

## CHARACTERIZATION OF CLOSED CONVEX SUBSETS OF $\mathbb{R}^n$

### Chebyshev Sets

A subset  $S$  of a metric space  $X$  is said to be a **Chebyshev set** if, for every  $x \in X$ , there is a unique point in  $S$  that is closest to  $x$ . Put differently,  $S$  is Chebyshev iff

$$|\{\omega \in S : d(x, \omega) = d(x, S)\}| = 1 \quad \text{for every } x \in X.$$

For example, the one-dimensional unit sphere  $S^1$  is not a Chebyshev subset of  $\mathbb{R}^2$ , for

$$\{\omega \in S^1 : d_2(\mathbf{0}, \omega) = d_2(\mathbf{0}, S^1)\} = S^1.$$

Similarly,  $N_{1, \mathbb{R}^2}(\mathbf{0})$  is not Chebyshev. (If  $x$  lies outside of  $N_{1, \mathbb{R}^2}(\mathbf{0})$ , then there is no point in  $N_{1, \mathbb{R}^2}(\mathbf{0})$  that is closest to  $x$ .) On the other hand, we have seen in Example D.5 that, given any positive integer  $n$ , every nonempty closed and convex subset of  $\mathbb{R}^n$  is, in fact, a Chebyshev subset of  $\mathbb{R}^n$ . It is remarkable that the converse of this also holds, that is, any Chebyshev set in  $\mathbb{R}^n$  is a nonempty closed and convex subset of  $\mathbb{R}^n$ . It is this result that we wish to prove in this section.

Let's first warm up with a few exercises.

**Exercise 1.** A subset  $S$  of a metric space  $X$  is said to be **proximal** if, for every  $x \in X$ , there is at least one point in  $S$  that is closest to  $x$ .

(a) Show that every proximal set is closed;

(b) A closed set need not be proximal (although this is the case in a Euclidean space). Let  $X := \{(x_m) \in \ell^2 : x_m = 0 \text{ for all but finitely many } m\}$ . Consider  $X$  as a metric subspace of  $\ell^2$ , and show that  $S := \{(x_m) \in X : \sum_{i=1}^{\infty} \frac{1}{2^i} x_i = 0\}$  is a closed subset of  $X$  which is not proximal.

(c) Every convex proximal subset of  $\mathbb{R}^n$  is Chebyshev.

**Exercise 2.** (Efimov-Stechkin) Let  $S$  be a subset of a metric space  $X$  and  $x \in X$ . A sequence  $(y^m) \in S^\infty$  is said to be a **minimizing sequence for  $x$  in  $S$** , if  $d(x, y^m) \rightarrow d(x, S)$ . In turn,  $S$  is said to be **approximately compact** if, for every  $x \in X$ , every minimizing sequence for  $x$  in  $S$  has a subsequence that converges in  $S$ .

(a) Show that every approximately compact set is proximal.

(b) Show that  $\{(x_m) \in \ell^2 : \sum_{i=1}^{\infty} |x_i| = 1\}$  is a proximal subset of  $\ell^2$  which is not approximately compact.

### Projection Operators

First, let us note that the projection operator  $\mathbf{p}_S$  onto any Chebyshev subset  $S$  of  $\mathbb{R}^n$  is well-defined via the equation

$$d_2(x, \mathbf{p}_S(x)) = d_2(x, S), \quad x \in X.$$

Remark G.4 shows that this operator is nonexpansive, *provided that  $S$  is convex*. It is not readily clear if the same is true for any Chebyshev subset  $S$  of  $\mathbb{R}^n$ , but we can at least say the following right away.

**Lemma 1.** *The projection operator onto any Chebyshev subset of  $\mathbb{R}^n$  is continuous.*

*Proof.* Take any Chebyshev subset  $S$  of  $\mathbb{R}^n$ , and let  $(x^m)$  be a convergent sequence in  $\mathbb{R}^n$  with  $x := \lim x^m$ . Note first that

$$d_2(\mathbf{0}, \mathbf{p}_S(x^m)) \leq d_2(\mathbf{0}, x) + d_2(x, x^m) + d_2(x^m, S), \quad m = 1, 2, \dots$$

It follows that  $\{\mathbf{p}_S(x^1), \mathbf{p}_S(x^2), \dots\}$  is a bounded set in  $\mathbb{R}^n$ . (Right?) Now, to derive a contradiction, suppose  $\lim \mathbf{p}_S(x^m) = \mathbf{p}_S(x)$  is false. Then, since  $\{\mathbf{p}_S(x^1), \mathbf{p}_S(x^2), \dots\}$  is bounded, there must exist a convergent subsequence  $(x^{m_k})$  of  $(x^m)$  such that  $y := \lim \mathbf{p}_S(x^{m_k}) \neq \mathbf{p}_S(x)$ .<sup>1</sup> But, by continuity of the maps  $d_2(\cdot, S)$  and  $d_2$ ,

$$d_2(x, \mathbf{p}_S(x)) = d_2(x, S) = \lim_{k \rightarrow \infty} d_2(x^{m_k}, S) = \lim_{k \rightarrow \infty} d_2(x^{m_k}, \mathbf{p}_S(x^{m_k})) = d_2(x, y).$$

Since  $S$  is Chebyshev, this implies  $\mathbf{p}_S(x) = y$ , a contradiction. ■

As another preliminary, we would like to make note of the following observation: If  $S$  is a Chebyshev subset of  $\mathbb{R}^n$  and  $x$  is any point in  $\mathbb{R}^n$ , then the nearest point  $\mathbf{p}_S(x)$  in  $S$  to  $x$  is also the nearest point in  $S$  to any point on the line segment between  $\mathbf{p}_S(x)$  and  $x$ . We prove this next.

**Lemma 2.** *Let  $S$  be a Chebyshev subset of  $\mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . Then,*

$$\mathbf{p}_S(\lambda x + (1 - \lambda)\mathbf{p}_S(x)) = \mathbf{p}_S(x), \quad 0 \leq \lambda \leq 1.$$

*Proof.* Suppose the claim is false, that is, there exists a  $(\lambda, y) \in (0, 1) \times S$  with

$$d_2(\lambda x + (1 - \lambda)\mathbf{p}_S(x), y) < d_2(\lambda x + (1 - \lambda)\mathbf{p}_S(x), \mathbf{p}_S(x)).$$

Then, by the triangle inequality,

$$\begin{aligned} d_2(x, y) &< d_2(x, \lambda x + (1 - \lambda)\mathbf{p}_S(x)) + d_2(\lambda x + (1 - \lambda)\mathbf{p}_S(x), \mathbf{p}_S(x)) \\ &= (1 - \lambda)d_2(x, \mathbf{p}_S(x)) + \lambda d_2(x, \mathbf{p}_S(x)) \\ &= d_2(x, \mathbf{p}_S(x)) \\ &= d_2(x, S) \end{aligned}$$

which is impossible. ■

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<sup>1</sup>Wait, why? Because the closure of  $\{\mathbf{p}_S(x^1), \mathbf{p}_S(x^2), \dots\}$  is closed and bounded, and hence, it is compact by the Heine-Borel Theorem. Conclusion: Every subsequence of  $(\mathbf{p}_S(x^m))$  has a convergent subsequence. Therefore, if every convergent subsequence of  $(\mathbf{p}_S(x^m))$  converged to  $\mathbf{p}_S(x)$ , it would follow that every subsequence of  $(\mathbf{p}_S(x^m))$  has a subsequence that converges to  $\mathbf{p}_S(x)$ , which is just another way of saying  $\lim \mathbf{p}_S(x^m) = \mathbf{p}_S(x)$ .

## Motzkin's Characterization of Convex Sets

We are now prepared to prove that every Chebyshev subset of  $\mathbb{R}^n$  is convex. Originally proved by Theodore Motzkin in 1935, this is one of the gems of convex analysis.

**Motzkin's Theorem.** *For any positive integer  $n$ , a nonempty closed subset  $S$  of  $\mathbb{R}^n$  is Chebyshev if, and only if, it is convex.<sup>2</sup>*

Combining this result with Exercise 1 yields the characterization we promised above.

**Corollary.** *A subset  $S$  of  $\mathbb{R}^n$  is Chebyshev if, and only if, it is nonempty, closed and convex.*

Thanks to Motzkin's Theorem, we can also strengthen Lemma 1 to the following:

**Corollary.** *The projection operator onto any Chebyshev subset of  $\mathbb{R}^n$  is nonexpansive.*

*Proof.* Apply Motzkin's Theorem and Remark G.4. ■

Exercise 3. Give an example of a metric  $d$  such that a convex set in  $(\mathbb{R}^2, d)$  is not Chebyshev.

Exercise 4. Give an example of a metric  $d$  such that a non-convex set in  $(\mathbb{R}^2, d)$  is Chebyshev.

The rest of this handout is devoted to the proof of Motzkin's Theorem. Given Example D.5, all we need to do here is, then, to show that a Chebyshev set  $S$  in  $\mathbb{R}^n$  is convex. The crux of the argument is contained in the following fact: For any point  $x$  in  $\mathbb{R}^n$ , the nearest point  $\mathbf{p}_S(x)$  in  $S$  to  $x$  is also the nearest point in  $S$  to any point on the ray that begins at  $\mathbf{p}_S(x)$  and passes through  $x$ .

**Lemma 3.** *Let  $S$  be a Chebyshev subset of  $\mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . Then,*

$$\mathbf{p}_S(\lambda x + (1 - \lambda)\mathbf{p}_S(x)) = \mathbf{p}_S(x), \quad \lambda \geq 1. \quad (1)$$

Motzkin's Theorem is easily proved by using Lemma 3. Let's see this first. Take any Chebyshev subset  $S$  of  $\mathbb{R}^n$ , and pick any two vectors  $x$  and  $y$  in  $S$ . For any given

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<sup>2</sup>A major open problem in approximation theory is if the role of  $\mathbb{R}^n$  can be replaced with an arbitrary Hilbert space in this statement. (This is *Klee's problem*.) While there are many partial answers to this query – for instance, it is known that an arbitrary pre-Hilbert space would not do – the status of the problem is open at present. (See Deutsch (2002) for more on this.)

$0 < \lambda < 1$ , we wish to show that  $z := \lambda x + (1 - \lambda)y$  belongs to  $S$ . To this end, fix an arbitrary positive real number  $\theta$ , and notice that  $(1 + \theta)z - \theta \mathbf{p}_S(z) = z + \theta(z - \mathbf{p}_S(z))$ . Since  $x \in S$ , then, Lemma 3 maintains that

$$d_2(z + \theta(z - \mathbf{p}_S(z)), \mathbf{p}_S(z)) \leq d_2(z + \theta(z - \mathbf{p}_S(z)), x)$$

that is,

$$(1 + \theta)^2 \sum_{i=1}^n (z_i - \mathbf{p}_S(z)_i)^2 \leq \sum_{i=1}^n (z_i - x_i + \theta(z_i - \mathbf{p}_S(z)_i))^2$$

where we denote that  $i$ th component of  $\mathbf{p}_S(z)$  as  $\mathbf{p}_S(z)_i$ . Let's open this up:

$$(1 + \theta)^2 \sum_{i=1}^n (z_i - \mathbf{p}_S(z)_i)^2 \leq \sum_{i=1}^n (z_i - x_i)^2 + 2\theta \sum_{i=1}^n (z_i - x_i)(z_i - \mathbf{p}_S(z)_i) + \theta^2 \sum_{i=1}^n (z_i - \mathbf{p}_S(z)_i)^2$$

that is,

$$(1 + 2\theta) \sum_{i=1}^n (z_i - \mathbf{p}_S(z)_i)^2 \leq \sum_{i=1}^n (z_i - x_i)^2 + 2\theta \sum_{i=1}^n (z_i - x_i)(z_i - \mathbf{p}_S(z)_i).$$

Now divide both sides by  $\theta$  and let  $\theta \rightarrow \infty$  to get

$$(d_2(z - \mathbf{p}_S(z), \mathbf{0}))^2 \leq (z - x)(z - \mathbf{p}_S(z)).$$

We can obviously replace  $x$  in this inequality by  $y$  (or by any element of  $S$  for that matter), so we also have

$$(d_2(z - \mathbf{p}_S(z), \mathbf{0}))^2 \leq (z - y)(z - \mathbf{p}_S(z)).$$

Aha! If we multiply the former inequality with  $\lambda$  and the latter with  $1 - \lambda$ , and add them up, we get

$$\begin{aligned} (d_2(z - \mathbf{p}_S(z), \mathbf{0}))^2 &\leq (z - (\lambda x + (1 - \lambda)y))(z - \mathbf{p}_S(z)) \\ &= (z - z)(z - \mathbf{p}_S(z)) \\ &= 0. \end{aligned}$$

But this means that  $z = \mathbf{p}_S(z)$ , that is,  $z \in S$ , as we sought.

It remains to establish Lemma 3, which is a far more delicate matter. We shall attack the problem by first transforming it into a fixed point problem.<sup>3</sup> Here is the argument.

*Proof of Lemma 3.* Let us suppose that the assertion of Lemma 3 is false. Then, there exists an  $x'$  in  $\mathbb{R}^n \setminus S$  such that

$$I := \{\lambda \geq 1 : \mathbf{p}_S(\lambda x' + (1 - \lambda)\mathbf{p}_S(x')) = \mathbf{p}_S(x')\} \neq [1, \infty).$$

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<sup>3</sup>To the best of my knowledge, this proof is due to Roger Webster.

But, by Lemma 2,  $I$  has to be an interval with left-end point 1. (Yes?) By continuity of  $\mathbf{p}_S$  (Lemma 1), in turn,  $I$  must contain its supremum. It follows that  $I = [1, \alpha]$  for some real number  $\alpha \geq 1$ . Define  $x := \alpha x' + (1 - \alpha)\mathbf{p}_S(x')$ . Then,  $\mathbf{p}_S(x) = \mathbf{p}_S(x')$  and

$$\mathbf{p}_S(\lambda x + (1 - \lambda)\mathbf{p}_S(x)) \neq \mathbf{p}_S(x) \quad \text{for any } \lambda > 1. \quad (2)$$

(This just shows that if Lemma 3 is false, then there is a point  $x$  in  $\mathbb{R}^n \setminus S$  such that, on the line that starts at  $\mathbf{p}_S(x)$  and passes through  $x$ , every point that is not on the line segment between  $x$  and  $\mathbf{p}_S(x)$  has a projection onto  $S$  different than  $\mathbf{p}_S(x)$ .)

The rest is magic! Let  $\beta := d_2(x, S)$ , which is a positive number (as  $S$  is closed (being proximal) and  $x$  lies outside  $S$ ). Also define

$$K := \left\{ \omega \in \mathbb{R}^n : d_2(x, \omega) \leq \frac{\beta}{2} \right\},$$

which is a nonempty, closed, bounded and convex set in  $\mathbb{R}^n$  that is disjoint from  $S$ . Now consider the function  $\Phi : K \rightarrow \mathbb{R}^n$  defined by

$$\Phi(\omega) := x + \frac{\beta}{2} \frac{x - \mathbf{p}_S(\omega)}{d_2(x, \mathbf{p}_S(\omega))}.$$

Since  $x \notin S$ , this function is well-defined, and, by Lemma 1, it is continuous. (Yes?) Furthermore, for any  $\omega \in K$ , we have

$$d_2(x, \Phi(\omega)) = d_2\left(\frac{\beta}{2} \frac{x - \mathbf{p}_S(\omega)}{d_2(x, \mathbf{p}_S(\omega))}, \mathbf{0}\right) = \frac{\beta}{2d_2(x, \mathbf{p}_S(\omega))} d_2(x, \mathbf{p}_S(\omega)) = \frac{\beta}{2}$$

so that  $\Phi(\omega) \in K$ . Conclusion:  $\Phi(K) \subseteq K$ . It then follows from the Brouwer Fixed Point Theorem that  $z = \Phi(z)$  for some  $z \in K$ . But this means that  $z = x + \theta(x - \mathbf{p}_S(z))$ , where  $\theta := \beta/2d_2(x, \mathbf{p}_S(z)) > 0$ . Then

$$x = \frac{1}{1 + \theta} z + \frac{\theta}{1 + \theta} \mathbf{p}_S(z),$$

that is,  $x$  lies on the line segment that joins  $z$  and  $\mathbf{p}_S(z)$ . So, by Lemma 2,  $\mathbf{p}_S(z) = \mathbf{p}_S(x)$ , and hence  $z = (1 + \theta)x + \theta\mathbf{p}_S(x)$ . Since  $\theta > 0$ , this contradicts (2). ■

**Exercise 5.** Let  $S$  be a Chebyshev subset of  $\mathbb{R}^n$  and  $x \in \mathbb{R}^n \setminus S$ . Prove that the following are equivalent without invoking Motzkin's Theorem:

(i)  $S$  is convex; (ii)  $\mathbf{p}_S$  is nonexpansive; and (iii) (1) holds for all  $x \in \mathbb{R}^n \setminus S$ .

\***Exercise 6.** (Motzkin-Straus-Valentine) Let  $S$  be a subset of  $\mathbb{R}^n$  such that for every  $x \in X$  there is a unique point in  $S$  that is farthest away from  $x$ . Prove that  $S$  is a singleton.

## References

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