Appendix 1
A Primer on Metric Spaces

This appendix, which is basically a short summary of the coverage of Chapters C, D and E of our companion volume Ok (2007), aims to introduce some basic notions from the real analysis of metric spaces. It is more than likely that you are familiar with most of this material, so the pace of our exposition is quite fast. In particular, we provide complete proofs of the results presented here only when they are either fairly short or are not covered fully in Ok (2007).

After going through a few standard examples, we talk about basic notions of the theory of metric spaces, such as open and closed sets, boundary of sets, convergence of sequences, and products of metric spaces. We then discuss the completeness of a metric space, and use this notion to characterize the exponential function as the only differentiable real function on $\mathbb{R}$ that equals to its derivative (and vanishes at 0). We also prove the Banach Fixed Point Theorem here, as we use this result in the body of the text. We then move to discuss compact metric spaces, but keep the discussion short, as most students of economic theory are duly familiar with the notion of compactness. Instead, we put our emphasis on separable metric spaces (as such spaces figure extensively in various parts of this monograph).

The second main part of our discussion pertains to the notion of continuity. We review here the basic characterizations of continuity, and the notions of homeomorphisms and isometries. We also talk briefly about the stronger notions of continuity (such as uniform and Lipschitz continuity) and weaker notions of continuity (such as upper and lower semicontinuity). Many of the results we cover here (such as Weierstrass' Theorem and the fact that uniform continuity of a real map on a compact set is equivalent to its continuity) are likely to be familiar to you. We also establish a few (somewhat more advanced) results here that we need for our coverage of probability theory. In particular, we prove here that every separable metric space can be embedded in the Hilbert cube, and that every complete and separable metric space is homeomorphic to the intersection of countably many open subsets of the Hilbert cube. We also show that every bounded lower semicontinuous real map can be approximated from below by continuous functions. The appendix is concluded by means of a quick discussion of continuity of correspondences (set-valued maps) and the Maximum Theorem of Claude Berge.

1 Metric Spaces

1.1 Basics

Let us first recall the formal definition of a metric space.

Definition. Let X be any nonempty set. A function $d : X \times X \to \mathbb{R}_+$ that satisfies the following properties is called a **distance function** (or a **metric**) on X: For any $\omega, \nu, \varsigma \in X$,

(i) $d(\omega, \nu) = 0$ iff $\omega = \nu$,

(ii) (*Symmetry*) $d(\omega, \nu) = d(\nu, \omega)$,

(iii) (*Triangle Inequality*) $d(\omega, \nu) \leq d(\omega, \varsigma) + d(\varsigma, \nu)$.

If d is a distance function on X, we say that (X, d) is a **metric space**, and refer to the elements of X as **points** in (X, d).

Notation. When the metric under consideration is apparent from the context, it is customary to dispense with the notation (X, d), and refer to X as a metric space. We also adhere to this convention here (and spare the notation d for a generic metric on X). But when we feel that there is a danger of confusion, or we endow X with a non-generic metric, then we shall revert back to the more descriptive notation.

Convention. We often talk as if a metric space (X, d) were indeed a set when referring to properties that apply only to X. For instance, when we say that Y is a subset of the metric space (X, d), we mean simply that $Y \subseteq X$. The phrase "a point in the metric space X" is similarly interpreted.

Example 1. [1] Given any $n \in \mathbb{N}$, $(\mathbb{R}^n, d_p)$ is a metric space for each $1 \leq p < \infty$, where $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$d_p((a_1, ..., a_n), (b_1, ..., b_n)) := \left(\sum_{i=1}^{n} |a_i - b_i|^p \right)^{\frac{1}{p}}.$$

It is easy to see that each d_p satisfies the first two axioms of being a distance function. We omit the verification of triangle inequality for d_p, which is, in fact, not a trivial matter. (See Ok (2007), Section C.1.)

Convention. When we refer to $\mathbb{R}^n$ in this book without specifying a particular metric, we mean $(\mathbb{R}^n, d_2)$, the so-called **n-dimensional Euclidean space**.

[2] Define $\varphi : \overline{\mathbb{R}} \to [-1,1]$ by $\varphi(-\infty) := -1$, $\varphi(\infty) := 1$, and $\varphi(t) := \frac{t}{1+|t|}$ for all $t \in \mathbb{R}$. The standard metric d on $\overline{\mathbb{R}}$ is defined by $d(s,t) := |\varphi(s) - \varphi(t)|$. This makes $\overline{\mathbb{R}}$ a metric space that is identical to $[-1,1]$, for φ is a bijection from $\overline{\mathbb{R}}$ onto $[-1,1]$ that leaves the distance between any two points intact: $d(s,t) = d_1(\varphi(s), \varphi(t))$ for any $s, t \in \overline{\mathbb{R}}$. $\square$

Example 2. For any $1 \leq p < \infty$, we define

$$\ell^p := \left\{ (a_m) \in \mathbb{R}^\infty : \sum_{i=1}^{\infty} |a_i|^p < \infty \right\},$$

and metrize this set by means of the map $d_p : \ell^p \times \ell^p \to \mathbb{R}_+$ defined by

$$d_p\left((a_m), (b_m)\right) := \left(\sum_{i=1}^{\infty} |a_i - b_i|^p \right)^{\frac{1}{p}}.$$

(When we speak of ℓ^p as a metric space, we always have the metric d_p in mind.) Of course, we need to check that d_p is a distance function on ℓ^p, but we omit doing this here. (See Ok (2007), Section C.1.) $\square$

Example 3. [1] For any nonempty set T, by $\mathbf{B}(T)$ we mean the set of all bounded real functions defined on T, that is,

$$\mathbf{B}(T) := \left\{ \varphi \in \mathbb{R}^T : \sup\{|\varphi(t)| : t \in T\} < \infty \right\}.$$

We will always think of this space as metrized by the **sup-metric** $d_\infty : \mathbf{B}(T) \times \mathbf{B}(T) \to \mathbb{R}_+$ defined by

$$d_\infty(\varphi, \phi) := \sup\{|\varphi(t) - \phi(t)| : t \in T\}.$$

It is easy to see that d_∞ is real-valued and satisfies the first two requirements of being a distance function. As for the triangle inequality, all we need is to invoke the corresponding property of the absolute value function (Exercise 12). After all, if $\varphi, \phi, \psi \in \mathbf{B}(T)$, then

$$\begin{aligned} |\varphi(t)-\phi(t)| &\leq |\varphi(t)-\psi(t)|+|\psi(t)-\phi(t)| \\ &\leq \sup\{|\varphi(t)-\psi(t)| : t\in T\}+\sup\{|\psi(t)-\phi(t)| : t\in T\} \\ &= d_\infty(\varphi,\psi)+d_\infty(\psi,\phi) \end{aligned}$$

for *any* $t \in T$, and it follows that $d_\infty(\varphi,\phi) \leq d_\infty(\varphi,\psi)+d_\infty(\psi,\phi)$.

Note. $\mathbf{B}(\mathbb{N})$ is the metric space of all bounded real sequences. (Yes?) This space is very important and denoted as ℓ^∞. By definition, the distance between any two members (a_m) and (b_m) of this space is $d_\infty((a_m),(b_m)) := \sup\{|a_m-b_m| : m\in\mathbb{N}\}$.

[2] Since every continuous function on the bounded closed interval $[a,b]$ is bounded (Exercise 22), we can use d_∞ to metrize $\mathbf{C}[a,b]$. That is, $\mathbf{C}[a,b]$ is the metric space of all continuous real maps on $[a,b]$ such that the distance between any two of its members φ and ϕ is $d_\infty(\varphi,\phi)$.

We can use the sup-metric also to metrize $\mathbf{C}^1[a,b]$, but we will see later (in Exercise 30) that it is a better idea to metrize this set instead by using the distance function $d_{\infty,\infty} : \mathbf{C}^1[a,b]\times\mathbf{C}^1[a,b]\to\mathbb{R}_+$, where

$$d_{\infty,\infty}(\varphi,\phi) := d_\infty(\varphi,\phi)+d_\infty(\varphi',\phi').$$

(*Quiz.* Prove that $d_{\infty,\infty}$ is a distance function.) It is this metric that we have in mind when talking about $\mathbf{C}^1[a,b]$ as a metric space. □

Example 4. [1] (*Finite Products of Metric Spaces*) For any given $n\in\mathbb{N}$, let (X_i,d_i) be a metric space, $i=1,...,n$. Let $X := \mathsf{X}^n X_i$, and define the map $\boldsymbol{\rho} : X\times X\to\mathbb{R}_+$ by

$$\boldsymbol{\rho}((\omega_1,...,\omega_n),(\nu_1,...,\nu_n)) := \sum_{i=1}^n d_i(\omega_i,\nu_i).$$

Evidently, $\boldsymbol{\rho}$ is a distance function on X. The metric space $(X,\boldsymbol{\rho})$ is referred to as the **product** of the metric spaces $(X_i,d_i), i=1,...,n$. (For instance, $(\mathbb{R}^n,d_1)$ is the product of n many $(\mathbb{R},d_1)$s.)

[2] (*Countably Infinite Products of Metric Spaces*) Let (X_i, d_i) be a metric space, $i = 1, 2, ...$ Let $X := \mathsf{X}^\infty X_i$, and define the map $\boldsymbol{\rho} : X \times X \to \mathbb{R}_+$ by

$$\boldsymbol{\rho}((\omega^1, \omega^2, ...), (\nu^1, \nu^2, ...)) := \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, d_i(\omega^i, \nu^i)\}.$$

Then $\boldsymbol{\rho}$ is a distance function on X. (Check!) Again, we refer to the metric space $(X, \boldsymbol{\rho})$ as the **product** of the metric spaces (X_i, d_i), $i = 1, 2, ...$[1] □

If X is a metric space and $\emptyset \neq Y \subset X$, we can view Y as a metric space in its own right by using the distance function induced by d on Y. More precisely, we make Y a metric space by means of the distance function $d|_{Y\times Y}$. We then say that $(Y, d|_{Y\times Y})$, or simply Y, is a **metric subspace** of X. For instance, we think of any interval, say $[0, 1]$, as a metric subspace of $\mathbb{R}$. This means simply that the distance between any two elements s and t of $[0, 1]$ is calculated by viewing s and t as points in $\mathbb{R}$: $d_1(s, t) = |s - t|$. Similarly, $\mathbf{C}[a, b]$ (but not $\mathbf{C}^1[a, b]$) is a metric subspace of $\mathbf{B}[a, b]$ for any real numbers a and b with $a < b$.

Convention. Throughout this book, when we consider a nonempty subset S of a Euclidean space $\mathbb{R}^n$ as a metric space without explicitly mentioning a particular metric, it should be understood that we view S as a metric subspace of $\mathbb{R}^n$.

We now review a number of fundamental concepts regarding metric spaces.

Definition. Let X be a metric space. For any $\omega \in X$ and $\varepsilon > 0$, we define the **ε-neighborhood of ω in X** as the set

$$N_{\varepsilon,X}(\omega) := \{\nu \in X : d(\omega, \nu) < \varepsilon\}.$$

In turn, a **neighborhood of x in X** is any subset of X that contains at least one ε-neighborhood of x in X.

The notion of ε-neighborhoods plays a major role in real analysis mainly through the following definition.

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[1] The "use" of metrizing X in this manner will become clear soon enough. If this is not so at the end of Section 5, then please consult on Ok (2006), Section C.8.

Definition. A subset S of X is said to be **open in** X (or an **open subset of** X) if for every $\omega \in S$, there exists an $\varepsilon > 0$ such that $N_{\varepsilon,X}(\omega) \subseteq S$. A subset S of X is said to be **closed in** X (or a **closed subset of** X) if $X \backslash S$ is open in X.

Notation. In this text, the class of all open subsets of a metric space X is denoted by $\mathcal{O}_X$ and that of all closed subsets of X by $\mathcal{C}_X$.

Because an ε-neighborhood of a point is inherently connected to the underlying metric space, so does the notions of open and closed sets. Consequently, changing the metric on a given set, or concentrating on a metric subspace of the original metric space, would in general yield different classes of open (and hence closed) sets.

Definition. Let X be a metric space and $S \subseteq X$. The largest open set in X that is contained in S (that is, the $\supseteq$-maximum of the class of all open subsets of X contained in S) is called the **interior** of S (*relative to* X), and is denoted by $int_X(S)$. On the other hand, the **closure** of S (*relative to* X) – denoted by $cl_X(S)$ – is defined as the smallest closed set in X that contains S (that is, the $\supseteq$-minimum of the class of all closed subsets of X that contain S). The **boundary** of S (*relative to* X) – denoted by $bd_X(S)$ – is defined as

$$bd_X(S) := cl_X(S) \backslash int_X(S).$$

Let X be a metric space, and Y a metric subspace of X. For any subset S of Y, we may think of the interior of S as lying in X or in Y. (And yes, these may well be quite different!) It is for this reason that we use the notation $int_X(S)$, instead of $int(S)$, to mean the interior of S relative to the metric space X.

Example 5. [1] In any metric space X, the sets X and $\varnothing$ are both open and closed, that is, $\{\emptyset, X\} \subseteq \mathcal{O}_X \cap \mathcal{C}_X$. On the other hand, for any $\omega \in X$ and $\varepsilon > 0$, the set $N_{\varepsilon,X}(\omega)$ is open in X, and the set $\{\omega\}$ is closed in X. (Proofs?)

[2] In $\mathbb{R}$, the interval $(0,1)$ is open, $[0,1]$ is closed, and $[0,1)$ is neither open nor closed. But observe that the structure of the mother metric space is crucial for the validity of these statements. For instance, the set $[0,1)$ is open in the metric space

$\mathbb{R}_+$. (*Note.* $0 \in int_{\mathbb{R}_+}([0,1))$ and $0 \in bd_{\mathbb{R}_+}([0,1)) = \{1\}$.) More generally, the following fact is true:

> **Exercise 1**. Given any metric space X, let Y be a metric subspace of X, and take any $S \subseteq Y$. Show that S is open in Y iff $S = O \cap Y$ for some $O \in \mathcal{O}_X$, and it is closed in Y iff $S = T \cap Y$ for some $T \in \mathcal{C}_X$.

[3] Is $int_X(S)$ well-defined for any subset S of a metric space X? (Perhaps there is no *largest* open subset of X that is contained in S!) The answer is yes, for the union of any family of open sets is open in a metric space. (Yes?) Thus $int_X(S)$ is well-defined, since, thanks to this property, we have $int_X(S) = \bigcup\{O \in \mathcal{O}_X : O \subseteq S\}$.[2] By contrast, the intersection of any family of closed subsets of X is closed – why? – and hence $cl_X(S)$ is well-defined for any $S \in 2^X$. We have $cl_X(S) = \bigcap\{C \in \mathcal{C}_X : S \subseteq C\}$.

Warning. While the intersection of a *finite* family of open sets is open – why? – the intersection of an arbitrary family of open sets need not be open in general. (*Example.* $(-1,1) \cap (-\frac{1}{2}, \frac{1}{2}) \cap \cdots = \{0\}$.) Similarly, the union of a *finite* family of closed sets is closed, but an arbitrary union of closed sets need not be closed.

[4] Let S be a subset of a metric space X. Then S is closed in X iff $cl_X(S) = S$, and it is open in X iff $int_X(S) = S$. Also, $\omega \in bd_X(S)$ iff the sets $S \cap N_{\varepsilon,X}(\omega)$ and $(X \backslash S) \cap N_{\varepsilon,X}(\omega)$ are nonempty for any $\varepsilon > 0$. (Proofs?) $\square$

The notion of closedness (and hence openness) of a set in a metric space can be characterized by means of the sequences that reside in that space. Let us first recall what it means for a sequence to converge in a metric space.

Definition. Let X be a metric space, $\omega \in X$, and $(\omega^m) \in X^\infty$.[3] We say that (ω^m) **converges to** ω if for any $\varepsilon > 0$, there exists a real number M such that $d(\omega^m, \omega) < \varepsilon$ for all $m \geq M$. (*Note.* This is the same thing as saying $d(\omega^m, \omega) \to 0$.) In this case,

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[2] *Corollary.* $\omega \in int_X(S)$ iff $N_{\varepsilon,X}(\omega) \subseteq S$ for some $\varepsilon > 0$.

[3] In this book, as a notational convention, I denote a generic sequence in a given (abstract) metric space X by (ω^m), (ν^m) etc.. (This convention becomes particularly useful when, for instance, the terms of (ω^m) are sequences themselves.) The generic real (or extended real) sequences are denoted as (a_m), (b_m), etc., and generic sequences of real functions are denoted as (φ_m), (ϕ_m), etc..

we say that (ω^m) **converges in** X, or that it is **convergent in** X, we refer to ω as the **limit** of (ω^m), and write either $\omega^m \to \omega$ or $\lim \omega^m = \omega$.[4]

A sequence (ω^m) in a metric space X thus converges to a point ω in X, if for any $\varepsilon > 0$, all but finitely many terms of the sequence (ω^m) belong to $N_{\varepsilon,X}(\omega)$. One way of thinking about this intuitively is viewing the sequence (ω^m) as "staying in $N_{\varepsilon,X}(\omega)$ eventually" no matter how small ε is. Equivalently, we have $\omega^m \to \omega$ iff for every open neighborhood O of ω in X, there is an $M > 0$ such that $\omega^m \in O$ for all $m \geq M$.

Example 6. Take any $n \in \mathbb{N}$, and let $((a_1^m, ..., a_n^m))$ be a sequence in $\mathbb{R}^n$. It is easy to show that this sequence converges to the real n-vector $(a_1, ..., a_n)$ iff (a_i^m) converges to a_i for each $i = 1, ..., n$. One thus says that convergence in $\mathbb{R}^n$ is *coordinatewise.* (The same conclusion obtains in $(\mathbb{R}^n, d_p)$ for any $1 \leq p < \infty$ as well.) $\square$

Example 7. (*Products of Metric Spaces, Again*) Let $n \in \mathbb{N}$, and let $(X, \boldsymbol{\rho})$ be the product of n many metric spaces (X_i, d_i). Then, a sequence $((\omega_1^m, ..., \omega_n^m))$ in X converges to some $(\omega_1, ..., \omega_n) \in X$ iff $\omega_i^m \to \omega_i$ for each i. Thus, in a finite product metric space convergence is coordinatewise.

The same conclusion is true in the case of countably infinite product metric spaces as well. (This is what's so good about the way we metrize the product of countably infinitely many metric spaces as in Example 4.) That is, if $(X, \boldsymbol{\rho})$ is the product of the metric spaces (X_i, d_i), $i = 1, 2, ...$, then a sequence $((\omega_1^m, \omega_2^m, ...))$ in X converges to some $(\omega_1, \omega_2, ...) \in X$ iff $\omega_i^m \to \omega_i$ for each $i = 1, 2, ...$ The "only if" part of this assertion is straightforward. To see the "if" part, suppose $\omega_i^m \to \omega_i$ for each i, and fix any $\varepsilon > 0$. We can obviously find a $k \in \mathbb{N}$ such that $\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots < \frac{\varepsilon}{2}$ (Exercise 21). Since $d_i(\omega_i^m, \omega_i) \to 0$ for each i, there is also an $M > 0$ such that

$$d_i(\omega_i^m, \omega_i) < \frac{\varepsilon}{2\left(\frac{1}{2} + \cdots + \frac{1}{2^k}\right)} \quad \text{for all } m \geq M \text{ and } i = 1, ..., k.$$

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[4] A sequence (ω^m) in a metric space X can converge to at most one limit. Suppose $\omega^m \to \omega$ and $\omega^m \to \nu$. Then, $d(\omega, \nu) \leq d(\omega, \omega^m) + d(\omega^m, \nu)$ for each $m \in \mathbb{N}$ while the right-hand side of this inequality converges to 0 as $m \to \infty$. (Or, written more compactly, $d(\omega, \nu) \leq d(\omega, \omega^m) + d(\omega^m, \nu) \to 0$.) This is possible only if $\omega = \nu$.

But then, for each $m \geq M$,

$$\sum_{i=1}^{k} \frac{1}{2^i} \min\{1, d_i(\omega_i^m, \omega_i)\} < \frac{\varepsilon}{2}$$

and

$$\sum_{i=k+1}^{\infty} \frac{1}{2^i} \min\{1, d_i(\omega_i^m, \omega_i)\} < \frac{\varepsilon}{2}$$

and hence $\boldsymbol{\rho}((\omega^m), \omega) < \varepsilon$. □

Here is the sequential characterization of closed sets we promised above.

Proposition 1. *Let X be a metric space and $S \subseteq X$. Then, $S \in \mathcal{C}_X$ if, and only if, every sequence in S that converges in X converges to a point in S.*

Proof. Take any $S \in \mathcal{C}_X$ and $(\omega^m) \in S^\infty$ with $\omega^m \to \omega$ for some $\omega \in X$. If $\omega \in X \backslash S$, then we can find an $\varepsilon > 0$ with $N_{\varepsilon,X}(\omega) \subseteq X \backslash S$, because $X \backslash S \in \mathcal{O}_X$. But since $d(\omega^m, \omega) \to 0$, there is a large enough $M \in \mathbb{N}$ such that $\omega^M \in N_{\varepsilon,X}(\omega)$, contradicting that all terms of the sequence (ω^m) lies in S. Conversely, suppose $S \notin \mathcal{C}_X$. Then $X \backslash S$ is not open in X, so we can find an $\omega \in X \backslash S$ such that every ε-neighborhood around ω intersects S. Thus, for any $m = 1, 2, \ldots$, there is an $\omega^m \in N_{\frac{1}{m},X}(\omega) \cap S$. But then $(\omega^m) \in S^\infty$ and $\omega^m \to \omega$, and yet $\omega \notin S$. Thus if S were not closed in X, there would exist at least one sequence in S converging to a point outside of S. ■

Exercise 2. For any subset S of a metric space X, show that the following are equivalent:
(a) $\omega \in cl_X(S)$;
(b) Every open neighborhood of ω in X intersects S;
(c) There exists a sequence in S which converges to ω.

1.2 Complete Metric Spaces

Complete metric spaces play a significant role in probabilistic analysis. These are the metric spaces in which any sequence whose terms eventually get arbitrarily close to one another is, per force, convergent. Let us first give a name to such sequences.

Definition. A sequence (ω^m) in a metric space X is called a **Cauchy sequence** if for any $\varepsilon > 0$, there exists an $M > 0$ with $d(\omega^k, \omega^l) < \varepsilon$ for all $k, l \geq M$.

For instance, $(1, \frac{1}{2}, \frac{1}{3}, ...)$ is a Cauchy sequence in $\mathbb{R}$, for the terms of this sequence get closer and closer toward its tail. More formally, $(1, \frac{1}{2}, \frac{1}{3}, ...)$ is Cauchy because $\left|\frac{1}{k} - \frac{1}{l}\right| \leq \left|\frac{1}{k}\right| + \left|\frac{1}{l}\right| \to 0$ (as $k, l \to \infty$).[5] On the other hand, $((-1)^m)$ is not a Cauchy sequence in $\mathbb{R}$, for $|(-1)^m - (-1)^{m+1}| = 2$ for all $m \in \mathbb{N}$.

A Cauchy sequence is necessarily *bounded*, that is, the distance between the terms of any such sequence cannot be arbitrarily large. Indeed, if (ω^m) is a Cauchy sequence in a metric space X, then by choosing an integer $M \geq 2$ such that $d(\omega^k, \omega^l) < 1$ for all $k, l \geq M$, we obtain $\{\omega^m : m \in \mathbb{N}\} \subseteq N_{\delta,X}(\omega^M)$ where $\delta := \max\{1, d(\omega^1, \omega^M), ..., d(\omega^{M-1}, \omega^M)\}$. Moreover, if $(\omega^m) \in X^\infty$ is convergent, then it is Cauchy, because

$$d(\omega^k, \omega^l) \leq d\left(\omega^k, \lim_{m\to\infty} \omega^m\right) + d\left(\lim_{m\to\infty} \omega^m, \omega^l\right) \to 0 \quad \text{(as } k, l \to \infty).$$

By contrast, a Cauchy sequence need not converge. (*Example.* $(1, \frac{1}{2}, \frac{1}{3}, ...)$ is a non-convergent Cauchy sequence in the metric space $(0, 1]$.) Yet if (ω^m) is Cauchy, and it has a convergent subsequence, say (ω^{m_k}), then (ω^m) must converge. For, $\omega^{m_k} \to \omega$ implies $d(\omega^m, \omega) \leq d(\omega^m, \omega^{m_k}) + d(\omega^{m_k}, \omega) \to 0$ (as $m, k \to \infty$). We proved:

Proposition 2. *Let X be a metric space and $(\omega^m) \in X^\infty$. If (ω^m) converges in X, then it is Cauchy. Conversely, if (ω^m) is Cauchy and it has a subsequence that converges in X, then it is convergent.*

We are now ready for the following fundamental definition.

Definition. A metric space X is said to be **complete** if every Cauchy sequence in X converges to a point in X.

We have just seen that $(0, 1]$ is not complete. Another example of an incomplete metric space is $\mathbb{Q}$ (viewed as a metric subspace of $\mathbb{R}$). Indeed, by Lemma 1, for any $a \in \mathbb{R}\backslash\mathbb{Q}$ we can find an $(r_m) \in \mathbb{Q}^\infty$ with $r_m \to a$. Then, (r_m) is Cauchy, and yet it does not converge in $\mathbb{Q}$. But don't despair, we are not short of complete metric spaces.

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[5] **Notation.** For any real number a and double sequence $(a_{kl}) \in \mathbb{R}^{\infty\times\infty}$, I write $a_{kl} \to a$ (as $k, l \to \infty$) to mean that for any $\varepsilon > 0$, there is an $M > 0$ such that $|a_{kl} - a| < \varepsilon$ for all $k, l \geq M$. (The notation $k, l \geq M$ is self-explanatory.)

Example 8. [1] $\mathbb{R}$ *is complete.* (*Proof.* If $(a_m) \in \mathbb{R}^\infty$ is Cauchy, then it is bounded, and hence, by the Bolzano-Weierstrass Theorem and Proposition 2, it converges.)

[2] $\mathbb{R}^n$ *is complete*, $n = 1, 2, \ldots$. (*Proof.* Exercise.)

[3] ℓ^p *is complete,* $1 \leq p < \infty$. (*Proof.* See Ok (2007), Section C.5.2.)

[4] $\mathbf{B}(T)$ *is complete for any nonempty set* T. Let (φ_m) be a Cauchy sequence in $\mathbf{B}(T)$. It is plain that, for each $t \in T$, $(\varphi_m(t))$ is a Cauchy sequence in $\mathbb{R}$, so $\varphi \in \mathbb{R}^T$ is well-defined by $\varphi(t) := \lim \varphi_m(t)$. Naturally, this map is a good candidate for the limit of (φ_m).[6] Let us first show that $\varphi \in \mathbf{B}(T)$. Fix any $\varepsilon > 0$, and pick any $M > 0$ such that $d_\infty(\varphi_k, \varphi_l) < \varepsilon$ for all $k, l \geq M$. Then, for any $l \geq M$,

$$|\varphi(t) - \varphi_l(t)| = \lim_{k\to\infty} |\varphi_k(t) - \varphi_l(t)| \leq \lim_{k\to\infty} d_\infty(\varphi_k, \varphi_l) \leq \varepsilon \quad \text{for all } t \in T \qquad (1)$$

(where we used implicitly the continuity of the absolute value function). So $|\varphi(t)| \leq |\varphi_l(t)| + \varepsilon$ for all $t \in T$, and hence, since $\varphi_l \in \mathbf{B}(T)$, we have $\varphi \in \mathbf{B}(T)$. Moreover, (1) gives us also that $d_\infty(\varphi, \varphi_l) \leq \varepsilon$ for all $l \geq M$. Conclusion: $d_\infty(\varphi, \varphi_m) \to 0$.

[5] The completeness property is inherited by products of metric spaces. That is: *The product of countably many complete metric spaces is complete.* It is enough to prove this assertion in the case of countably infinite products. (Why?) Let $(X, \boldsymbol{\rho})$ be the product of the metric spaces (X_i, d_i), $i = 1, 2, \ldots$, and take any Cauchy sequence (ω^m) in this space. Then (ω_i^m) must be Cauchy in (X_i, d_i) for each i. (Why?) Using the completeness of each (X_i, d_i) along with the fact that convergence in $(X, \boldsymbol{\rho})$ is coordinatewise (Exercise 12), completes the proof. □

There is a tight connection between the closedness of a set and its completeness as a metric subspace. Indeed, a complete subspace of a metric space is necessarily closed. We even have a partial converse of this fact.

Proposition 3. *Let X be a metric space, and Y a metric subspace of X. If Y is complete, then $Y \in \mathcal{C}_X$. Conversely, if $Y \in \mathcal{C}_X$ and X is complete, then Y is complete.*

Proof. Let Y be complete, and take any $(\omega^m) \in Y^\infty$ that converges in X. Then (ω^m) is Cauchy, and thus $\lim \omega^m \in Y$. It follows from Proposition 1 that $Y \in \mathcal{C}_X$. To

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[6] **Warning.** The fact that $\varphi_m(t) \to \varphi(t)$ for each t does *not* imply in general that $d_\infty(\varphi_m, \varphi) \to 0$. The latter statement is true in the present case, thanks to the "Cauchyness" of (φ_m).

prove the second assertion, assume that X is complete, and $Y \in \mathcal{C}_X$. If $(\omega^m) \in Y^\infty$ is Cauchy, then by completeness of X, it must converge in X. But since Y is closed in X, $\lim \omega^m$ belongs to Y (Proposition 1). It follows that Y is complete. ■

Corollary 1. *A metric subspace Y of a complete metric space X is complete if, and only if, $Y \in \mathcal{C}_X$.*

Exercise 3. Show that $\mathbf{C}^1[a,b]$ is a complete metric space for any real numbers a and b with $a \leq b$. (*Note.* $(\mathbf{C}^1[a,b], d_\infty)$ is not complete.)

Exercise 4. (The Cantor-Fréchet Intersection Theorem) The diameter of a bounded set S in a metric space X is defined as

$$\mathit{diam}(S) := \sup\{d(\omega,\nu) : \omega,\nu \in S\}.$$

Prove: X is complete iff for any sequence (S_m) of nonempty closed subsets of X with

$$S_1 \supseteq S_2 \supseteq \cdots \qquad \text{and} \qquad \mathit{diam}(S_m) \to 0, \tag{2}$$

we have $\bigcap^\infty S_i \neq \emptyset$ (so that $|\bigcap^\infty S_i| = 1$).

1.3 Application: The Exponential Function

Other than the polynomials, we use only two types of special real functions in this book: the exponential and logarithmic functions. These functions are customarily introduced by using integral calculus (as in Ok (2007), Section A.4.4), but instead, we derive these functions here by using the theory of metric spaces.

Proposition 4. *There exists a unique differentiable self-map φ on $\mathbb{R}$ such that $\varphi' = \varphi$ and $\varphi(0) = 1$.*

Proof. Define the sequence (φ_m) of polynomials on $\mathbb{R}$ by

$$\varphi_m(t) := 1 + \sum_{i=1}^{m} \frac{t^i}{i!},$$

and let $\varphi_{m,a} := \varphi_m|_{[-a,a]}$ for any $a > 0$ and $m \in \mathbb{N}$.

Claim. $(\varphi_m(t))$ is convergent for any $t \in \mathbb{R}$, and $(\varphi_{m,a})$ is a Cauchy sequence in $\mathbf{C}^1[-a,a]$ for any $a > 0$.

Proof of Claim. Fix any $a > 0$. That each $\varphi_{m,a}$ is continuously differentiable is obvious. Moreover, for any $-a \leq t \leq a$ and $k, l \in \mathbb{N}$ with $k > l$, we have

$$\left|\varphi_{k,a}(t) - \varphi_{l,a}(t)\right| \leq \sum_{i=l+1}^{k} \frac{|t|^i}{i!} \leq \frac{a^{l+1}}{(l+1)!}\left(1 + \frac{a}{l} + \cdots + \left(\frac{a}{l}\right)^{k-l-1}\right).$$

(Why?) So, choosing $l > 2a$, we have

$$d_\infty(\varphi_{k,a}, \varphi_{l,a}) < \frac{a^{l+1}}{(l+1)!}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{k-l-1}\right) < \frac{2a^{l+1}}{(l+1)!}.$$

Thus $d_\infty(\varphi_{k,a}, \varphi_{l,a}) \to 0$ (as $k, l \to \infty$), and since $a > 0$ is arbitrary here, we find that $(\varphi_m(t))$ is convergent for each $t \in \mathbb{R}$. (Yes?) Moreover, since $\varphi'_{m+1} = \varphi_m$ for all $m \in \mathbb{N}$, we have $d_\infty(\varphi'_{k,a}, \varphi'_{l,a}) \to 0$ (as $k, l \to \infty$). Conclusion: $d_{\infty,\infty}(\varphi_{k,a}, \varphi_{l,a}) \to 0$ (as $k, l \to \infty$). $\|$

For any given $a > 0$, the metric space $\mathbf{C}^1[-a, a]$ is complete (Exercise 3). By the Claim above, then, we have $d_{\infty,\infty}(\varphi_{m,a}, \varphi_a) \to 0$ for some $\varphi_a \in \mathbf{C}^1[-a, a]$. Since $\varphi'_{m+1} = \varphi_m$ for all $m \in \mathbb{N}$, this means that we have both $d_\infty(\varphi_{m,a}, \varphi_a) \to 0$ and $d_\infty(\varphi_{m,a}, \varphi'_a) \to 0$. Conclusion: $\varphi_a = \varphi'_a$ and $\varphi_a(0) = 1$ for any $a > 0$.

Now we use the first part of the Claim above to define the self-map φ on $\mathbb{R}$ by $\varphi(t) := \lim \varphi_m(t)$. Since $d_\infty(\varphi_{m,a}, \varphi_a) \to 0$ for any $a > 0$, it is obvious that $\varphi|_{[-a,a]} = \varphi_a$ for any $a > 0$. Since $\mathbb{R} = \bigcup\{[-a, a] : a > 0\}$, then, φ is differentiable and satisfies $\varphi = \varphi'$ and $\varphi(0) = 1$.

To prove the uniqueness assertion, let φ and ϕ be two differentiable self-maps on $\mathbb{R}$ such that $\varphi' = \varphi$, $\phi' = \phi$, and $\varphi(0) = 1 = \phi(0)$. Define $\psi := \varphi - \phi$. We wish to show that $\psi = 0$. Observe first that $\psi' = \varphi' - \phi' = \varphi - \phi = \psi$ and $\psi(0) = 0$. By induction, then, we have $\psi \in \mathbf{C}^k(\mathbb{R})$ and $\psi^{(k)} = \psi$ for any $k \in \mathbb{N}$. Now pick an arbitrary $t \in \mathbb{R}\backslash\{0\}$, and let I be the closed interval with endpoints 0 and t. Since ψ is continuous, it is bounded on I, that is,

$$K := \sup\{|\psi(s)| : s \in I\} < \infty.$$

Then, for any $k \in \mathbb{N}$, using Taylor's Theorem and the fact that $\psi^{(k)}(0) = \psi(0) = 0$, we find a real number c_k between 0 and t such that

$$\psi(t) = \frac{\psi^{(k)}(c_k)}{k!}t^k \leq \frac{K}{k!}t^k.$$

(Yes?) But $\lim \frac{t^k}{k!} = 0$ – why? – so letting $k \to \infty$ in the previous inequality yields $\psi(t) = 0$. ■

The unique differentiable self-map found in Proposition 4 is called the **exponential function**, and its value at any $t \in \mathbb{R}$ is denoted as e^t.[7] All properties of this map can be deduced from Proposition 4. For instance, we have $e^t \neq 0$ for any $t \in \mathbb{R}$. Indeed, if $e^t = 0$ for some $t \in \mathbb{R}$, then, for any $k \in \mathbb{N}$, using Taylor's Theorem and the fact that the derivative of the exponential function at 0 is 1 (as is known from Proposition 4), we find a $c_k \in \mathbb{R}$ between 0 and t such that $1 = e^0 = \frac{e^{c_k}}{k!} |t|^k$. Since the exponential function is continuous, and hence bounded, on the closed interval with endpoints 0 and t, it then follows that $1 \leq \frac{K}{k!} |t|^k$ for some $K > 0$, and letting $k \to \infty$ yields a contradiction. Conclusion: *The exponential function vanishes nowhere.* Thanks to the "baby" Intermediate Value Theorem (Exercise 23), we thus have $e^t > 0$ for all $t \in \mathbb{R}$. (Why?) But then the derivative of the exponential function (of any order) is strictly positive everywhere on $\mathbb{R}$, so this map is strictly increasing and convex.

The following property of the exponential function is basic:

$$e^{s+t} = e^s e^t \quad \text{for all } s, t \in \mathbb{R}. \tag{3}$$

(*Proof.* For any fixed $t \in \mathbb{R}$, define the self-map ϕ on $\mathbb{R}$ by $\phi(s) := \frac{e^{s+t}}{e^t}$. Clearly, ϕ is differentiable, $\phi' = \phi$, and $\phi(0) = 1$. By Proposition 4, then, $\phi(s) = e^s$ for any $s \in \mathbb{R}$, and hence (3).)[8] Finally, note that

$$\lim_{t \to -\infty} e^t = 0 \quad \text{and} \quad \lim_{t \to \infty} e^t = \infty. \tag{4}$$

So, by the "baby" Intermediate Value Theorem, the range of the map $t \mapsto e^t$ is $\mathbb{R}_{++}$.[9]

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[7] As you know, e^1 is denoted simply as e – this number is known as *Euler's number.* (*Note.* $e = 1 + \sum^{\infty} \frac{1}{i!}$.)

[8] I can now give you a justification for the notation e^t. Now, e is a real number, right? So e times e is another well-defined number. And what is the value of the exponential function at 2? Well, by (3), it equals exactly e times e! By induction, for any $k \in \mathbb{N}$, the value of the exponential function at k and $-k$ are found to be $\prod^k e$ and $\prod^k \frac{1}{e}$, respectively; hence the notation e^k.

[9] We have $e = 1 + \sum^{\infty} \frac{1}{i!} > 2$, and hence $e^k > 2^k$ and $e^{-k} < \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Letting $k \to \infty$ yields (4). (Why?)

Given that the exponential function is a continuous, concave and strictly increasing bijection from $\mathbb{R}$ onto $\mathbb{R}_{++}$, it has a continuous, convex and strictly increasing inverse from $\mathbb{R}_{++}$ onto $\mathbb{R}$ – the so-called **logarithmic function**. As you know, the value of this map at any $t > 0$ is denoted as $\ln t$. By definition, we have $\ln e^s = s$ for any $s \in \mathbb{R}$ and $e^{\ln t} = t$ for any $t > 0$. All properties of this map can be deduced from these equations. In particular, the logarithmic function is differentiable (of any order), we have $\frac{d}{dt} \ln t = \frac{1}{t}$ for any $t > 0$, while $\ln st = \ln s + \ln t$ and $\ln \frac{s}{t} = \ln s - \ln t$ for any $s, t > 0$. (Proofs?)

1.4 Application: The Banach Fixed Point Theorem

Definition. A self-map φ on a metric space X is said to be a **contraction** (or a **contractive self-map**) if there is a real number $0 < K < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq K d(x, y) \quad \text{for all } x, y \in X.$$

(The infimum of the set of all such K is called the **Lipschitz constant of** φ.)

For instance, the self-map $t \mapsto \alpha t$ on $\mathbb{R}$ is a contraction (with Lipschitz constant $|\alpha|$) iff $|\alpha| < 1$. More generally, a differentiable real function φ on a nonempty open subset O of $\mathbb{R}$ is a contraction, provided that $\sup\{|\varphi'(t)| : t \in O\} < 1$. (*Proof.* Apply the Mean Value Theorem.)[10]

A remarkable result of metric space theory says that, when it is defined on a complete metric space, a contraction must map a point to itself, that is, it must have a *fixed point.* In fact more is true, such a contraction must have a *unique* fixed point.

The Banach Fixed Point Theorem. *Let X be a complete metric space. If $\varphi \in X^X$ is a contraction, then there exists a unique $\omega^* \in X$ such that $\varphi(\omega^*) = \omega^*$.*

Proof. Assume that $\varphi \in X^X$ is a contraction with Lipschitz constant K. Pick any $\omega^0 \in X$ and define $(\omega^m) \in X^\infty$ recursively as $\omega^{m+1} := \varphi(\omega^m)$, $m = 0, 1, \ldots.$ By

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[10] The notion of contraction is intimately linked to the metric of the underlying space X. A self-map on X which is not a contraction, could become a contraction if we remetrized X appropriately. (See Ok (2006), Section C.7.2, for an illustration of this technique.)

induction, one can show that, for any $k > l$,

$$d(\omega^k, \omega^l) \leq d(\omega^k, \omega^{k-1}) + \cdots + d(\omega^{l+1}, \omega^l) \leq (K^{k-1} + \cdots + K^l) d(\omega^1, \omega^0)$$

so that $d(\omega^k, \omega^l) < \frac{K^l}{1-K} d(\omega^1, \omega^0)$. (Verify!) It follows that (ω^m) is a Cauchy sequence. (Yes?) Since X is complete, then, we have $\omega^m \to \omega^*$ for some $\omega^* \in X$. But

$$\begin{aligned} d(\varphi(\omega^*), \omega^*) &\leq d(\varphi(\omega^*), \omega^{m+1}) + d(\omega^{m+1}, \omega^*) \\ &= d(\varphi(\omega^*), \varphi(\omega^m)) + d(\omega^{m+1}, \omega^*) \\ &\leq K d(\omega^*, \omega^m) + d(\omega^{m+1}, \omega^*), \end{aligned}$$

so letting $m \to \infty$ yields $d(\varphi(\omega^*), \omega^*) = 0$, that is, $\varphi(\omega^*) = \omega^*$. Moreover, if $\varphi(\omega) = \omega$ for some $\omega \in X \backslash \{\omega^*\}$, then $d(\omega, \omega^*) = d(\varphi(\omega), \varphi(\omega^*)) < d(\omega, \omega^*)$, a contradiction. ■

Exercise 5. Let X be a complete metric space. Prove: For any self-map φ on X such that φ^m is a contraction for some $m \in \mathbb{N}$, there exists a unique $\omega^* \in X$ such that $\varphi(\omega^*) = \omega^*$.

The Banach Fixed Point Theorem is often used to establish that a unique solution to a given equation exists. For instance, suppose we are interested in solving an equation like $g(a_1, ..., a_n) = \mathbf{0}$, where g is a self-map on a closed subset X of a Euclidean space $\mathbb{R}^n$ such that $\mathbf{0} \in X$. (Here $\mathbf{0}$ is the real n-vector $(0, ..., 0)$.) An indirect way of checking if there is a unique solution to this equation is to check if $\mathrm{id}_X - g$ is a contraction. In fact, there is quite a bit of leeway in how we may choose to reduce the problem of solving this equation to a fixed point problem. For instance, if ϕ is an injective self-map on X such that $\phi(\mathbf{0}) = \mathbf{0}$ and $\mathrm{id}_X - \phi \circ g$ is a contraction, then, by the Banach Fixed Point Theorem, there is a unique fixed point of $\mathrm{id}_X - \phi \circ g$, and hence a unique solution to our equation. (Why?) Here is another illustration.

Example 9. Let h be a self-map on $\mathbb{R}_+$, and $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ a bounded function such that there exists a number $0 < K < 1$ with

$$|H(t, a) - H(t, b)| < K |a - b| \quad \text{for all } t \geq 0 \text{ and } a, b \in \mathbb{R}.$$

We wish to show that there exists a unique bounded function $U : \mathbb{R}_+ \to \mathbb{R}$ with

$$U(t) = H(t, U(h(t))) \quad \text{for all } t \geq 0. \tag{5}$$

To this end, define the self-map φ on $\mathbf{B}(\mathbb{R}_+)$ by $\varphi(U) := H(\cdot, U(h(\cdot)))$. Then φ is a contraction, because, for any $U, V \in \mathbf{B}(\mathbb{R}_+)$ and $t \geq 0$, we have

$$\begin{aligned} |\varphi(U)(t) - \varphi(V)(t)| &= |H(t, U(h(t))) - H(t, V(h(t)))| \\ &\leq K\,|U(h(t)) - V(h(t))| \\ &\leq K d_\infty(U, V) \end{aligned}$$

so that $d_\infty(\varphi(U), \varphi(V)) \leq K d_\infty(U, V)$. Since $\mathbf{B}(\mathbb{R}_+)$ is complete, then, the Banach Fixed Point Theorem ensures that there is a unique $U \in \mathbf{B}(\mathbb{R}_+)$ with $U = \varphi(U)$. $\square$

1.5 Compact Metric Spaces

We now go back to our review of the theory of metric spaces.

Definition. Let X be a metric space. A class $\mathcal{O}$ of subsets of X is said to **cover** X if $X \subseteq \bigcup \mathcal{O}$. If $\mathcal{O} \subseteq \mathcal{O}_X$ and $\mathcal{O}$ covers X, then we say that $\mathcal{O}$ is an **open cover** of S. A metric space X is said to be **compact** if every open cover of X has a finite subset that also covers X. A subset S of X is said to be **compact in** X (or a **compact subset of** X) if it is a compact metric subspace of X.

Evidently, any finite subset of a metric space is compact. In fact, it makes sense to regard the compactness property as a generalization of the notion of finiteness which is suitable for infinite sets. In many cases where finiteness makes life easier (such as in optimization problems), compactness does the same thing.[11] In particular, every compact set S in a metric space X is **bounded**, that is, *diam*$(S) < \infty$. In fact, more generally, such a set S has to be **totally bounded**, that is, for any $\varepsilon > 0$, there is a finite subset T of S such that $\{N_{\varepsilon,X}(\omega) : \omega \in T\}$ covers S.[12]

Before we continue further, let us put on record the following two characterizations of the compactness property.

Theorem 1. *For any metric space X, the following statements are equivalent:*

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[11] This issue is explored in some detail in Ok (2006), Section C.3.2.

[12] True, a subset of a Euclidean space is bounded iff it is totally bounded, but it is easy to find examples of bounded, but not totally bounded, sets, say, in ℓ^∞.

(a) *X is compact;*

(b) *X is* **sequentially compact**, *that is, every sequence in X has a subsequence that converges to a point in X;*

(c) *X is complete and totally bounded.*

The proof of this important result is somewhat involved, so we omit it here. (See Ok (2007), Sections C.4 and C.5.4.) But it is imperative that you start right away to regard compactness and sequential compactness as equivalent properties for metric spaces. We rely on this equivalence throughout this text.

Example 10. [1] *For any $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$, the cube $[a, b]^n$ is compact.* (This is the *Heine-Borel Theorem.*) To see this, take any sequence $((a_1^m, ..., a_n^m))$ in $[a, b]^n$. Then, by the Bolzano-Weierstrass Theorem, (a_1^m) has a convergent subsequence in $[a, b]$, say, $(a_1^{m_k})$. Similarly, $(a_2^{m_k})$ has a convergent subsequence in $[a, b]$. Continuing this way, we obtain a subsequence of $((a_1^m, ..., a_n^m))$ that converges in $[a, b]^n$. Thus: $[a, b]^n$ is sequentially compact.[13]

[2] *For any $n \in \mathbb{N}$ and $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$, $i = 1, ..., n$, the prism $\mathsf{X}^n[a_i, b_i]$ is compact.* Using Example 7 and Proposition 1, it is easy to show that $\mathsf{X}^n[a_i, b_i]$ is a closed subset of some cube $[a, b]^n$. In view of the Heine-Borel Theorem, therefore, the following observation establishes our assertion.

Observation 1. *Any closed subset S of a compact metric space X is compact.*

Proof. Since X is sequentially compact by Theorem 1, any given sequence in S must have a subsequence that converges somewhere in X. But S is closed in X, so by Proposition 1, the limit of this subsequence belongs to S. Thus S is sequentially compact, and hence, compact.[14] $\|$

[3] *The interval $(0, 1)$ is not compact.* Indeed, the sequence $(\frac{1}{m})$ does not have a sequence that converges to a point in $(0, 1)$, so $(0, 1)$ is not sequentially compact.[15]

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[13] Proving the Heine-Borel Theorem without Theorem 1 is harder. See Ok (2006), Section C.3.1.

[14] *Quiz.* Prove Observation 1 without using Theorem 1.

[15] An argument that is not based on Theorem 1 may be sketched as follows: The collection $\mathcal{O} := \{(\frac{1}{m}, 1) : m \in \mathbb{N}\}$ is an open cover of $(0, 1)$. But the inf of any finite subset of $\mathcal{O}$ is bounded away from 0, so no such subset can possibly cover $(0, 1)$ entirely.

[4] $\mathbb{R}$ *is not compact, but* $\overline{\mathbb{R}}$ *is.* (Recall Example 1.[2].) Indeed, the sequence (m) does not have a sequence that converges to a real number. On the other hand, if (a_m) is a sequence in $\overline{\mathbb{R}}$, then it has a monotonic subsequence (a_{m_k}) by Proposition 9, and since $-\infty \leq a_{m_k} \leq \infty$ for each k, this subsequence must converge in $\overline{\mathbb{R}}$. (Why?)

[5] The compactness property is inherited by products of metric spaces. That is: *The product of countably many compact metric spaces is compact.* (See Ok (2007), Section C.8.2, for a proof.) □

The following is a very useful fact about compact metric spaces.

Proposition 5. *Any compact subset of a metric space X is closed and bounded.*

Proof. Let S be a compact subset of X. Then S is totally bounded, and hence bounded. To see that S is closed, let $(\omega^m) \in S^\infty$ converge to a point in X. By Theorem 1, (ω^m) must have a subsequence that converges to a point x in S. But any subsequence of a convergent sequence must converge to the limit of the mother sequence, and hence $\omega^m \to \omega$. Applying Proposition 1 completes the proof. ■

It is worth noting that the converse of Proposition 5 holds in any Euclidean space. This result is also called the *Heine-Borel Theorem* in the real analysis folklore.

Proposition 6. (Heine-Borel) *Given any $n \in \mathbb{N}$, a subset S of $\mathbb{R}^n$ is compact if, and only if, it is closed and bounded.*

Proof. Due to its boundedness, S must lie within some cube $[a, b]^n$. (Why?) Now apply the Heine-Borel Theorem and Observation 1. ■

Contrary to what Proposition 6 may suggest, a closed and bounded set need not be compact in an arbitrary metric space. That is, compactness is, in general, a stronger property than closedness and boundedness put together; it often introduces significantly more structure to the analysis. For instance, the metric subspace $(0, 1)$ of $\mathbb{R}$ is a closed and bounded metric space – recall that any metric space is closed in itself – but it is, of course, not compact.[16]

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[16] Not convinced? Here is another example. Let $\mathbf{e}^1 := (1, 0, 0, ...)$, $\mathbf{e}^2 := (0, 1, 0, ...)$, etc., and check that $S := \{(\mathbf{e}^m) : m \in \mathbb{N}\}$ is a closed and bounded subset of ℓ^2 which is not compact.

The Cantor-Fréchet Intersection Theorem (Exercise 4) tells us in what type of a situation a class of closed subsets of a complete metric space is sure to have a nonempty intersection. We conclude this section by showing how one may improve that result within the realm of compact metric spaces.

Example 11. A class $\mathcal{A}$ of sets is said to have the **finite intersection property** if $\bigcap \mathcal{B} \neq \emptyset$ for any finite (nonempty) subclass $\mathcal{B}$ of $\mathcal{A}$. Suppose that X is a metric space and $\mathcal{A} \subseteq \mathcal{C}_X$. Question: If $\mathcal{A}$ has the finite intersection property, is $\bigcap \mathcal{A} \neq \emptyset$?

The answer depends on the structure of X. If X is finite, it is obviously affirmative, for then $\mathcal{A}$ is itself a finite class. In keeping with the intuition that "compactness" and "finiteness" are parallel concepts, the answer is also affirmative when X is compact. Indeed, if $\bigcap \mathcal{A} = \emptyset$, then $X = X \backslash (\bigcap \mathcal{A}) = \bigcup \{X \backslash A : A \in \mathcal{A}\}$, so $\{X \backslash A : A \in \mathcal{A}\}$ is an open cover of X. So, if X is compact, there is a finite subset $\mathcal{B}$ of $\mathcal{A}$ with $\{X \backslash A : A \in \mathcal{B}\} = X$, which implies $\bigcap \mathcal{B} = \emptyset$, contradicting the finite intersection property of $\mathcal{A}$. We proved: *A class of closed sets in a compact metric space which has the finite intersection property has a nonempty intersection.* $\square$

1.6 Separable Metric Spaces

As it is an analogue of the property of "finiteness," one may be tempted to view the compactness property as particular to "small" metric spaces. This is actually a problematic viewpoint,[17] and moreover, compactness is often too limiting – even $\mathbb{R}$ is not compact – and we need another way of thinking about the size of metric spaces. The notion of *separability* is one such notion.

Definition. Let X be a metric space and $Y \subseteq X$. If $cl_X(Y) = X$, then Y is said to be **dense in** X (or a **dense subset of** X). In turn, X is said to be **separable** if it contains a countable dense set.

Thanks to Exercise 2, it is readily observed that a set $Y \subseteq X$ is dense in a metric

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[17] If you view a given compact metric space as "small," then in all probability you would view a metric *sub*space of that space also as "small." Compactness does not allow for this. (*Example.* $[0,1]$ is compact, but $(0,1)$ is not.) Besides, even a countable set – which is set-theoretically "small" – need not be compact. (*Example.* $\mathbb{Q}$.)

space X iff any point in X can be approached by means of a sequence in Y. So, X is a separable metric space iff it contains a countable set Y such that $\omega \in X$ iff $\omega^m \to \omega$ for some $(\omega^m) \in Y^\infty$. Intuitively, then, there is sense in regarding a separable metric space as a space that is "small." After all, in such a space, there is a countable set which is "almost" equal to the entire space.

Example 12. [1] A countable metric space is separable.

[2] $\mathbb{R}$ *is separable.* (*Proof.* Apply Lemma 1.)

[3] $\mathbb{R}^n$ *is separable*, $n = 1, 2, ...$ (*Proof.* $\mathbb{Q}^n$ is a countable dense set in $\mathbb{R}^n$.)

[4] ℓ^p *is separable for any* $1 \leq p < \infty$. Indeed, it can be shown that the set of the sequences in $\ell^p \cap \mathbb{Q}^\infty$ all but finitely many components of which are zero, is dense in ℓ^p, for any $1 \leq p < \infty$. (See Ok (2007), Section C.2.2.)

[5] $\mathbf{C}[a, b]$ *is separable for any* $-\infty < a \leq b < \infty$. This is a corollary of the following famous result.

The Weierstrass Approximation Theorem. *The set of all polynomials defined on* $[a, b]$ *is dense in* $\mathbf{C}[a, b]$. *That is,* $cl_{\mathbf{C}[a,b]}(\mathbf{P}[a, b]) = \mathbf{C}[a, b]$.

A proof of this theorem is given in Chapter E by means of a curious application of the Weak Law of Large Numbers. Presently, it is enough to see that the separability of $\mathbf{C}[a, b]$ follows readily from this theorem and the fact that the set of all polynomials on $[a, b]$ with rational coefficients is dense in $\mathbf{P}[a, b]$.[18]

[6] *Every compact metric space* X *is separable.* (*Proof.* $\{N_{\frac{1}{m},X}(\omega) : \omega \in X\}$ is an open cover of X for each $m \in \mathbb{N}$. So, if X is compact, there is a finite $S_m \subseteq X$ with $\{N_{\frac{1}{m},X}(\omega) : \omega \in S_m\} = X$ (for each m), and hence $S_1 \cup S_2 \cup \cdots$ is dense in X.)

[7] *The product of countably many separable metric spaces is separable.* (*Proof.* See Ok (2007), Section C.8.2.) $\square$

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[18] Observe that the cardinality of this set is equal to that of $\mathbb{Q} \cup \mathbb{Q}^2 \cup \cdots$ (and not $\mathbb{Q}^\infty$). Since a countable union of countable sets is countable (Proposition 4), the set of all polynomials on $[a, b]$ with rational coefficients is thus countable.

The following result is sometimes useful in deducing the separability of a metric space from that of another. It also supplies further credence for thinking of a separable metric space as "small."

Proposition 7. *Any metric subspace of a separable metric space is separable.*

Proof. Let X be any metric space, and Y a countable dense subset of X. Take any metric subspace Z of X. Define

$$Y_m := \{\nu \in Y : N_{\frac{1}{m},X}(\nu) \cap Z \neq \emptyset\}, \qquad m = 1, 2, ...$$

(*Note.* Each Y_m is nonempty, thanks to the denseness of Y.) Now pick an arbitrary $\varsigma^m(\nu) \in N_{\frac{1}{m},X}(\nu) \cap Z$ for each $m \in \mathbb{N}$ and $\nu \in Y_m$, and define

$$W := \{\varsigma^m(\nu) : \nu \in Y_m \text{ and } m \in \mathbb{N}\}.$$

Clearly, W is a countable subset of Z. (Yes?) Now take any $\varsigma \in Z$. By denseness of Y, for each $m \in \mathbb{N}$ we can find a $\nu^m \in Y$ with $d(\varsigma, \nu^m) < \frac{1}{m}$. So $\varsigma \in N_{\frac{1}{m},X}(\nu^m) \cap Z$, and hence $\nu^m \in Y_m$. Therefore, $d(\varsigma, \varsigma^m(\nu^m)) \leq d(\varsigma, \nu^m) + d(\nu^m, \varsigma^m(\nu^m)) < \frac{2}{m}$. So, any point in Z is in fact a limit of some sequence in W, that is, W is dense in Z. ■

Separable metric spaces are useful precisely because in such spaces all open sets can be described in terms of a (fixed) countable class of open sets. For instance, all open subsets of $\mathbb{R}^n$ can be expressed by using the members of the countable collection $\mathcal{O} := \{N_{\varepsilon,\mathbb{R}^n}(\omega) : \omega \in \mathbb{Q}^n \text{ and } \varepsilon \in \mathbb{Q}_{++}\}$. Indeed, by using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, it can easily be verified that any open subset of U equals the union of sets in $2^U \cap \mathcal{O}$. Thus, there is sense in which all open subsets of $\mathbb{R}^n$ are *generated* by a given countable collection of open sets in $\mathbb{R}^n$. The following result generalizes this observation to the case of an arbitrary separable metric space.

Proposition 8. *A metric space X is separable if, and only if, there exists a countable $\mathcal{O} \subseteq \mathcal{O}_X$ such that*

$$U = \bigcup\{O \in \mathcal{O} : O \subseteq U\} \quad \text{for any } U \in \mathcal{O}_X.$$

Proof. We only prove here the "only if" part, leaving the other direction as an exercise. Assume then that X is a separable metric space, pick any countable dense subset Y of X, and define $\mathcal{O} := \{N_{\varepsilon,X}(\nu) : \nu \in Y \text{ and } \varepsilon \in \mathbb{Q}_{++}\}$. This is a countable class as it is in one-to-one correspondence with $Y \times \mathbb{Q}_{++}$. Now pick any $U \in \mathcal{O}_X$, and take any $\omega \in U$. We claim that $\omega \in O$ for some $O \in \mathcal{O}$ with $O \subseteq U$. Since U is open in X, there is an $\varepsilon \in \mathbb{Q}_{++}$ such that $N_{\varepsilon,X}(\omega) \subseteq U$. But $cl_X(Y) = X$, so there exists a $\nu \in Y$ with $d(\omega, \nu) < \frac{\varepsilon}{2}$. But then $\omega \in N_{\frac{\varepsilon}{2},X}(\nu) \subseteq N_{\varepsilon,X}(\omega) \subseteq U$. (Yes?) This proves that $U \subseteq \bigcup\{O \in \mathcal{O} : O \subseteq U\}$. The converse containment is obvious. ■

Many interesting properties of a metric space are defined through the notion of an open set. For instance, if we know all open subsets of a metric space, then we know all closed subsets of this space. (Why?) Similarly, if we know all open subsets of a metric space, then we know which sequences in this space are convergent and which ones are not. (Why?) Consequently, a lot can be learned about the general structure of a metric space X by investigating $\mathcal{O}_X$. But the significance of this observation would be limited, if, in a sense, we had "too many" open sets lying around. In the case of a separable metric space this is not the case, for such a space has *only* a countable number of open sets "that matter" in the sense that all other open subsets of this space can be described using only these open sets. This is the gist of Proposition 8.

Exercise 5. (a) Let X be a metric space such that there exists an $m \in \mathbb{N}$ and an uncountable $S \subseteq X$ such that $d(\omega, \nu) \geq \frac{1}{m}$ for all distinct $\omega, \nu \in S$. Show that X cannot be separable.
(b) Show that ℓ^∞ is not a separable metric space.

2 Continuity

2.1 Continuous Functions

Let us now talk a bit about the "continuity" of maps between two metric spaces.

Definition. Let (X, d) and (Y, d_Y) be two metric spaces. We say that the map $\varphi : X \to Y$ is **continuous at** $\omega \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varphi(N_{\delta,X}(\omega)) \subseteq N_{\varepsilon,Y}(\varphi(\omega)),$$

that is,

$$d(\omega, \nu) < \delta \quad \text{implies} \quad d_Y(\varphi(\omega), \varphi(\nu)) < \varepsilon$$

for every $\nu \in X$. If φ is not continuous at ω, then it is said to be **discontinuous at** ω. For any nonempty $S \subseteq X$, we say that φ is **continuous on** S, if it is continuous at every $\omega \in S$. In turn, φ is said to be **continuous**, if it is continuous on X.

Notation. The set of all continuous maps from the metric space X into the metric space Y is denoted by $\mathbf{C}(X, Y)$, but if $Y = \mathbb{R}$ here, we write $\mathbf{C}(X)$ for $\mathbf{C}(X, \mathbb{R})$.

Notation. Some authors prefer to write $f : (X, d) \to (Y, d_Y)$ to make it clear that the continuity properties of f depend both on d and d_Y. Since it leads to somewhat cumbersome notation, we often refrain from doing this, but it is advisable that you view the notation $f : X \to Y$ (or $f \in Y^X$) as $f : (X, d) \to (Y, d_Y)$ throughout this section. After a while, this will become automatic anyway.

Example 13. [1] $\mathrm{id}_X \in \mathbf{C}(X, X)$ for any metric space X, for we have $\mathrm{id}_X(N_{\varepsilon,X}(x)) = N_{\varepsilon,X}(x) = N_{\varepsilon,X}(\mathrm{id}_X(x))$ for all $x \in X$ and $\varepsilon > 0$. Similarly, any constant function on a metric space is continuous.

[2] Consider the map $\varphi : \ell^\infty \to \mathbb{R}$ defined by $\varphi((a_m)) := \sup\{|a_m| : m \in \mathbb{N}\}$. For any (a_m) and (b_m) in ℓ^∞, we have $|\varphi((a_m)) - \varphi((b_m))| \leq d_\infty((a_m), (b_m))$. (Prove!) This implies that, for any $\varepsilon > 0$ and $(a_m) \in \ell^\infty$, we have $\varphi(N_{\varepsilon,\ell^\infty}((a_m))) \subseteq N_{\varepsilon,\mathbb{R}_+}(\varphi((a_m)))$. Thus: $\varphi \in \mathbf{C}(\ell^\infty)$.

[3] Let S be a nonempty subset of a metric space X. The distance between S and a point $\omega \in X$ is defined as

$$d(\omega, S) := \inf\{d(\omega, \varsigma) : \varsigma \in S\}.$$

Thus the function $\varphi \in \mathbb{R}_+^X$ defined by $\varphi(\omega) := d(\omega, S)$, measures the distance of any given point in X from the set S. It is important to note that this function is continuous. Indeed, for each $\omega, \nu \in X$, the triangle inequality yields

$$\varphi(\omega) = d(\omega, S) \leq \inf\{d(\omega, \nu) + d(\nu, \varsigma) : \varsigma \in S\} = d(\omega, \nu) + \varphi(\nu),$$

and similarly, $\varphi(\nu) \leq d(\nu, \omega) + \varphi(\omega)$. So $|\varphi(\omega) - \varphi(\nu)| \leq d(\omega, \nu)$ for all $\omega, \nu \in X$, and hence $\varphi \in \mathbf{C}(X)$. (*Corollary.* $d(\nu, \cdot)$ and $d(\cdot, \nu)$ belong to $\mathbf{C}(X)$ for any $\nu \in X$.) $\square$

The following characterization of the continuity property is basic.

Proposition 9. *For any metric spaces X and Y, and $\varphi \in Y^X$, the following statements are equivalent:*

(a) *φ is continuous;*

(b) *$\varphi^{-1}(O) \in \mathcal{O}_X$ for every $O \in \mathcal{O}_Y$;*

(c) *$\varphi^{-1}(S) \in \mathcal{C}_X$ for every $S \in \mathcal{C}_Y$;*

(d) *For any $\omega \in X$ and $(\omega^m) \in X^\infty$, $\omega^m \to \omega$ implies $\varphi(\omega^m) \to \varphi(\omega)$.*[19]

Proof. We leave it as an exercise to prove that (a) implies (b), and (b) implies (c). The equivalence of (b) and (c), on the other hand, follows from the fact that $X \backslash \varphi^{-1}(A) = \varphi^{-1}(Y \backslash A)$ for any $A \subseteq X$. Finally, assume (d), and take any $\omega \in X$ and $\varepsilon > 0$. We wish to find a $\delta > 0$ such that $\varphi(N_{\delta,X}(\omega)) \subseteq N_{\varepsilon,Y}(\varphi(\omega))$. If there is no such δ, then we can find a $(\nu^m) \in Y^\infty$ with $\nu^m \in \varphi(N_{\frac{1}{m},X}(\omega)) \backslash N_{\varepsilon,Y}(\varphi(\omega))$ for each m. Clearly, $\nu^m = \varphi(\omega^m)$ for some $\omega^m \in N_{\frac{1}{m},X}(\omega)$, $m = 1, 2, \ldots$. But it is obvious that $\omega^m \to \omega$ so, by (d), $\nu^m \to \varphi(\omega)$. This implies that there exists an integer M such that $\nu^m \in N_{\varepsilon,X}(\varphi(\omega))$ for all $m \geq M$, contradicting the choice of ν^m. Conclusion: (d) implies (a). ■

Depending on the nature of the problem at hand, any one of the viewpoints provided by Proposition 9 may prove more useful than the others.

Example 14. In the following examples X, Y and Z are arbitrary metric spaces.

[1] If $\varphi \in \mathbf{C}(X, Y)$ and $\phi \in \mathbf{C}(\varphi(X), Z)$, then $\phi \circ \varphi \in \mathbf{C}(X, Z)$. (Here we consider $\varphi(X)$ as a subspace of Y.) The proof follows easily from either the sequential or the open set characterization of continuity.

[2] For any $n \in \mathbb{N}$, take any metric spaces W_i, $i = 1, \ldots, n$, let W be the product of these spaces, and take any $\varphi_i : X \to W_i$, $i = 1, \ldots, n$. Now define the map $\varphi : X \to W$ by $\varphi(\omega) := (\varphi_1(\omega), \ldots, \varphi_n(\omega))$. Then, φ is continuous iff each φ_i is continuous. (*Proof.* Apply the sequential characterization of continuity and Example 7.)

[3] For any $n \in \mathbb{N}$, take any $\varphi_i \in \mathbf{C}(X)$, $i = 1, \ldots, n$, and $\phi \in \mathbf{C}(\mathbb{R}^n)$. Define the map $\psi \in \mathbb{R}^X$ by $\psi(\omega) := \phi(\varphi_1(\omega), \ldots, \varphi_n(\omega))$. Then we have $\psi = \phi \circ \varphi$, where

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[19] If $\varphi(\omega^m) \to \varphi(\omega)$ holds for any $(\omega^m) \in X^\infty$ that converges to $\omega \in X$, then φ is continuous at ω. Thus the sequential characterization of continuity applies *locally* as well.

$\varphi : X \to \mathbb{R}^n$ is defined by $\varphi(\omega) := (\varphi_1(\omega), ..., \varphi_n(\omega))$. It follows from the findings of [1] and [2] that ψ is continuous.

Similarly, if W is the product of the metric spaces W_i, $i = 1, ..., n$, and $(\phi, \varphi_i) \in \mathbf{C}(\mathbb{R}^n) \times \mathbf{C}(W_i)$, $i = 1, ..., n$, then $\psi \in \mathbb{R}^X$ defined by $\psi(\omega^1, ..., \omega^n) := \phi(\varphi_1(\omega^1), ..., \varphi_n(\omega^n))$, is a continuous function. (Why?)

[4] *A continuous function is determined by its values on a dense set.* That is, if S is a dense subset of X, and $\varphi, \phi \in \mathbf{C}(X, Y)$ satisfy $\varphi|_S = \phi|_S$, then, $\varphi = \phi$. For, denseness of S in X implies that, for any $\omega \in X$ there exists an $(\omega^m) \in S^\infty$ with $\omega^m \to \omega$ so that $\varphi(\omega) = \lim \varphi(\omega^m) = \lim \phi(\omega^m) = \phi(\omega)$.

[5] Let $\varphi, \psi \in \mathbf{C}(X)$. Then, $\varphi + \psi$, $|\varphi|$ and $\varphi\psi$ are continuous, while $\frac{\varphi}{\psi}$ is continuous provided that it is well-defined. (*Proof.* Apply the sequential characterization of continuity and Exercise 13.) $\square$

Example 15. In the following examples X and Y are arbitrary metric spaces.

[1] If $\varphi \in Y^X$ is an open injection, that is, φ is an injection that maps every open subset of X onto an open subset of Y, then φ^{-1} is a continuous function on $\varphi(X)$. This fact follows immediately from the open set characterization of continuity.

[2] The set $\{x \in X : \varphi(x) \geq a\}$ is closed for any $\varphi \in \mathbf{C}(X)$ and $a \in \mathbb{R}$. Since $\{x : \varphi(x) \geq a\} = \varphi^{-1}([a, \infty))$, this is immediate from the closed set characterization of continuity. $\square$

Exercise 6. Let X be a metric space, and $\varphi \in \mathbf{C}(X)$. Prove that if $\varphi(\omega) > 0$ for some $\omega \in X$, then there exists an $O \in \mathcal{O}_X$ such that $\varphi(\nu) > 0$ for all $\nu \in O$.

Exercise 7. Let S be a nonempty closed subset of a metric space X. Prove that $\omega \in S$ iff $d(\omega, S) = 0$.

Exercise 8. For any metric space X, show that the metric d of X belongs to $C(X \times X)$.

Exercise 9. For any $n \in \mathbb{N}$, a metric space X, and $\varphi_i \in \mathbf{C}(X)$, $i = 1, ..., n$, show that $|\varphi_1|$, $\sum^n \varphi_i$, $\prod^n \varphi_i$, $\max\{\varphi_1, ..., \varphi_n\}$ and $\min\{\varphi_1, ..., \varphi_n\}$ all belong to $\mathbf{C}(X)$.

2.2 Homeomorphisms

If φ is a bijection from a metric space X onto another metric space Y, and both φ and φ^{-1} are continuous, then it is called a **homeomorphism** between X and Y. If there exists such a bijection, then we say that X and Y are **homeomorphic** (or that "X is homeomorphic to Y"). If φ is not necessarily surjective, but $\varphi : X \to \varphi(X)$ is a homeomorphism, then φ is called an **embedding** (from X into Y). If there exists such an injection, we say that "X can be embedded in Y."

Two homeomorphic spaces are indistinguishable from each other insofar as their neighborhood structures are concerned. If X and Y are homeomorphic, then corresponding to every $O \in \mathcal{O}_X$, there is a $\varphi(O) \in \mathcal{O}_Y$, and conversely, corresponding to every $U \in \mathcal{O}_Y$, there is a $\varphi^{-1}(U) \in \mathcal{O}_X$. (*Proof.* Apply Proposition 9.) Therefore, loosely speaking, Y possesses any property that X possesses so long as this property is defined in terms of open sets. (Such a property is called a *topological property.*[20])

For instance, compactness is a topological property. (Why?) It follows that neither $(0,1)$ nor $\mathbb{R}_+$ is homeomorphic to $[0,1]$. Yet $[0,1)$ is homeomorphic to $\mathbb{R}_+$. (*Proof.* $t \mapsto \frac{t}{1-t}$ is a homeomorphism between $[0,1)$ and $\mathbb{R}_+$.) Thus, neither completeness nor boundedness need to be preserved by a homeomorphism – these are *not* topological properties.[21] So bear in mind that there are important senses in which two homeomorphic metric spaces may be of different character. If, however, $\varphi \in Y^X$ is a homeomorphism that preserves the distance between any two points, that is, if

$$d_Y(\varphi(x), \varphi(y)) = d(x, y) \quad \text{for all } x, y \in X,$$

then we may conclude that the spaces (X, d) and (Y, d_Y) are indistinguishable as metric spaces – one is merely a relabelling of the other. In this case, X and Y are said to be **isometric**, and we say that φ is an **isometry** between them. For instance, $\overline{\mathbb{R}}$ and $[-1,1]$ are isometric (Example 1.[2]).

Exercise 10. For any $a, b \in R$ with $a < b$, show that $\mathbf{C}[0,1]$ and $\mathbf{C}[a,b]$ are isometric.

2.3 The Hilbert Cube

When the set $[0,1]^\infty$ is metrized by the product metric, the resulting metric space is called the **Hilbert cube**. But note well: The Hilbert cube is not a metric subspace

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[20] Formally, a property for metric spaces is referred to as a **topological property** if it is invariant under any homeomorphism, that is, whenever this property is true for X, it must also be true for any other metric space that is homeomorphic to X.

[21] In fact, *every* metric space (X, d) is homeomorphic to a bounded metric space. (*Proof.* Compare (X, d) with $(X, \frac{d}{1+d})$.)

of ℓ^∞. True, for any sequence (ω^m) in $[0,1]^\infty$, $d_\infty(\omega^m,\omega) \to 0$ implies $\omega_i^m \to \omega_i$ for each i, and hence, $\boldsymbol{\rho}(\omega^m,\omega) \to 0$ (Example 7). Yet the converse implication is false. For instance, for the sequence (ω^m) in $[0,1]^\infty$ with

$$\omega^m = \left(\tfrac{1}{m}, \tfrac{1}{m-1}, ..., \tfrac{1}{2}, 1, 0, 0, ...\right), \qquad m = 2, 3, ...,$$

we have $\omega_i^m \to 0$ for each i, but $d_\infty(\omega^m,(0,0,...)) = 1$ for all m. Thus $\boldsymbol{\rho}(\omega^m,(0,0,...)) \to 0$ and yet (ω^m) is not even Cauchy in ℓ^∞. Conclusion: *A convergent sequence in* $([0,1]^\infty, d_\infty)$ *is also convergent in* $[0,1]^\infty$*, but not conversely.* It follows that a closed subset of $[0,1]^\infty$ is closed in $([0,1]^\infty, d_\infty)$, but not conversely. (Why?) That is, there are fewer closed (and hence open) sets in the Hilbert cube $[0,1]^\infty$ than in $([0,1]^\infty, d_\infty)$.

Exercise 11. Metrize the set $\mathsf{X}^\infty[0,\frac{1}{i}]$ by the product metric, and show that the resulting metric space is homeomorphic to the Hilbert cube.

The Hilbert cube $[0,1]^\infty$ is compact (Example 10.[5]), and hence separable (Example 12.[6]). In fact, and this is mainly why the Hilbert cube is of interest to us, there is a sense in which $[0,1]^\infty$ "includes" all separable metric spaces.

Proposition 10. *Every separable metric space X can be embedded in the Hilbert cube* $[0,1]^\infty$.

Proof. Define the metric D on X by $D(\omega,\omega') := \min\{1, d(\omega,\omega')\}$. Let $Y = \{\nu^1,\nu^2,...\}$ be a countable dense subset of X, and define the map $\varphi : X \to [0,1]^\infty$ by $\varphi(\omega) := (D(\nu^1,\omega), D(\nu^2,\omega), ...)$. We wish to show that φ is a homeomorphism from X onto $\varphi(X)$. That φ is a continuous injection is easily proved; we leave this as an exercise. To show that $\varphi^{-1} \in \mathbf{C}(\varphi(X), X)$, take any $\omega, \omega^1, \omega^2, ...$ in X such that $\varphi(\omega^m) \to \varphi(\omega)$. The proof will be complete if we can establish that $\omega^m \to \omega$. (Yes?)

Fix an arbitrarily small $0 < \varepsilon < 1$. Since Y is dense in X, we can find a $k \in \mathbb{N}$ such that $d(\nu^k,\omega) < \frac{\varepsilon}{4}$. Since $\boldsymbol{\rho}(\varphi(\omega^m),\varphi(\omega)) \to 0$, we have $D(\nu^i,\omega^m) \to D(\nu^i,\omega)$ for each $i = 1, 2, ...$ (Example 7). So, there exists an $M > 0$ such that $d(\nu^k,\omega^m) < d(\nu^k,\omega) + \frac{\varepsilon}{2}$ for all $m \geq M$. (Why?) Then

$$d(\omega^m,\omega) \leq d(\omega^m,\nu^k) + d(\nu^k,\omega) < 2d(\nu^k,\omega) + \tfrac{\varepsilon}{2} < \varepsilon$$

for all $m \geq M$. Conclusion: $\omega^m \to \omega$. ■

2.4 Polish Spaces

A metric space that is both complete and separable is often referred to as a **Polish space** (because these spaces were first studied by Polish mathematicians Sierpinski, Kuratowski, Tarski and others). Such spaces make frequent appearence in probabilistic analysis, primarily due to the following fact.

Proposition 11. *Every Polish space X is homeomorphic to the intersection of a countable class of open subsets of the Hilbert cube.*

Proof. Define $\varphi : X \to [0,1]^\infty$ as in the proof of Proposition 10. We already know that $\varphi : X \to \varphi(X)$ is a homeomorphism, so our task here is only to prove that $\varphi(X) = \bigcap \mathcal{O}$ for some countable $\mathcal{O} \subseteq \mathcal{O}_{[0,1]^\infty}$. To this end, for each $m \in \mathbb{N}$ and $\omega \in X$, we pick an open subset $O_{m,\omega}$ of $[0,1]^\infty$ such that

$$\varphi(\omega) \in O_{m,\omega}, \quad \mathit{diam}(O_{m,\omega}) < \tfrac{1}{m} \text{ and } \mathit{diam}(\varphi^{-1}(O_{m,\omega})) < \tfrac{1}{m}.$$

(We can do this because both φ and φ^{-1} are continuous.[22]) Now define

$$O_m := \bigcup \{O_{m,\omega^m} : \omega \in X\} \text{ and } \mathcal{O} := \{O_m : m = 1, 2, ...\}.$$

It is plain that $\varphi(X) \subseteq \bigcap \mathcal{O}$. To prove the converse containment, take any $\zeta \in \bigcap \mathcal{O}$. Then, for any m, there is an $O_{m,\omega^m} \in X$ such that $\zeta \in O_{m,\omega^m}$. (To simplify notation, we let $U_m := O_{m,\omega^m}$ in what follows.) Since $\varphi(\omega^m) \in U_m$, we have $\boldsymbol{\rho}(\zeta, \varphi(\omega^m)) < \frac{1}{m}$ for each m, and hence $\varphi(\omega^m) \to \zeta$, and it follows that ζ belongs to the closure of $\varphi(X)$ in $[0,1]^\infty$.

Now fix any positive integers k and l, and note that $U_k \cap U_l$ is an open subset of $[0,1]^\infty$ that contains ζ. Since ζ is in the closure of $\varphi(X)$, therefore, this set must intersect $\varphi(X)$ – yes? – that is, $\zeta_* \in U_k \cap U_l$ for some $\zeta_* \in \varphi(X)$. Then $\varphi^{-1}(\zeta_*) \in \varphi^{-1}(U_k) \cap \varphi^{-1}(U_l)$ and

$$d(\omega^k, \omega^l) \leq d(\omega^k, \varphi^{-1}(\zeta_*)) + d(\varphi^{-1}(\zeta_*), \omega^l) < \tfrac{1}{k} + \tfrac{1}{l}.$$

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[22] I'm using the open set characterization of continuity here, of course. Let's see. Fix any x and m, and let A be the $\frac{1}{2m}$-neighborhood of $\varphi(x)$ in $[0,1]^\infty$. Then $\varphi(x) \in A$ and $\mathit{diam}(A) < \frac{1}{m}$. Now, since $x \in \varphi^{-1}(A)$ and $\varphi^{-1}(A)$ is open (Proposition 20), we can obviously find an open neighborhood B of x such that $B \subseteq \varphi^{-1}(A)$ and $\mathit{diam}(B) < \frac{1}{m}$. Then, $O := \varphi(B)$ is an open neighborhood of $\varphi(x)$ in $[0,1]^\infty$ (again, by Proposition 20) such that $\mathit{diam}(\varphi^{-1}(O)) = \mathit{diam}(B) < \frac{1}{m}$ and $O \subseteq \varphi(\varphi^{-1}(A)) = A$.

It follows that (ω^m) is a Cauchy sequence in X, so $\omega^m \to \omega$ for some $\omega \in X$. So we have $\varphi(\omega^m) \to \zeta$ and $\varphi(\omega^m) \to \varphi(\omega)$, and hence $\zeta = \varphi(\omega)$, establishing that $\zeta \in \varphi(X)$, as we sought. ■

2.5 Stronger Notions of Continuity

The notion of continuity is an inherently local one. If $\varphi \in \mathbf{C}(X, Y)$ is continuous, we know that, for any $\omega \in X$, "the images of points nearby ω under φ are close to $\varphi(\omega)$," but we don't know if the word "nearby" in this statement depends on ω or not. A global property would make this statement independently of ω by regulating the behavior of φ on its entire domain.

Definition. Let X and Y be two metric spaces. We say that $\varphi \in Y^X$ is **uniformly continuous** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(\omega, \nu) < \delta \quad \text{implies} \quad d_Y(\varphi(\omega), \varphi(\nu)) < \varepsilon$$

for every $\omega, \nu \in X$, that is, $\varphi(N_{\delta,X}(\omega)) \subseteq N_{\varepsilon,Y}(\varphi(\omega))$ for all $\omega \in X$.

Obviously, a uniformly continuous function is continuous, but not conversely. (*Example.* The map $t \mapsto \frac{1}{t}$ is not uniformly continuous on $\mathbb{R}_{++}$.)

> **Exercise 12**. Let X and Y be two metric spaces and $\varphi \in Y^X$. Show that if $(\omega^m) \in X^\infty$ is Cauchy and φ is uniformly continuous, then $(\varphi(\omega^m)) \in Y^\infty$ is Cauchy. (*Corollary.* If there exists a bijection $\varphi \in Y^X$ such that both φ and φ^{-1} are uniformly continuous and X is complete, then Y must be complete as well.) Would this be true if φ was only known to be continuous?

Sometimes one needs to work with a continuity notion that is even stronger than uniform continuity. For instance, the following notion is used quite frequently in applications.

Definition. Let X and Y be two metric spaces. We say that a function $\varphi \in Y^X$ is **Lipschitz continuous** if there exists a $K > 0$ such that

$$d_Y(\varphi(\omega), \varphi(\nu)) \leq K d(\omega, \nu) \quad \text{for all } \omega, \nu \in X.$$

(The infimum of the set of all such Ks is called the **Lipschitz constant** of φ. If $K = 1$ here, then φ is called **nonexpansive**.)

We have already seen some examples of nonexpansive and Lipschitz continuous functions. For instance, all maps studied in Example 13 are nonexpansive. Moreover, any concave real map on a compact interval is Lipschitz continuous (Proposition 11), but it does not have to be nonexpansive.[23]

2.6 Weaker Notions of Continuity

The following weaker notions of continuity play an important role in the theory of real functions and optimization theory.

Definition. Let X be any metric space, and $\varphi \in \mathbb{R}^X$. We say that φ is **upper semicontinuous at** $\omega \in X$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(\omega, \nu) < \delta \quad \text{implies} \quad \varphi(\nu) \leq \varphi(\omega) + \varepsilon$$

for each $\nu \in X$. It is called **lower semicontinuous at** ω, if $-\varphi$ is upper semicontinuous at ω. The function φ is said to be **upper (lower) semicontinuous**, if it is upper (lower) semicontinuous at each $\omega \in X$.

In a manner of speaking, if $\varphi \in \mathbb{R}^X$ is upper semicontinuous at x, then the images of points nearby x under φ do not exceed $\varphi(x)$ "too much," while there is no restriction about how far these images can fall below $\varphi(x)$. Similarly, if φ is lower semicontinuous at x, then the images of points nearby x under φ do not fall below $\varphi(x)$ "too much," but they can still be vastly greater than $\varphi(x)$. So, a real function on a metric space is continuous iff it is both upper and lower semicontinuous.

Here is a useful characterization of upper semicontinuous functions which parallels the characterization of continuous functions given in Proposition 9.

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[23] It is often easy to check for the Lipschitz continuity of a differentiable self-map φ on $\mathbb{R}$. Any such φ is Lipschitz continuous if its derivative is bounded, and it is nonexpansive if $\sup\{|\varphi'(t)| : t \in \mathbb{R}\} \leq 1$. (*Proof.* Apply the Mean Value Theorem.)

Proposition 12. *For any metric space X, and $\varphi \in \mathbb{R}^X$, the following statements are equivalent:*

(a) *φ is upper semicontinuous;*

(b) *$\varphi^{-1}((-\infty, a)) \in \mathcal{O}_X$ for every $a \in \mathbb{R}$;*

(c) *$\varphi^{-1}([a, \infty)) \in \mathcal{C}_X$ for every $a \in \mathbb{R}$;*

(d) *For any $\omega \in X$ and $(\omega^m) \in X^\infty$, $\omega^m \to \omega$ implies $\varphi(\omega) \geq \limsup \varphi(\omega^m)$.*

Exercise 13. Prove Proposition 12.

By using the fact that $\inf\{\varphi(\omega) : \omega \in X\} = -\sup\{-\varphi(\omega) : \omega \in X\}$ for any $\varphi \in \mathbb{R}^X$, one can easily recover from Proposition 12 the corresponding characterizations for lower semicontinuous functions. For example, φ is lower semicontinuous iff $\varphi^{-1}((a, \infty))$ is open for every $a \in \mathbb{R}$ iff $\varphi(\lim \omega_m) \leq \liminf \varphi(\omega_m)$ for any convergent sequence (ω_m) in X.

2.7 Approximation by Continuous Functions

When, in an application, the function of interest is known only to be semicontinuous, and not necessarily continuous, we may wish to resort to a (pointwise) approximation of that function by means of continuous maps. The following result shows that this is possible to do in a variety of circumstances.

Proposition 13. (Baire) *Let X be a metric space, and $\varphi : X \to \mathbb{R}$ a lower semicontinuous map. If φ is bounded from below, then there exists a sequence (φ_m) in $\mathbf{C}(X)$ such that $\varphi_m \nearrow \varphi$.*

Proof. Assume that φ is bounded from below, that is, $\inf \varphi(X) > -\infty$. It is then without loss of generality to assume $\varphi \geq 0$ (for otherwise we would work with $\varphi - \inf \varphi(X)$ instead of φ). For a fixed, but arbitrary, positive integer m, we define the map $\varphi_m : X \to \mathbb{R}$ by

$$\varphi_m(\omega) := \inf\{\varphi(\nu) + md(\omega, \nu) : \nu \in X\}. \tag{6}$$

This function is in fact Lipschitz continuous (with Lipschitz constant m). To see this, take any $\omega, \omega' \in X$, and note that, by (6) and the triangle inequality,

$$\varphi_m(\omega) \leq \varphi(\nu) + md(\omega, \nu) \leq \varphi(\nu) + m\left(d(\omega, \omega') + d(\omega', \nu)\right),$$

and hence $\varphi_m(\omega) - md(\omega, \omega') \leq \varphi(\nu) + md(\omega', \nu)$ for any $\nu \in X$. It follows that

$$\varphi_m(\omega) - md(\omega, \omega') \leq \inf\{\varphi(\nu) + md(\omega', \nu) : \nu \in X\} = \varphi_m(\omega').$$

Interchanging the roles of ω and ω' in this argument, then, we find

$$|\varphi_m(\omega) - \varphi_m(\omega')| \leq md(\omega, \omega'),$$

as we sought.

It is obvious from (6) that $0 \leq \varphi_m \leq \varphi_{m+1} \leq \varphi$ for each $m \in \mathbb{N}$, so we have $\lim \varphi_m \leq \varphi$. It remains to prove the converse inequality. To this end, suppose a is a real number with $a < \varphi(\omega)$ for some arbitrarily fixed $\omega \in X$. Since, by Proposition 12 (adapted to lower semicontinuous maps), $\varphi^{-1}((a, \infty))$ is open, there exists an $\varepsilon > 0$ such that $a < \varphi(\nu)$ for all $\nu \in N_{\varepsilon,X}(\omega)$. Then, for any $m \in \mathbb{N}$,

$$a < \varphi(\nu) + md(\omega, \nu) \quad \text{for all } \nu \in N_{\varepsilon,X}(\omega),$$

and since $\varphi \geq 0$,

$$m\varepsilon \leq \varphi(\nu) + md(\omega, \nu) \quad \text{for all } \nu \in X \backslash N_{\varepsilon,X}(\omega).$$

Therefore, so long as m is greater than $\frac{a}{\varepsilon}$, we have $a \leq \varphi(\nu) + md(\omega, \nu)$ for all $\nu \in X$, that is, $a \leq \varphi_m(\omega)$. Conclusion: For any $a \in \mathbb{R}$,

$$a < \varphi(\omega) \quad \text{implies} \quad a \leq \lim \varphi_m.$$

As this is possible only if $\varphi \leq \lim \varphi_m$, we are done. ■

Needless to say, we may rephrase this result for upper semicontinuous maps as follows: Every upper semicontinuous real map φ on a metric space can be approximated from above by continuous maps, provided that φ is bounded from above.

2.8 Continuity and Compactness

We now turn to the investigation of the properties of continuous functions defined on compact metric spaces. A fundamental observation in this regard is the following:

Proposition 14. *Let X and Y be two metric spaces, and $\varphi \in \mathbf{C}(X, Y)$. If S is a compact subset of X, then $\varphi(S)$ is a compact subset of Y.*

Exercise 14. Prove Proposition 14.

Exercise 15. (The Homeomorphism Theorem) Prove: If X is a compact metric space and $\varphi \in Y^X$ is a continuous bijection, then φ is a homeomorphism.

The famous Weierstrass' Theorem on the optimum of continuous real functions is an immediate consequence of Proposition 24. Alternatively, we can deduce it from its generalization to the case of semicontinuous real maps.

Proposition 15. (Baire) *Let X be a compact metric space, and $\varphi \in \mathbb{R}^X$. If φ is upper semicontinuous, then there exists an $\omega \in X$ with $\varphi(\omega) = \sup \varphi(X)$. If φ is lower semicontinuous, then there exists a ν with $\varphi(\nu) = \inf \varphi(X)$.*

Proof. Clearly, there exists a sequence $(\omega^m) \in X^\infty$ such that $\varphi(\omega^m) \nearrow \sup \varphi(X)$ (Exercise 16). By Theorem 1, there exists a subsequence (ω^{m_k}) of this sequence which converges to some $\omega \in X$. But then, if φ is upper semicontinuous, Proposition 12 implies $\sup \varphi(X) \geq \varphi(\omega) \geq \limsup \varphi(\omega^{m_k}) = \lim \varphi(\omega^m) = \sup \varphi(X)$. ■

In words, an upper semicontinuous function always assumes its maximum (but not necessarily its minimum) over a compact set. Thus, if you are interested in the maximization of an upper semicontinuous function over a compact set, then you're assured of the *existence* of a solution to your maximization problem. By contrast, lower semicontinuity is the relevant property in the case of minimization problems. (See Ok (2007), Section D.4 for more on this.)

Since a real map is continuous iff it is both upper and lower semicontinuous, the following famous result is a special case of Proposition 15.

Weierstrass' Theorem. *If X is a compact metric space and $\varphi \in \mathbf{C}(X)$, then there exist $\omega, \nu \in X$ with $\varphi(\omega) = \sup \varphi(X)$ and $\varphi(\nu) = \inf \varphi(X)$.*

Also worth noting is that combining Propositions 13 and 15 yields the following result on approximation by continuous functions.

Corollary 2. *If X is a compact metric space, and $\varphi : X \to \mathbb{R}$ a lower semicontinuous map, then there exists a sequence (φ_m) in $\mathbf{C}(X)$ such that $\varphi_m \nearrow \varphi$.*

These results show that (semi)continuous maps are particularly well-behaved on a compact metric space. Our final result in this subsection supports this point further.

Proposition 16. (Heine) *Let X and Y be two metric spaces and $\varphi \in \mathbf{C}(X,Y)$. If X is compact, then φ is uniformly continuous.*

Proof. Suppose X is compact but φ is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that we can find two sequences (ω^m) and (ν^m) in X with

$$d(\omega^m, \nu^m) < \tfrac{1}{m} \quad \text{and} \quad d_Y(\varphi(\omega^m), \varphi(\nu^m)) \geq \varepsilon, \quad m = 1, 2, ... \tag{7}$$

(Why?) By Theorem 1, there exists a subsequence (ω^{m_k}) of (ω^m) which converges to some $\omega \in X$. Then, since the first part of (7) guarantees that $\nu^{m_k} \to \omega$, we have $\lim \varphi(\omega^{m_k}) = \varphi(\omega) = \lim \varphi(\nu^{m_k})$ (Proposition 9). Thus $d_Y(\varphi(\omega^M), \varphi(\nu^M)) < \varepsilon$ for some $M \in \mathbb{N}$ large enough, contradicting the second part of (7). ■

2.9 CB(T) and Uniform Convergence

For any given metric space T, an important metric subspace of $\mathbf{B}(T)$ consists of all continuous *and* bounded functions on T – we denote this subspace by $\mathbf{CB}(T)$ throughout this text. When T is compact, Weierstrass' Theorem implies $\mathbf{CB}(T) = \mathbf{C}(T)$, so we can, and will, think of $\mathbf{C}(T)$ as metrized by the sup-metric.

Proposition 17. *$\mathbf{CB}(T)$ is complete for any metric space T.*

Proof. By Proposition 3 and Example 8.[4], it is enough to show that $\mathbf{CB}(T)$ is closed in $\mathbf{B}(T)$. Let (φ_m) be a sequence in $\mathbf{CB}(T)$ and let $d_\infty(\varphi_m, \varphi) \to 0$ for some $\varphi \in \mathbf{B}(T)$. We wish to show that φ is continuous (Proposition 1). Take any $\varepsilon > 0$, and observe that there exists an $M \in \mathbb{N}$ with $\sup\{|\varphi(\varsigma) - \varphi_M(\varsigma)| : \varsigma \in T\} < \frac{\varepsilon}{3}$. Thus,

$$|\varphi(\varsigma) - \varphi_M(\varsigma)| < \tfrac{\varepsilon}{3} \quad \text{for all } \varsigma \in T. \tag{8}$$

(Notice that this statement is a "global" one, for it holds for *all* $\varsigma \in T$ thanks to the sup-metric.) Now take an arbitrary $\omega \in T$. By continuity of φ_M, there exists a $\delta > 0$ such that $\varphi_M(N_{\delta,T}(\omega)) \subseteq N_{\frac{\varepsilon}{3},\mathbb{R}}(\varphi_M(\omega))$. But then, for any $\nu \in N_{\delta,T}(\omega)$,

$$|\varphi(\omega) - \varphi(\nu)| \leq |\varphi(\omega) - \varphi_M(\omega)| + |\varphi_M(\omega) - \varphi_M(\nu)| + |\varphi_M(\nu) - \varphi(\nu)| < \varepsilon$$

where we used (8) twice. ■

In general, "convergence" in $\mathbf{CB}(T)$ has very desirable properties. If we know that a sequence (φ_m) in $\mathbf{CB}(T)$ converges to some $\varphi \in \mathbf{B}(T)$, that is, if

$$\lim_{m\to\infty} (\sup\{|\varphi_m(\omega) - \varphi(\omega)| : \omega \in T\}) = 0, \tag{9}$$

Proposition 17 assures us that φ is continuous. In fact, we have not used in the proof of this result the boundedness of φ_m and φ, all that mattered there was (9) being satisfied by these maps. It follows that, for any sequence (φ_m) in $\mathbf{C}(T)$, and any $\varphi \in \mathbb{R}^T$, (9) implies that $\varphi \in \mathbf{C}(T)$. This suggests the following convergence concept for sequences of real functions.

Definition. Let T be a metric space and (φ_m) a sequence in $\mathbb{R}^T$. If (9) holds for some $\varphi \in \mathbb{R}^T$, then φ is called the **uniform limit** of (φ_m). In this case, we say that (φ_m) **converges to φ uniformly**, and write $\varphi_m \to \varphi$ **uniformly**.

Uniform convergence is the same as convergence relative to d_∞ for bounded maps, and it is a "global" notion of convergence.[24] This leads to the informal principle that "uniform convergence preserves good behavior." Let's illustrate.

Example 16. In the following examples, T stands for an arbitrary metric space.

[1] If $\varphi \in \mathbb{R}^T$, and (φ_m) is a sequence of continuous real maps on T with $\varphi_m \to \varphi$ uniformly, then φ is continuous. (*Proof.* Use the argument we gave for Proposition 17.)

[2] If (φ_m) is a sequence of bounded real maps on T and $\varphi_m \to \varphi$ uniformly for some $\varphi \in \mathbb{R}^T$, then φ is bounded. (*Proof.* (φ_m) is a Cauchy sequence in $\mathbf{B}(T)$.)

[3] (*Interchanging Limits*) Let (ω^k) be a sequence in T such that $\omega^k \to \omega$, and let (φ_m) be a sequence in $\mathbf{C}(T)$ such that $\varphi_m \to \varphi$ uniformly. Then, the numbers

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[24] If φ is the uniform limit of a sequence (φ_m) in $\mathbb{R}^{[0,1]}$, what this means is that, for any $\varepsilon > 0$, eventually the entire graph of each member of the sequence lies within the ε-strip around the graph of φ, that is, $\varphi - \varepsilon < \varphi_m < \varphi + \varepsilon$ holds for all m large enough. This statement is "global" in the sense that it concerns the entire domain of φ, not a particular point in $[0, 1]$.

$\lim_{k\to\infty}\lim_{m\to\infty}\varphi_m(\omega^k)$ or $\lim_{m\to\infty}\lim_{k\to\infty}\varphi_m(\omega^k)$ exist and equal. Indeed, since $\varphi \in \mathbf{C}(T)$ by part [1], we have

$$\lim_{k\to\infty}\lim_{m\to\infty}\varphi_m(\omega^k) = \lim_{k\to\infty}\varphi(\omega^k) = \varphi(\omega) = \lim_{m\to\infty}\varphi_m(\omega) = \lim_{m\to\infty}\lim_{k\to\infty}\varphi_m(\omega^k).$$

This fact provides relief surprisingly often. $\square$

In passing, we note that uniform convergence is more demanding than **pointwise convergence** of a sequence (φ_m) of real functions on a nonempty set T, which, by definition, requires only that

$$\lim |\varphi_m(\omega) - \varphi(\omega)| = 0 \quad \text{for all } \omega \in T. \tag{10}$$

Put differently, $\varphi_m \to \varphi$ **pointwise** if for any $\varepsilon > 0$ and $\omega \in T$, there exists an $M > 0$ (which may depend on *both* ε *and* ω) such that $|\varphi_m(\omega) - \varphi(\omega)| < \varepsilon$. (This situation is denoted simply as $\varphi_m \to \varphi$.) Clearly, equation (9), being a "global" (i.e. ω-independent) statement, implies (10) which is a "local" (ω-dependent) condition. Of course, the converse of this implication does not hold.[25]

2.10 The Separability of $\mathbf{C}(T)$

Recall that $\mathbf{C}[a,b]$ is separable, thanks to the Weierstrass Approximation Theorem (Example 12.[5]). It turns out that what is essential here is the compactness of the interval $[a,b]$, and nothing more. Indeed, by using a generalization of the Weierstrass Approximation Theorem – the famous Stone-Weierstrass Theorem – one can show that the set of all continuous real maps on a compact metric space T is a separable metric subspace of $\mathbf{B}(T)$. (See Ok (2007), Section D.6.3.)

Proposition 18. *$\mathbf{C}(T)$ is separable for any compact metric space T.*[26]

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[25] *Example.* Consider the sequence (φ_m) in $\mathbf{C}[0,1]$ defined by $\varphi_m(t) := t^m$ for each m. Then $\varphi_m \to \mathbf{1}_{\{1\}}$, but we do not have $\varphi_m \to \mathbf{1}_{\{1\}}$ uniformly, for $d_\infty(\varphi_m, \mathbf{1}_{\{1\}}) = 1$ for each m. Moreover, interchanging the limit operations is not warranted in this example. If, say, $(t_k) := (0, \frac{1}{2}, ..., 1-\frac{1}{k}, ...)$, then $\lim_{k\to\infty}\lim_{m\to\infty}\varphi_m(t_k) = 0 \neq 1 = \lim_{m\to\infty}\lim_{k\to\infty}\varphi_m(t_k)$.

[26] This is a first best result in the sense that if $C(T)$ is separable, then T must be a compact.

2.11 Continuous Correspondences

In economic theory and optimization, one often deals with functions that map a given point in a metric space to a *subset* of another metric space. A detailed discussion of the theory and applications of such maps – the so-called *correspondences* – is provided in Ok (2007), Chapter E. Here we briefly recall the basic elements of this theory.

A **correspondence** Γ from a nonempty set X into another nonempty set Y – denoted as $\Gamma : X \rightrightarrows Y$ – is a map from X into $2^Y \backslash \{\emptyset\}$. (*Note.* $\Gamma(x)$ is a nonempty subset of Y for each $x \in X$.) Here X is called the **domain** of Γ, and Y the **codomain** of Γ. For any $S \in 2^X$, we let

$$\Gamma(S) := \bigcup\{\Gamma(x) : x \in S\}.$$

(*Note.* $\Gamma(\emptyset) = \emptyset$.) The set $\Gamma(X)$ is called the **range** of Γ, and if $\Gamma(X) \subseteq X$, then Γ is called a **self-correspondence** on X.

Every function can be thought of as a single-valued correspondence – one that maps every point in its domain to a singleton. It is thus reasonable to ask if we can extend our ordinary notion of "continuity" to the realm of correspondences in a way that is consistent with our analysis of continuous functions. There are at least two reasonable ways of doing this. We take each of these in turn.

Definition. For any two metric spaces X and Y, a correspondence $\Gamma : X \rightrightarrows Y$ is said to be **upper hemicontinuous at** $\omega \in X$ if, for every $O \in \mathcal{O}_Y$ with $\Gamma(\omega) \subseteq O$, there exists a $\delta > 0$ such that $\Gamma(N_{\delta,X}(\omega)) \subseteq O$. We say that Γ is **upper hemicontinuous** if it is upper hemicontinuous at each $\omega \in X$.

Loosely speaking, upper hemicontinuity of Γ at ω says that a small perturbation of ω does not cause the image set $\Gamma(\omega)$ to "suddenly" get large. This property surely reduces to ordinary continuity when Γ is single-valued. (Yes?) However, there are important differences regarding the implications of these properties. Most notably, while a continuous function maps compact sets to compact sets (Proposition 12), this is not true for upper hemicontinuous correspondences. (*Example.* Consider the correspondence $\Gamma : [0, 1] \rightrightarrows \mathbb{R}_+$ defined by $\Gamma(t) := \mathbb{R}_+$.) But, if every value of an upper hemicontinuous correspondence is compact – in this case we refer to the correspondence as **compact-valued** – all goes well.

Proposition 19. *Let X and Y be two metric spaces. If $\Gamma : X \rightrightarrows Y$ is a compact-valued upper hemicontinuous correspondence, then $\Gamma(S)$ is compact in Y for any compact subset S of X.*

> Exercise 16. Prove Proposition 19.

One can use this fact to prove the following sequential characterization for upper hemicontinuity. (See Ok (2007), Section E.2.1, for a proof.)

Proposition 20. *Let X and Y be two metric spaces. Then, $\Gamma : X \rightrightarrows Y$ is upper hemicontinuous at $\omega \in X$ if for any $(\omega^m) \in X^\infty$ and $(\nu^m) \in Y^\infty$ with $\omega^m \to \omega$ and $\nu^m \in \Gamma(\omega^m)$ for each m, there exists a subsequence of (ν^m) that converges to a point in $\Gamma(\omega)$. If Γ is compact-valued, then the converse is also true.*

> Exercise 17. Consider the self-correspondence Γ on $[0,1]$ defined as: $\Gamma(0) := (0,1]$ and $\Gamma(t) := (0,t)$ for all $0 < t \leq 1$. Is Γ upper hemicontinuous? Does Γ satisfy the sequential property considered in Proposition 20?
>
> **Exercise 18**. (The Closed-Graph Property) Let X and Y be two metric spaces. A correspondence $\Gamma : X \rightrightarrows Y$ is said to have a closed graph if for any convergent $(\omega^m) \in X^\infty$ and $(\nu^m) \in Y^\infty$, we have $\lim \nu^m \in \Gamma(\lim \omega^m)$ whenever $\nu^m \in \Gamma(\omega^m)$ for each $m = 1, 2, ...$ Prove:
> **(a)** If Γ has a closed graph, then it need not be upper hemicontinuous. But if Γ has a closed graph and Y is compact, then it is upper hemicontinuous.
> **(b)** If Γ is upper hemicontinuous, then it need not have a closed graph. But if Γ is upper hemicontinuous and closed-valued, then it has a closed graph.

While upper hemicontinuity of a correspondence Γ guarantees that the image set $\Gamma(\omega)$ does not "explode" due to a small perturbation of ω, it allows for it to "implode." Consider the self-correspondence Γ on $[0,1]$ defined by $\Gamma(t) := \{0\}$ for any $0 \leq t < \frac{1}{2}$ and $\Gamma(t) := [0,1]$ for any $\frac{1}{2} \leq t \leq 1$. This correspondence is upper hemicontinuous, but it hardly seems "continuous" at $\frac{1}{2}$ as the image of t under Γ implodes dramatically the moment we decrease its value from $\frac{1}{2}$. Our next continuity concept outrules precisely this sort of a thing.

Definition. For any metric spaces X and Y, a correspondence $\Gamma : X \rightrightarrows Y$ is said to be **lower hemicontinuous at** $\omega \in X$, if for every $O \in \mathcal{O}_Y$ with $\Gamma(\omega) \cap O \neq \emptyset$, there exists a $\delta > 0$ such that $\Gamma(\nu) \cap O \neq \emptyset$ for all $\nu \in N_{\delta,X}(\omega)$. Γ is called **lower hemicontinuous** if it is lower hemicontinuous at each $\omega \in X$.

While it also reduces to our usual notion of continuity in the single-valued case, lower hemicontinuity is logically independent of upper hemicontinuity. Indeed, the self-correspondence Γ considered above is upper, but not lower, hemicontinuous, at $\frac{1}{2}$. Conversely, if we changed the value of this correspondence at $\frac{1}{2}$ to $\{0\}$, then the resulting correspondence would be lower, but not upper, hemicontinuous at $\frac{1}{2}$.

The following result, a proof of which can be found in Ok (2007), Section E.2.3, provides a sequential characterization of lower hemicontinuity, which is often useful in applications.

Proposition 21. *Let X and Y be two metric spaces. Then, $\Gamma : X \rightrightarrows Y$ is lower hemicontinuous at $\omega \in X$ if, and only if, for any $(\omega^m) \in X^\infty$ with $\omega^m \to \omega$ and any $\nu \in \Gamma(\omega)$, there exists a $(\nu^m) \in Y^\infty$ such that $\nu^m \to \nu$ and $\nu^m \in \Gamma(\omega^m)$ for each m.*

Here is our final product.

Definition. Let X and Y be two metric spaces. A correspondence $\Gamma : X \rightrightarrows Y$ is said to be **continuous at** $\omega \in X$ if it is both upper and lower hemicontinuous at x. It is called **continuous** if it is continuous at each $\omega \in X$.

Exercise 19. Let X be a metric space and $\varphi \in \mathbf{C}(X)$. Show that the correspondence $\Gamma : X \rightrightarrows \mathbb{R}$, defined by $\Gamma(\omega) := [0, \varphi(\omega)]$, is continuous.

2.12 The Maximum Theorem

Let T be any nonempty set and $\varphi \in \mathbb{R}^T$. The canonical optimization problem is to find the maximum value that φ attains on a nonempty subset S of T. (*Note.* φ is the **objective function** of the problem, and S is its **constraint set**.) To "solve" this problem means to identify all $\nu \in S$ with $\varphi(\nu) \geq \varphi(S)$, that is, to compute

$$\arg\max\{\varphi(\omega) : \omega \in S\} := \{\nu \in S : \varphi(\nu) \geq \varphi(\omega) \text{ for all } \omega \in S\}.$$

This set is referred to as the **solution set** of the problem.

Now take any nonempty set Θ, call it the **parameter space**, and suppose both the constraint set and the objective function of our optimization problem depends on the parameter $\theta \in \Theta$. The constraint of the problem would then be modeled by means of a correspondence of the form $\Gamma : \Theta \rightrightarrows T$, the **constraint correspondence** of the problem. This leads to the following formulation: *For each* $\theta \in \Theta$, *maximize* $\varphi(\omega, \theta)$ *such that* $\omega \in \Gamma(\theta)$. Clearly, the solution set $\arg\max\{\varphi(\omega, \theta) : \omega \in \Gamma(\theta)\}$ of the problem depends in this case on the value of θ.

Assume next that T is a metric space, and $\varphi \in \mathbf{C}(T)$. If $\Gamma : \Theta \rightrightarrows T$ is compact-valued, we can then think of the solution to our problem as a correspondence from Θ into T. For, in this case, Weierstrass' Theorem makes sure that $\sigma : \Theta \rightrightarrows X$ is well-defined by $\sigma(\theta) := \arg\max\{\varphi(\omega, \theta) : \omega \in \Gamma(\theta)\}$. Naturally enough, σ is called the **solution correspondence** of the problem.

Understanding how the continuity properties of σ depend on those of φ and Γ is of primary importance. A major result in this regard is the following:

The Maximum Theorem. (Berge) *Let* Θ *and* X *be two metric spaces,* $\Gamma : \Theta \rightrightarrows X$ *a compact-valued correspondence, and* $\varphi \in \mathbf{C}(X \times \Theta)$. *Define*

$$\sigma(\theta) := \arg\max\{\varphi(\omega, \theta) : \omega \in \Gamma(\theta)\} \qquad \text{for all } \theta \in \Theta \tag{11}$$

and

$$\varphi^*(\theta) := \max\{\varphi(\omega, \theta) : \omega \in \Gamma(\theta)\} \qquad \text{for all } \theta \in \Theta, \tag{12}$$

and assume that Γ *is continuous at* $\theta \in \Theta$. *Then:*

(a) $\sigma : \Theta \rightrightarrows X$ *is compact-valued and upper hemicontinuous at* θ,

(b) $\varphi^* : \Theta \to \mathbb{R}$ *is continuous at* θ.

You can consult on Ok (2007), Section E.3, for a proof of this theorem.

Remark 1. [1] The *lower* hemicontinuity of σ does not follow from the hypotheses of the Maximum Theorem. (*Example.* If $X := \Theta := [0, 1] =: \Gamma(\theta)$ for all $\theta \in \Theta$, and $\varphi \in \mathbf{C}([0, 1]^2)$ is defined by $\varphi(t, \theta) := t\theta$, the solution correspondence σ is not lower hemicontinuous at 0.) But if, in addition to the assumptions of the Maximum Theorem, it is the case that there is a *unique* maximum of $\varphi(\cdot, \theta)$ on $\Gamma(\theta)$ for each θ, then σ must be a continuous function. (Yes?)

[2] Let $X := \Theta := \mathbb{R}$, and define $\varphi \in \mathbf{C}(X \times \Theta)$ by $\varphi(t, \theta) := t$. Consider the following correspondences from X into Θ:

$$\Gamma_1(\theta) := \begin{cases} [0,1], & \theta = 0 \\ \{0\}, & \theta \neq 0 \end{cases} \quad \text{and} \quad \Gamma_2(\theta) := \begin{cases} \{0\}, & \theta = 0 \\ [0,1], & \theta \neq 0 \end{cases} .$$

If $\sigma_i(\theta)$ is defined by (11) with Γ_i playing the role of Γ, then σ_i is not upper hemicontinuous at 0, $i = 1, 2$. Thus the continuity of Γ can be replaced with neither upper nor lower hemicontinuity in the Maximum Theorem. $\square$