. By Corollary 7.2, there is a countable set $\mathcal{V}$ of continuous real functions on X which represents $\succsim$. If $\mathcal{V}$ is finite, then the proof is completed by using the trick we used at the end of Example 6.1. We thus assume that $\mathcal{V}$ is countably infinite. Let us enumerate $\mathcal{V}$ as $\{v_1, v_2, ...\}$, and define

$$u_i := \frac{2^{-i} v_i}{1 + \|v_i\|_\infty}, \qquad i = 1, 2, ...$$

Since $\mathcal{V}$ represents $\succsim$, we have

$$x \succsim y \;\text{ iff }\; u_i(x) \geq u_i(y) \;\text{ for each }\; i = 1, 2, ... \tag{7}$$

while

$$x \succ y \;\text{ implies }\; x \succsim y \;\text{ and }\; u_i(x) > u_i(y) \;\text{ for some }\; i = 1, 2, ... \tag{8}$$

Moreover, as each $|u_i|$ is bounded above by 2^{-i}, the Weierstrass M-test ensures that $u : X \to \mathbb{R}$ is well-defined by

$$u(x) := \sum_{i=1}^{\infty} u_i(x)$$

as a continuous function on X. In turn, (7) and (8) guarantee that u is strictly $\succsim$-increasing.

Combining Theorem 7.2.1 and Levin's Theorem yields the sharpening of Theorem 7.2.1 that we are after:

Theorem 7.2.3. Let X be a compact Hausdorff topological space, and $\succsim$ a preference relation on X. If $\succsim$ is closed-continuous, then it can be properly represented by a set of continuous utility functions.

Proof. By Theorem 7.2.1, there exists a set $\mathcal{U}$ of continuous functions on X which represents $\succsim$. By Levin's Theorem, there exists a strictly $\succsim$-increasing real function f on X. Then

$$\{u + \lambda f : u \in \mathcal{U} \;\text{ and }\; \lambda > 0\}$$

properly represents $\succsim$.

> **Note.** With a slightly more involved argument, one can show that X can be taken to be a locally compact and separable metric space (such as an arbitrary closed subspace of a Euclidean space) in Levin's Theorem. Since Evren and Ok (2009) prove that Theorem 7.1 is valid when X is such a space, it follows that one can take X in Theorem 7.3 to be a locally compact and separable metric space.

The next step on the agenda should be to look for ways of improving this theorem towards obtaining a suitable continuous finite multi-utility representation theorem. A natural conjecture is that this can be done by strengthening the upper semicontinuity hypothesis to continuity of $\succsim$. Unfortunately, this conjecture is false, and, in fact, there does not seem to be a natural way of deriving a continuous finite multi-utility representation theorem, at least, not by using the standard methods of utility theory. We conclude our presentation with an example that aims to illustrate just how elusive this representation problem is.

Example 7.2.1. Define $\succcurlyeq$ on $[0, 2]$ as

$$b \succcurlyeq a \quad \text{iff} \quad \begin{array}{rl} \text{either} & 0 \leq a, b \leq 1 \text{ and } b \leq a, \\ \text{or} & 1 \leq a, b \leq 2 \text{ and } a \leq b, \\ \text{or} & b = 2. \end{array}$$

Then, $\succcurlyeq$ is a closed-continuous partial order on $[0, 2]$, and hence, by Theorem 7.2.1, there exists a continuous multi-utility representation for it. Moreover, $\succcurlyeq$ is near-complete. In fact, $\{u, v\}$ represents $\succcurlyeq$, where u and v are real maps on $[0, 2]$, defined as

$$u(t) := \begin{cases} 1 - t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } 1 < t < 2, \\ 1, & \text{if } t = 2 \end{cases} \qquad \text{and} \qquad v(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq 1, \\ t - 1, & \text{if } 1 < t \leq 2. \end{cases}$$

Notice that v is continuous and u is upper semicontinous. Moreover, u is continuous everywhere but at 2.

Surprisingly, there does not exist a finite set $\mathcal{U} \subseteq \mathbf{C}[0, 2]$ that represents $\succcurlyeq$. Indeed, if $\mathcal{U}$ was such a set, then, for every non-constant member u of $\mathcal{U}$ we would have $u(2) > u(1)$, and hence, by using finiteness of $\mathcal{U}$, we could find large enough $s \in [0, 1)$ and $t \in [1, 2)$ such that $u(t) > u(s)$ for all non-constant $u \in \mathcal{U}$, which would contradict $\mathcal{U}$ representing $\succcurlyeq$.

References

R. Aumann, Utility theory without the completeness axiom, *Econometrica* **30** (1962), 445-62.

K. Basu and T. Mitra, Aggregating infinite utility streams with intergenerational equity: The impossibility of being Paretian, *Econometrica* **71** (2003), 1557-63.

A. Beardon, Debreu's gap theorem, *Econ. Theory*, **2** (1992), 150-52.

R. Bowen, A new proof of a theorem in utility theory, *Int. Econ. Rev.* **9** (1968), 374.

M. Campión, J. Candeal, E. Induráin and G. Mehta, Representable topologies and locally connected spaces, *Topology and App.* **154** (2007), 2040-2049.

G. Debreu, Representation of a preference ordering by a numerical function, *in* "Decision Processes" (R. Thrall, C. Coombs, and R. Davis, Eds.), pp. 159-65, Wiley, New York, 1954.

G. Debreu, Continuity properties of paretian utility, *Int. Econ. Rev.* **5** (1964), 285-93.

R. Dilworth, A decomposition theorem for partially ordered sets, *Ann. Math.* **51** (1950), 161-6.

J. Dubra and F. Echenique, Monotone preferences over information, *Topics in Theoretical Econ.* **1** (2001), Article 1.

S. Eilenberg, Ordered topological spaces, *Amer. J. Math.* **63** (1941), 39-45.

P. Erdös and G. Szerekes, A combinatorial problem in geometry, *Comp. Math.* **2** (1935), 463–70.

O. Evren and E. A. Ok, On the Multi-Utility Representation of Preference Relations, *J. Econ. Theory,* forthcoming.

Y. Jaffray, Existence of a continuous utility function: an elementary proof, *Econometrica* **43** (1975), 981-3.

V. Levin, Measurable utility theorems for closed and lexicographic preference relations, *Soviet Math. Dokl.* **27** (1983), 639-43.

M. Majumdar and A. Sen, A note on representing partial orderings, *Rev. Econ. Stud.* **43** (1976), 543-45.

M. Mandler, Cardinality versus Ordinality: A Suggested Compromise. *Amer. Econ. Rev.* **96** (2006), 1114-36.

G. Mehta, Existence of an order-preserving function on a normally preordered space. *Bull. Austral. Math Soc.* **34** (1986), 141-7.

L. Mirsky and H. Perfect, Systems of representatives, *J. Math. Anal.* **15** (1966), 520-68.

L. Nachbin, *Topology and Order*, Princeton: Van Nostrand, 1965.

E. A. Ok, Utility representation of an incomplete preference relation, *J. Econ. Theory* **104** (2002), 429-49.

B. Peleg, Utility functions for partially ordered topological spaces, *Econometrica* **38** (1970), 93-6.

T. Rader, The existence of a utility function to represent preferences, *Rev. Econ. Stud.* **30** (1963), 229-32.

M. Richter, Continuous and semi-continuous utility, *Int. Econ. Rev.* **21**, 293-9.

J. Sagi, Anchored preference relations, *J. Econ. Theory* **130** (2006), 283-95.

D. Sondermann, Utility representation for partial orders, *J. Econ. Theory* **23** (1980), 183-8.