FUNCTIONAL REPRESENTATION OF ROTUND-VALUED PROPER MULTIFUNCTIONS

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Abstract. The main result of the paper is a representation theorem for multifunctions that says that the graph of every weakly continuous multifunction (from a k-space into a Banach space) which has weakly compact and strictly convex values is the union of the graphs of a compact and convex set of continuous functions. This result is used to obtain a continuous extension theorem for a class of multifunctions defined on a closed subset of a metric space. We also consider the implications of our representation theorem for the characterization of certain classes of additive multifunctions and their selections.

1. Introduction

It is a set-theoretical fact that to any multifunction (set-valued map) one can assign a family of (single-valued) functions such that the value of the multifunction at any given point equals the set of the values of the members of the family at that point. Such a functional representation of a multifunction is not likely to be useful in applications, however, unless the representing family is found to be a "small" set that consists of functions with desirable properties (which would of course be inherited from the multifunction at hand). For instance, it is important to identify which sort of multifunctions Γ can be characterized as $\Gamma(\cdot) = \mathrm{cl}\{f_m(\cdot) : m = 1, 2, ...\}$ where each f_m satisfies some form of measurability/continuity/differentiability condition (cf. Michael (1956), Ioffe (1979), Spahn (1983), Bresson and Colombo (1992), and Dentcheva (2001)), or as $\mathrm{graph}(\Gamma) = \mathrm{cl}\bigcup\{\mathrm{graph}(f_m) : m = 1, 2, ...\}$ where each f_m satisfies a differentiability property (Dentcheva (2001)). We present in this paper a basic representation theorem in the similar vein; we show that the graph of a continuous multifunction with weakly compact and strictly convex values can be represented as the union of the graphs of a compact family of continuous functions, under mild restrictions on the underlying spaces. (It seems that this representation notion is considered only by Ekeland and Valadier (1971) in the literature.) To illustrate, we note the following (very) special case of our main result (Theorem 4).

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Theorem 1. *Let X be any metric space, and Y a topological linear space of finite dimension. A multifunction $\Gamma : X \rightrightarrows Y$ is continuous, and has compact and strictly convex values if, and only if, there exists a compact and convex set $\mathcal{G}$ in $\mathbf{C}(X,Y)$ such that $\{G(x) : G \in \mathcal{G}\}$ is a strictly convex set that equals $\Gamma(x)$ for all $x \in X$.*

One can easily see that the Michael selection theorem would allow one find a convex set $\mathcal{G}$ in $\mathbf{C}(X,Y)$ that represents Γ as in this theorem, but this approach would not guarantee the compactness of $\mathcal{G}$. Indeed, Michael's theorem applies to any lower hemicontinuous, compact and convex-valued multifunction but easy examples show that such a multifunction need not be representable in the sense of Theorem 1 by a compact family of continuous functions. By contrast, Theorem 1 (and the more general Theorem 4) applies to multifunctions which are, in addition, upper hemicontinuous and rotund-valued, but it warrants a representation by a compact set of continuous functions. Somewhat surprisingly, the rotund-valuedness property plays a crucial role in this regard. Roughly put, this is because the set of extreme points of a value of a compact and strictly convex-valued multifunction is particularly well-behaved; in particular it is compact, being equal to the boundary of that value. The exploitation of this simple observation lies at the heart of our approach which is significantly more geometric than those typically adopted in selection theory.

Finally, we note that functional representation theorems like Theorem 1 may prove useful in studying the structure of multifunctions. For instance, if X is a closed subset of a metric space W, by using Theorem 1 and the Tietze extension theorem, one can easily show that any multifunction that satisfies the conditions of Theorem 1 can be extended to a continuous, compact and convex-valued multifunction on W the values of which are contained in the convex hull of the range of the original multifunction. Admittedly, this particular result is not new; it is indeed a special case of the extension theorem of Antosiewicz and Cellina (1977). However, this is only because we are couching this introductory discussion in terms of the special case of Theorem 1. We show in Section 4 that new continuous extension theorems can be obtained very easily by using our main representation result (Theorem 4) along with a suitable generalization of Tietze's theorem. In Section 5, we provide further applications by exploring the implications of this result for the representation, selection and extension of additive multifunctions.

2. Preliminaries

Let X and Z be two topological spaces, and let Γ be a function from X into 2^Z. If $\Gamma(x) \neq \emptyset$ for any $x \in X$, we say that Γ is a *multifunction* from X to Z (or a *correspondence*, or a *set-valued map*), and use the expressions $\Gamma : X \rightrightarrows Z$ and $\Gamma : X \to 2^Z \backslash \{\emptyset\}$ interchangeably. We denote the set of all nonempty compact subsets of Z by $\mathbf{c}(Z)$, and say that Γ is *compact-valued* if $\Gamma(x) \in \mathbf{c}(Z)$ for all $x \in X$.

The *upper* and *lower inverse images* of a set $O \subseteq Z$ under Γ is defined as

$$\Gamma_u^{-1}(O) := \{x \in X : \Gamma(x) \subseteq O\} \quad \text{and} \quad \Gamma_\ell^{-1}(O) := \{x \in X : \Gamma(x) \cap O \neq \emptyset\},$$

respectively. We say that Γ is *upper* (resp. lower) *hemicontinuous* if upper (resp. lower) inverse image of every open set under Γ is open in X. Γ is called *continuous* if it is both upper and lower hemicontinuous. It is well-known that one can topologize $2^Z\backslash\{\emptyset\}$ in a way to express each of these continuity notions as the standard continuity property defined for $2^Z\backslash\{\emptyset\}$-valued functions. (See, for instance, Michael (1951), and Klein and Thomson (1984), pp. 72-79.) In particular, the *upper Vietoris topology* on $2^Z\backslash\{\emptyset\}$, defined as the topology generated by the basis $\{\{B \in 2^Z\backslash\{\emptyset\} : B \subseteq O\} : O$ is an open subset of $Z\}$, allows us to view upper hemicontinuity as a basic continuity condition. We state this simple fact as a lemma (Klein and Thomson (1984), Theorem 7.1.4).

Lemma 1. *For any topological spaces X and Z, the multifunction $\Gamma : X \rightrightarrows Z$ is upper hemicontinuous if and only if $\Gamma : X \to 2^Z\backslash\{\emptyset\}$ is continuous relative to the upper Vietoris topology.*

Another useful property of the upper Vietoris topology is that it declares the boundary operator on the compact subsets of a regular space continuous.

Lemma 2. *If Z is a regular topological space, the map* bd : $\mathbf{c}(Z) \to 2^Z\backslash\{\emptyset\}$ *defined by* $\text{bd}(S) := \partial S$ *is continuous relative to the upper Vietoris topology.*

Proof. Take any $S \in \mathbf{c}(Z)$ and let O be any open subset of Z such that $\text{bd}(S) \in \{B : \emptyset \neq B \subseteq O\} =: \mathcal{O}$. Since Z is regular, for any $z \in \partial S$, there exists an open subset U_z of Z such that $z \in U_z$ and $\text{cl}U_z \subseteq O$. But since S is compact, so is ∂S, and hence, given that $\{U_z : z \in \partial S\}$ is an open cover of ∂S, there exist finitely many $z_1, ..., z_n \in \partial S$ such that $\partial S \subseteq U_{z_1} \cup \cdots \cup U_{z_n} =: U$. Now define $\mathcal{U} := \{B : \emptyset \neq B \subseteq U\}$ which is an open set in $2^Z\backslash\{\emptyset\}$ that includes ∂S. Notice that if $B \in \mathcal{U}$, then $\partial B \subseteq \text{cl}B \subseteq \text{cl}U = \text{cl}U_{z_1} \cup \cdots \cup \text{cl}U_{z_n} \subseteq O$, so we have $\text{bd}(B) \in \mathcal{O}$. This establishes that $\text{bd}(\mathcal{U}) \subseteq \mathcal{O}$, and completes the proof. $\square$

Now assume that Z is a real topological linear space.[1] We denote the convex hull and the closed convex hull of a nonempty subset S of Z by $\text{co}(S)$ and $\overline{\text{co}}(S)$, respectively, while $\text{ext}(S)$ stands for the set of all extreme points of S. The set S is called *rotund* if its boundary does not contain a nondegenerate line segment (that is, $\lambda z + (1-\lambda)z' \in \partial S$ for all $0 \leq \lambda \leq 1$ holds only if $z = z'$), and it is called *strictly convex* if it is a convex rotund set. We denote the nonempty, compact and convex subsets of Z by $\mathbf{cc}(Z)$, and say that Γ is *compact and convex-valued* if $\Gamma(x) \in \mathbf{cc}(Z)$ for all $x \in X$. In turn, Γ is said to be *rotund-valued* (resp. *strictly convex-valued*) if $\Gamma(x)$ is a rotund (resp. strictly convex) set for each $x \in X$.

Before proceeding further, we need to recall the following elementary observation that points to a few useful geometric properties of strictly convex sets.

Lemma 3. *Let S be a nonempty strictly convex set in a topological linear space Z. Then every support functional for S supports S at a single point. Moreover, if S is not a singleton, then it must have a nonempty interior, and if it is closed, then we have* $\partial S = \text{ext}(S)$.

[1]All linear spaces that appear in this note are over the field of real numbers, and hence from now we will refrain from giving explicit reference to the scalar field under consideration.

Proof. To prove the first claim, let f be a continuous linear functional on Z such that $f(z) = f(z') \geq f(S)$ for some $z, z' \in S$. The linearity of f implies that $f(\text{co}\{z, z'\}) \geq f(S)$, which means that every point in $\text{co}\{z, z'\}$ is a support point of S. Since every support point of a set in a topological linear space is a boundary point of that set, we thus find that $\text{co}\{z, z'\} \subseteq \partial S$. Since S is rotund, this implies that $z = z'$.

To prove the first part of the second claim, take any $z, z' \in S$ with $z \neq z'$. If either z or z' belongs to $\text{int}(S)$, then there is nothing to prove, so say $z, z' \in \partial S$. By convexity, $z'' := \frac{1}{2}(z + z') \in S$, but by rotundity, $z'' \notin \partial S$. Thus $z'' \in \text{int}(S)$.

As for the last claim, note that an extreme point can never belong to the algebraic interior of a set whereas the (topological) interior of a convex set is always included in its algebraic interior. That $\text{ext}(S) \subseteq \partial S$ follows from these facts. To establish the converse, let $z \in \partial S \backslash \text{ext}(S)$. Since S is closed, $z \in S$, so there must exist distinct $u, v \in S$ such that $z \in \lambda u + (1 - \lambda)v$ for some $\lambda \in (0, 1)$. If $u \in \text{int}(S)$, then there exists an open neighborhood O of u such that $O \subseteq S$. But then $z \in \text{int}(\text{co}(\{v\} \cup O)) \subseteq S$ by convexity of S, and hence we get $z \notin \partial S$, a contradiction. Thus $u \in \partial S$. But we can apply the same argument to every member of $\text{co}\{u, z\} \backslash \{z\}$ and $\text{co}\{z, v\} \backslash \{z\}$, and conclude that $\text{co}\{u, v\} \subseteq \partial S$. Since u and v are distinct, this contradicts the rotundity of S. □

We denote the topological dual of a topological linear space Y by Y^*, and let $Y_0^* := Y^* \backslash \{0\}$. Also let Y_w stand for the linear space Y endowed with the weak topology, and recall that Y_w is a completely regular locally convex topological linear space.

Definition. Let X be a topological space and Y a topological linear space. A multifunction $\Gamma : X \rightrightarrows Y_w$ is said to be **proper** if it is a compact and convex-valued continuous multifunction.

Proper multifunctions that have rotund-values are the principal objects of analysis in this paper. Clearly, $\Gamma : X \rightrightarrows Y$ is such a multifunction iff $\Gamma(x)$ is a weakly compact and strictly convex set for each $x \in X$, and Γ is weakly continuous, that is, $\Gamma_u^{-1}(O)$ and $\Gamma_l^{-1}(O)$ are open in X for any weakly open subset O of Y. One can show that the set of all such multifunctions is an Abelian semigroup with identity under the usual set-addition. We stress that a proper multifunction need not have compact values. Indeed, compact-valuedness relative to the original topology of Y would be a severe restriction for rotund-valued proper multifunctions; by Riesz's lemma, such a multifunction is compact-valued iff Y is finite dimensional.

As for examples, we note that any continuous function, or any multifunction whose values are compact intervals in $\mathbb{R}$, is obviously a rotund-valued proper multifunction. Two further examples are given next.

Examples 1. (i) Assume that X is a topological space and Y a uniformly rotund Banach space (e.g. a Hilbert space). Let $\varepsilon \in \mathbf{C}(X, \mathbb{R}_{++})$ and $F \in \mathbf{C}(X, Y)$, and define $\Gamma : X \rightrightarrows Y$ by $\Gamma(x) := F(x) +\, \text{cl}N_{\varepsilon(x)}(0)$ where $N_{\varepsilon(x)}(0)$ is the $\varepsilon(x)$-neighborhood of 0. Then, Γ is a rotund-valued proper multifunction. For, every uniformly rotund Banach space is reflexive (the

Milman-Pettis theorem), and the closed unit ball is a weakly compact and strictly convex set in any rotund reflexive normed linear space. The proof of the rest of the claim is elementary.

(ii) Let X be a reflexive Banach space with a Fréchet-smooth dual, and let $M \in \mathbf{cc}(X_w)$. The *metric projection* onto M is the multifunction $\Gamma : X \rightrightarrows X_w$ defined by $\Gamma(x) := \{y \in M : \|x - y\| = \inf_{m \in M} \|x - m\|\}$. It is easy to show that $\Gamma(x)$ is a nonempty, closed, bounded and convex set for each x. Moreover, since M is weakly compact, so is $\Gamma(x)$, so we have $\Gamma(x) \in \mathbf{cc}(X_w)$ for each x. Finally, it is known that Γ is continuous under the geometric assumptions imposed on X (Oshman, 1971). Thus Γ is a proper multifunction. However, it is obvious that $\Gamma(x)$ must have an empty interior, so by Lemma 3, Γ is rotund-valued iff it is single-valued.

Finally, we review briefly the basic notions that we will borrow from the theory of function spaces. By a *k-space* (or a *compactly generated* space) we mean a Hausdorff topological space in which a set is closed if and only if the intersection of this set with every compact set in the space is compact. It is easy to check that any locally compact, or first countable, Hausdorff space is a k-space. The topology of such a space is determined through the sets it declares as compact, and hence is particularly suitable for the study of function space topologies such as the compact-open topology. For completeness, we recall that for any topological spaces X and Z, the *point-open topology* on $\mathbf{C}(X, Z)$ is the one generated by the subbasis $\{\{F \in \mathbf{C}(X, Z) : F(x) \in O\} : x \in X$ and O is an open set in $Z\}$, and the *compact-open topology* on $\mathbf{C}(X, Z)$ is the one generated by the subbasis $\{\{F \in \mathbf{C}(X, Z) : F(K) \in O\} : K \in \mathbf{c}(X)$ and O is an open set in $Z\}$. In what follows, when its topology is not explicitly mentioned, it should be understood that $\mathbf{C}(X, Z)$ is given the compact-open topology.

The following version of the Ascoli theorem is proved recently by Edwards (1999, Theorem 3.13).

Lemma 4. *Let X and Z be topological spaces with Z being Hausdorff. A subset $\mathcal{G}$ of $\mathbf{C}(X, Z)$ is compact if, and only if, (1) $\mathcal{G}$ is closed in $\mathbf{C}(X, Z)$ relative to the point-open topology, and (2) $K \in \mathbf{c}(X)$ implies $\bigcup\{G(K) : G \in \mathcal{G}\} \in \mathbf{c}(Z)$.*

The final result of this section concerns the closed convex hulls of compact families of functions relative to the compact-open topology.

Lemma 5. *Let X be any k-space, and Z a Hausdorff locally convex topological linear space such that $\overline{\mathrm{co}}(S) \in \mathbf{c}(Z)$ for any $S \in \mathbf{c}(Z)$. If $\mathcal{F}$ is a compact subset of $\mathbf{C}(X, Z)$, then $\overline{\mathrm{co}}(\mathcal{F})$ is compact and*

$$\overline{\mathrm{co}}\{F(x) : F \in \mathcal{F}\} = \{F(x) : F \in \overline{\mathrm{co}}(\mathcal{F})\}, \qquad x \in X. \tag{2.1}$$

Proof.[2] Define $\mathcal{T}(x) := \{T(x) : T \in \mathcal{T}\}$ for any $x \in X$ and any subset $\mathcal{T}$ of $\mathbf{C}(X, Z)$. Fix $x \in X$, and let (F_α) be a net in $\mathrm{co}(\mathcal{F})$ with $F_\alpha \to F$ uniformly on compacta. In

[2] The present direct proof of this lemma was communicated to me by Laszlo Zsilinszky. We note that this result can also be proved by using the generalization of the Ascoli theorem (Theorem 4) of Bagley and Yang (1966).

particular, $F_\alpha(x) \to F(x)$, and given that $F_\alpha(x) \in \text{co}(\mathcal{F})(x) = \text{co}(\mathcal{F}(x)) \subseteq \overline{\text{co}}(\mathcal{F}(x))$, we find $F(x) \in \overline{\text{co}}(\mathcal{F}(x))$, that is, $\overline{\text{co}}(\mathcal{F})(x) \subseteq \overline{\text{co}}(\mathcal{F}(x))$. On the other hand, $\text{cl}(\overline{\text{co}}(\mathcal{F})(x))$ is obviously a closed set that includes $\overline{\text{co}}(\mathcal{F}(x))$, so:

$$\text{cl}(\overline{\text{co}}(\mathcal{F})(x)) = \overline{\text{co}}(\mathcal{F}(x)) = \overline{\text{co}}(\text{cl}(\mathcal{F}(x))). \tag{2.2}$$

Since $\mathcal{F}$ is compact, $\text{cl}(\mathcal{F}(x))$ is compact in Z by the Ascoli theorem, so by hypothesis and (2.2), we find that $\overline{\text{co}}(\mathcal{F})(x)$ is relatively compact in Z for each $x \in X$.

By the Ascoli theorem, then, the compactness of $\overline{\text{co}}(\mathcal{F})$ will be established if we can show that $\overline{\text{co}}(\mathcal{F})$ is an equicontinuous family (with respect to the uniformity generated on Z). But if P is a set of seminorms that induces the topology of Z, then for each $x \in X$, $p \in P$ and $\varepsilon > 0$, compactness, hence the equicontinuity of $\mathcal{F}$, implies that there exists an open neighborhood U of x such that $p(F(u) - F(x)) < \varepsilon$ for all $(F, u) \in \mathcal{F} \times U$. Thus, for any $F_1, ..., F_n \in \mathcal{F}$ and any $\lambda_1, ..., \lambda_n \geq 0$ with $\sum \lambda_i = 1$, we have, for each $x \in X$, $p \in P$ and $\varepsilon > 0$,

$$p\left(\sum \lambda_i F_i(u) - \sum \lambda_i F_i(x)\right) \leq \sum \lambda_i p\left(F_i(u) - F_i(x)\right) < \varepsilon.$$

Thus $\text{co}(\mathcal{F})$ is equicontinuous, and since point-closure of an equicontinuous family is equicontinuous, we conclude that $\overline{\text{co}}(\mathcal{F})$ is equicontinuous, hence compact.

Finally, fix any $x \in X$, define $\phi_x : \mathbf{C}(X, Z) \to Z$ by $\phi_x(F) := F(x)$, and note that this function is continuous since, for any $F \in \mathbf{C}(X, Z)$ and any open subset O of Z, $\phi_x(F) \in O$ implies $\phi_x(\{T \in \mathbf{C}(X, Z) : T(x) \in O\}) \subseteq O$. So since $\overline{\text{co}}(\mathcal{F})$ is compact, $\phi_x(\overline{\text{co}}(\mathcal{F})) = \overline{\text{co}}(\mathcal{F})(x)$ is compact, hence closed, in Z. Combining this observation with (2.2) yields (2.1). $\square$

3. Representation of Rotund-Valued Proper Multifunctions

The first main result of this section yields a generalization of the "only if" part of Theorem 1.

Theorem 2. *Let X be any topological space, and Y a Hausdorff locally convex topological linear space. If $\Gamma : X \rightrightarrows Y$ is a rotund-valued proper multifunction, then there exists a compact set $\mathcal{H}$ in $\mathbf{C}(X, Y_w)$ such that*

$$\Gamma(x) = \overline{\text{co}}\{H(x) : H \in \mathcal{H}\} \qquad \textit{for all } x \in X.$$

Proof. For each $f \in Y_0^*$, we define

$$H_f(x) := \arg\max\{f(w) : w \in \Gamma(x)\}, \qquad x \in X. \tag{3.1}$$

Since $\Gamma(x)$ is weakly compact, and f is weakly continuous, the Weierstrass theorem yields $H_f(x) \neq \emptyset$ for each $x \in X$, so by Lemma 3, $H_f(x)$ is a singleton for any $(x, f) \in X \times Y_0^*$. We may then identify $H_f(x)$ with the point it contains, and write $H_f(x) \in Y$. With this convention, (3.1) defines H_f as a function from X into Y for each $f \in Y_0^*$. Moreover, by the Bauer maximum principle, we have $H_f(x) \in \text{ext}(\Gamma(x))$ for all $(x, f) \in X \times Y_0^*$. Thus $\{H_f(x) : f \in Y_0^*\} \subseteq \text{ext}(\Gamma(x))$ for all $x \in X$. Conversely, if $y \in \text{ext}(\Gamma(x))$ for any x, then

$y \in \partial\Gamma(x)$, so since $\Gamma(x)$ has a nonempty interior (Lemma 3), we can apply the supporting hyperplane theorem to find an $f \in Y_0^*$ such that $f(y) \geq f(\Gamma(x))$. It follows that

$$\{H_f(x) : f \in Y_0^*\} = \text{ext}(\Gamma(x)), \qquad x \in X. \tag{3.2}$$

Moreover, that $\{H_f : f \in Y_0^*\} \subseteq \mathbf{C}(X, Y_w)$ follows readily from the Berge maximum theorem.

Now let $\mathcal{H}$ be the closure of $\{H_f : f \in Y_0^*\}$ in $\mathbf{C}(X, Y_w)$ relative to the point-open topology. Then, if $y = H(x)$ for some $(x, H) \in X \times \mathcal{H}$ there exists a net (f_α) in Y_0^* such that $H_{f_\alpha} \to H$ pointwise. Therefore, by (3.2), $(H_{f_\alpha}(x))$ is a net in $\partial\Gamma(x)$ that converges weakly to y. But since $\Gamma(x)$ is weakly compact and $\partial\Gamma(x)$ is closed, $\partial\Gamma(x)$ is weakly compact, so given that Y_w is Hausdorff, $\partial\Gamma(x)$ is weakly closed. So $y \in \partial\Gamma(x)$, and by Lemma 3 and (3.2), therefore, we find that $y \in \{H_f(x) : f \in Y_0^*\}$. Consequently, we have $\mathcal{H}(x) := \{H(x) : H \in \mathcal{H}\} \subseteq \{H_f(x) : f \in Y_0^*\}$ for all $x \in X$. Since the converse containment is trivial, we may conclude that

$$\mathcal{H}(x) = \text{ext}(\Gamma(x)), \qquad x \in X. \tag{3.3}$$

Since $\text{ext}(\Gamma(x)) = \partial\Gamma(x)$ (Lemma 3), and $\partial\Gamma(x)$ is weakly compact, the Krein-Milman theorem then implies that $\Gamma(x)$ is the weakly closed convex hull of $\mathcal{H}(x)$ for each x. But a convex subset of a real locally convex topological linear space is closed iff it is weakly closed, so we get $\Gamma(x) = \overline{\text{co}}(\mathcal{H}(x))$ for all $x \in X$, as is sought.

We next claim that $\mathcal{H}$ is compact relative to the compact-open topology. $\mathcal{H}$ is obviously closed in $\mathbf{C}(X, Y_w)$ relative to the point-open topology. Endow $\mathbf{c}(Y_w)$ by the upper Vietoris topology, and define the map $\text{bd} : \mathbf{c}(Y_w) \to 2^{Y_w}\backslash\{\emptyset\}$ by $\text{bd}(S) := \partial S$. Since Y_w is a regular space, bd is continuous by Lemma 2, and since Γ is upper hemicontinuous, $\Gamma : X \to 2^{Y_w}\backslash\{\emptyset\}$ is continuous by Lemma 1. Moreover, by hypothesis, the range of Γ is contained in $\mathbf{c}(Y_w)$. Consequently, $b_\Gamma := \text{bd} \circ \Gamma$ is a well-defined continuous function mapping X into $2^{Y_w}\backslash\{\emptyset\}$. By Lemma 1, this means that $b_\Gamma : X \rightrightarrows Y_w$ is upper hemicontinuous. In addition, for any x, $\Gamma(x)$, and hence $\partial\Gamma(x)$, is weakly compact; that is, $b_\Gamma : X \rightrightarrows Y_w$ is also compact-valued. But it is well-known that any compact-valued upper hemicontinuous multifunction maps compact sets to compact sets (Klein and Thompson (1984), Theorem 7.4.2), so $K \in \mathbf{c}(X)$ implies that $b_\Gamma(K) \in \mathbf{c}(Y_w)$. Consequently, by (3.3) and Lemma 3, we have

$$\begin{aligned} \bigcup_{H\in\mathcal{H}} H(K) &= \bigcup_{H\in\mathcal{H}} \bigcup_{x\in K} H(x) = \bigcup_{x\in K} \bigcup_{H\in\mathcal{H}} H(x) = \bigcup_{x\in K} \mathcal{H}(x) = \bigcup_{x\in K} \partial\Gamma(x) = \bigcup_{x\in K} b_\Gamma(x) \\ &= b_\Gamma(K) \\ &\in \mathbf{c}(Y_w) \end{aligned}$$

for any $K \in \mathbf{c}(X)$. By Lemma 4, therefore, $\mathcal{H}$ is compact in $\mathbf{C}(X, Y_w)$. $\square$

Remark 1. If Γ is not rotund-valued but $\Gamma(X)$ is compact and metrizable, then a result by Ekeland and Valadier (1971) shows that the conclusion of Theorem 2 remains valid. We should note that this result is proved by a method which is significantly different than the present one.

Theorem 2 has immediate applications to selection theory. For instance, it yields readily the following two corollaries.

Corollary 1. *Under the conditions of Theorem 2, for any* $(x_0, y_0) \in X \times Y$ *with* $y_0 \in \Gamma(x_0)$, *there exists a weakly continuous selection* G *of* Γ *such that* $y_0 = G(x_0)$.

Corollary 2. *Under the conditions of Theorem 2, there exists a set* $\mathcal{F}$ *of selections of* Γ *such that* $\mathcal{F}$ *is the convex closure of a compact set in* $\mathbf{C}(X, Y_w)$ *and* $\{F(x) : F \in \mathcal{F}\}$ *is dense in* $\Gamma(x)$ *for all* $x \in X$.

Remark 2. The famous Michael selection theorem states that a Hausdorff topological space is paracompact if and only if every closed and convex-valued lower hemicontinuous multifunction Γ from X into a Banach space Y has a continuous selection. Corollary 1 is in the same spirit with the "only if" part of this result. Its main advantage is that its assumptions on the domain and codomain of the multifunction are substantially weaker than those of Michael's theorem. It compensates for this by making more stringent assumptions on the multifunction (except that it asks Γ to be *weakly* lower hemicontinuous), and it delivers the existence of a *weakly* continuous selection. If Y is a finite dimensional Euclidean space, then the comparison of the results become clearer. In this case, Corollary 1 says that the paracompactness requirement in the "only if" part of the Michael selection theorem can be replaced with upper hemicontinuity, and compact and rotund-valuedness.

A similar remark also applies to Corollary 2. A well-known corollary of Michael's theorem is that, for any closed and convex-valued lower hemicontinuous multifunction Γ from a perfectly normal space into a separable Banach space Y, there exists a countable set $\mathcal{F}$ of continuous selections of Γ such that $\{F(x) : F \in \mathcal{F}\}$ is dense in $\Gamma(x)$ for each x (Michael (1956), Lemma 5.2). Corollary 2 delivers a similar result, by asking less from the underlying spaces and (essentially) more from the multifunction at hand.

We conclude by making note of the following converse to Theorem 2, which, in particular, yields the "if" part of Theorem 1.

Theorem 3. *Let* X *be a* k*-space,* Y *a Hausdorff topological linear space, and* $\mathcal{G}$ *a nonempty compact and convex subset of* $\mathbf{C}(X, Y_w)$ *such that* $\{G(x) : G \in \mathcal{G}\}$ *is rotund. Then, if* $\Gamma : X \rightrightarrows Y$ *is defined by* $\Gamma(x) := \{G(x) : G \in \mathcal{G}\}$, *then it is a rotund-valued proper multifunction.*

Proof. That Γ is strictly convex-valued is obvious. From the compactness of $\mathcal{G}$ and the Ascoli theorem it follows that $\mathcal{G}(x) = \Gamma(x)$ is weakly compact for each $x \in X$. To verify the continuity of Γ, take any open subset O of Y and observe that, if $x \in \Gamma_\ell^{-1}(O)$, then there exists a $G \in \mathcal{G}$ such that $G(x) \in O$. By continuity of G, therefore, there exists an open neighborhood U of x such that $G(U) \subseteq O$, and hence $U \subseteq \Gamma_\ell^{-1}(O)$, establishing that Γ is lower hemicontinuous. On the other hand, we have

$$\Gamma_\ell^{-1}(O) = \bigcap \{G^{-1}(O) : G \in \mathcal{G}\},$$

and since $\mathcal{G}$ is compact and O is open, by Gale's generalization of the Ascoli theorem (Gale (1950), Theorem 1), the set on the right hand side of this equation must be open in X. This proves that Γ is upper hemicontinuous. $\square$

The final result of this section will summarize the work done so far in the context of multifunctions whose domain is a k-space and whose codomain is a space in which the closed convex hull of every weakly compact set is weakly compact. First we need a definition.

Definition. A locally convex topological linear space Z is said to have the **C-property** if the set of all compact subsets of Z is closed under taking closed convex hulls, that is, $\overline{\text{co}}(S) \in \mathbf{c}(Z)$ for any $S \in \mathbf{c}(Z)$.

We are particularly interested here in those spaces Y such that Y_w has the C-property. The classic Krein-Šmulian theorem states that every Banach space is of this form. However, the class of spaces which satisfy the C-property relative to their weak topologies is of course richer than that of Banach spaces.

Theorem 4. *Let X be a k-space, and Y a Hausdorff locally convex topological linear space such that Y_w has the C-property. The following statements are equivalent*:

(a) $\Gamma : X \rightrightarrows Y$ *is rotund-valued proper multifunction*;

(b) *There exists an* $\mathcal{H} \in \mathbf{c}(\mathbf{C}(X, Y_w))$ *such that* $\text{co}\{H(x) : H \in \mathcal{H}\}$ *is a rotund set which is dense in* $\Gamma(x)$ *for each* $x \in X$;

(c) *There exists a* $\mathcal{G} \in \mathbf{cc}(\mathbf{C}(X, Y_w))$ *such that* $\{G(x) : G \in \mathcal{G}\}$ *is a rotund set which equals* $\Gamma(x)$ *for each* $x \in X$.

Proof. (a) $\Rightarrow$ (b) is a special case of Theorem 2, and (c) $\Rightarrow$ (a) is a special case of Theorem 3. To verify (b) $\Rightarrow$ (c), we let $\mathcal{G} := \overline{\text{co}}(\mathcal{H})$ and use Lemma 5 to verify that all the required properties are satisfied by $\mathcal{G}$. □

While it is not apparent from the statement of this result, the proof of Theorem 2 actually provides an easy method to "compute" the set $\mathcal{H}$ the existence of which is asserted in part (b). The following simple example illustrates this point.

Example 2. Assume that X is a k-space and Y a Hilbert space. Let $\varepsilon \in \mathbf{C}(X, \mathbb{R}_{++})$ and $F \in \mathbf{C}(X, Y)$, and define $\Gamma : X \rightrightarrows Y$ by $\Gamma(x) := F(x) +$ $\text{cl}N_{\varepsilon(x)}(0)$ (recall Example 1). Then the proof of Theorem 2 entails that $\Gamma(x) = \overline{\text{co}}\{H_f(x) : f \in S_{Y^*}\}$ for all $x \in X$, where $H_f : X \to Y$ is defined by $H_f(x) := \arg\max\{f(w) : w \in \Gamma(x)\}$. (Here S_{Y^*} denotes the unit sphere of Y^*.) By the Riesz representation theorem, this means that, for any (fixed) $x \in X$, we have $\Gamma(x) = \{h_y(x) : \|y\| = 1\}$, where $h_y(x)$ is the unique solution to the following optimization problem:

$$\text{Maximize } \langle w, y \rangle \qquad \text{such that } w \in Y \text{ and } \langle w - F(x), w - F(x) \rangle \leq \varepsilon(x)^2.$$

Defining the Lagrangian $L : Y \times \mathbb{R} \to \mathbb{R}$ by $L(w, \lambda) := \langle w, y \rangle + \lambda(\varepsilon(x)^2 - \langle w - F(x), w - F(x) \rangle)$, and applying the generalized Lagrange theorem, we see that there exists a nonzero λ such that

$$\nabla_w L(h_y(x), \lambda)(u) = \langle u, y \rangle - 2\lambda \langle h_y(x) - F(x), u \rangle = 0 \qquad \text{for all } u \in Y,$$

where $\nabla_w L(w, \lambda)$ is the Gateaux derivative of $L(\cdot, \lambda)$ at w. This entails that $\langle h_y(x) - F(x), u \rangle = \left\langle \frac{1}{2\lambda} y, u \right\rangle$ for all $u \in Y$, that is, every bounded linear functional on Y vanishes at $h_y(x) - F(x) -$

$\frac{1}{2\lambda}y$. By the Hahn-Banach theorem, then, we must have $h_y(x) - F(x) - \frac{1}{2\lambda}y = 0$. Combining this with the constraint of the problem and using the fact that $\langle y, y\rangle = 1$, we reach the following conclusion: Where $h_y \in Y^X$ is defined by $h_y(x) := F(x) + \varepsilon(x)y$ for each $y \in Y$, the closed convex hull of $\{h_y : \|y\| = 1\}$ is a compact set in $\mathbf{C}(X, Y_w)$ such that

$$\Gamma(x) = \overline{\mathrm{co}}\{h_y(x) : \|y\| = 1\}, \qquad x \in X.$$

4. Proper Extensions of Rotund-Valued Proper Multifunctions

An important topic of topological linear analysis concerns the continuous extendability of a given continuous Y-valued function defined on a closed subset X of a certain topological space W. The classical results in this regard are the Tietze extension theorem that states that such an extension is possible when Y is a finite dimensional Euclidean space and W is a metric space, and Dugundji's 1951 theorem that achieves a substantial generalization of this observation by showing that every locally convex topological linear space is an absolute retract. Since Theorem 4 shows that a given rotund-valued proper multifunction can be represented as a compact union of continuous functions, it enables one use Dugundji type extension theorems for studying the continuous (in fact, proper) extensions of multifunctions. Particularly useful in this regard is the following generalization of Dugundji's theorem obtained by Michael (1953).

The Dugundji-Michael Extension Theorem. *Let X be a nonempty closed subset of a metric space W, and Z a locally convex topological linear space. There exists a continuous linear function $\mathbf{e} : \mathbf{C}(X, Y) \to \mathbf{C}(W, Z)$ such that $\mathbf{e}(F)|_X = F$ and $\mathbf{e}(F)(W) \subseteq \mathrm{co}(F(X))$ for all $F \in \mathbf{C}(X, Y)$.*

The following result is an immediate corollary of this theorem and Theorem 4.

Theorem 5. *Let X be a nonempty closed subset of a metric space W, and Y a Hausdorff locally convex topological linear space such that Y_w has the C-property. If $\Gamma : X \rightrightarrows Y$ is a rotund-valued proper multifunction, then there exists a proper $\Gamma_* : W \rightrightarrows Y$ such that $\Gamma_*|_X = \Gamma$ and $\Gamma_*(W) \subseteq \mathrm{co}(\Gamma(X))$.*

Proof. By Theorem 4, there exists a $\mathcal{G} \in \mathbf{cc}(\mathbf{C}(X, Y_w))$ such that $\Gamma(x) = \{G(x) : G \in \mathcal{G}\}$ for all $x \in X$. We define $\Gamma_* : W \rightrightarrows Y$ by $\Gamma_*(w) := \{\mathbf{e}(G)(w) : G \in \mathcal{G}\}$, where $\mathbf{e} : \mathbf{C}(X, Y_w) \to \mathbf{C}(W, Y_w)$ is as found in the Dugundji-Michael extension theorem. Since $\mathbf{e}$ is continuous and $\mathcal{G}$ is compact, $\mathbf{e}(\mathcal{G})$ must be a compact subset of $\mathbf{C}(W, Y_w)$. This implies that Γ_* is weakly compact-valued and continuous (by the argument given in the proof of Theorem 3). That it is convex-valued, on the other hand, follows from the linearity of $\mathbf{e}$ and convexity of $\mathcal{G}$. Thus, Γ_* is a proper multifunction that extends Γ. Moreover, we have $\mathbf{e}(G)(W) \subseteq \mathrm{co}(G(X))$ for each $G \in \mathcal{G}$, so

$$\Gamma_*(W) = \bigcup_{G\in\mathcal{G}} \mathbf{e}(G)(W) \subseteq \bigcup_{G\in\mathcal{G}} \mathrm{co}(G(X)) \subseteq \mathrm{co} \bigcup_{G\in\mathcal{G}} G(X) = \mathrm{co}(\Gamma(X)),$$

and the proof is complete. $\square$

Antosiewicz and Cellina (1977) has proved that every closed and bounded-valued continuous multifunction Γ from a closed subset of a metric space to a normed linear space admits a continuous extension with closed and bounded values (the range of which is contained in the convex hull of the range of Γ). Theorem 5 differs from this result in that it demands further that the original multifunction be rotund-valued and proper, but it also ensures that the extension of Γ be proper. Moreover, this result applies to certain multifunctions whose codomain need not be metrizable. (Similar comparisons show that Theorem 5 is not nested in the extension theorems of Borges (1967) and Tolstonogov (1987).) Perhaps more importantly, the approach followed here reduces the multifunction extension problem to that of compactly extending a compact set of continuous functions, and since various Dugundji-Michael type extension theorems have been obtained in the literature, the latter problem may sometimes be easier to tackle. For example, Borges (1988) has shown that the Dugundji-Michael extension theorem remains valid if W is a topological space whose topology is finer than a metrizable topology (i.e. W is *submetrizable*) and ∂X is compact (or, if W is a normal topological space and ∂X is separably metrizable). Combining this fact with the proof of Theorem 5 shows that one can replace the requirement of metrizability of W in the statement of Theorem 5 with its submetrizability (resp. normality) and the compactness (resp. separable metrizability) of ∂X.

5. Rotund-Valued Proper Additive Multifunctions

One can utilize the results obtained above (especially the constructive nature of the proof of Theorem 2) to see which sort of properties of a multifunction are inherited by the functions that it is made of, and conversely. This is particularly easy if X is a convex cone in some linear space and the multifunction $\Gamma : X \rightrightarrows Y$ at hand is *additive*, that is, if $\Gamma(x+x') = \Gamma(x)+\Gamma(x')$ holds for all $x, x' \in X$. (By a *convex cone* we mean here a convex set X such that $\lambda X \subseteq X$ for any $\lambda \geq 0$.) In particular, the present approach yields readily the following additive selection theorem which complements the additive selection theorems of Gajda and Ger (1987) and Smajdor (1990).

Proposition 1. *Let X be a convex cone in a topological linear space, Y a Hausdorff locally convex topological linear space, and $\Gamma : X \rightrightarrows Y_w$ a rotund-valued proper multifunction. If Γ is additive and $y_0 \in \partial\Gamma(x_0)$ for some x_0 that belongs to the algebraic interior of X, then there exists a unique linear operator $L : \mathrm{span}(X) \to Y_w$ such that $L|_X$ is a continuous selection of Γ and $L(x_0) = y_0$.*

The proof of this result is facilitated by the following elementary observation concerning the linear extensions of an additive function defined on a convex cone.

Lemma 6. *Let X be a convex cone in a topological linear space W, and Z a topological linear space. For any continuous additive function $F \in Z^X$, there exists a unique linear operator $L \in Z^{\mathrm{span}(X)}$ such that $L|_X = F$. Moreover, in the special case where W and Z are Banach spaces and $\mathrm{span}(X)$ is closed, L must be continuous.*

Proof. The argument is largely standard, so we only give a sketch. By additivity of F, we have $F(mx') = mF(x')$ for all $(m, x') \in \mathbb{N}\times X$, so since for any positive integers k and l and any $x \in X$, we have $(k/l)x \in X$, we obtain $kF(x) = F(kx) = lF((k/l)x)$ for all such k, l and x. Thus $F(rx) = rF(x)$ for all $(r, x) \in \mathbb{Q}_{++}\times X$. Since F is continuous, it follows that F is positively homogeneous.

Now, obviously, $X - X = \text{span}(X)$, so for every $x \in \text{span}(X)$, there exists an $(a_x, b_x) \in X^2$ with $x = a_x - b_x$. We define $L \in Z^{\text{span}(X)}$ by $L(x) := F(a_x) - F(b_x)$. L is well-defined, for if $a - b = a' - b'$ for some $a, a', b, b' \in X$, we have $F(a + b') = F(a' + b)$, so we get $F(a) - F(b) = F(a') - F(b')$ by additivity of F. The additivity and positive homogeneity of F are easily shown to imply the linearity of L. The uniqueness claim is trivial.

Now assume the hypothesis of the second claim, and let N_m denote the m-neighborhood of 0 in $X - X$, $m = 1, 2, \ldots$. Since $X - X = \bigcup_{m\in\mathbb{N}} \text{cl}(X \cap N_m - X \cap N_m)$, the Baire category theorem entails that there exists a $k \in \mathbb{N}$ with $\text{int}(X \cap N_k - X \cap N_k) \neq \emptyset$. But then since $X\cap N_k - X\cap N_k$ is convex, $\text{int}(X\cap N_k - X\cap N_k)$ equals the algebraic interior of $X\cap N_k - X\cap N_k$, and hence $0 \in \text{int}(X \cap N_k - X \cap N_k)$. We conclude that there exists an open neighborhood O of 0 such that $O \subseteq X \cap N_k - X \cap N_k$, so

$$L(O) \subseteq L(X \cap N_k - X \cap N_k) = F(X \cap N_k) - F(X \cap N_k).$$

But since F is continuous, it is bounded on X, and hence $F(X \cap N_k)$ is a bounded subset of Y, and so is $L(O)$. Thus, the image of some neighborhood of 0 in $X - X$ is bounded in Y, which is equivalent to say that L is continuous. □

In what follows, we will adopt the notation introduced in the proof of Theorem 2.

Proof of Proposition 1. By (3.2), there exists an $f \in Y_0^*$ such that $H_f \in \mathbf{C}(X, Y_w)$ is a selection of Γ such that $y_0 = H_f(x_0)$. Since, for any $x, x' \in X$, we have $f(H_f(x)) \geq f(\Gamma(x))$ and $f(H_f(x')) \geq f(\Gamma(x'))$, we get

$$\begin{aligned} f(H_f(x) + H_f(x')) &= f(H_f(x)) + f(H_f(x')) \\ &\geq f(\Gamma(x)) + f(\Gamma(x')) \\ &= f(\Gamma(x) + \Gamma(x')) \\ &= f(\Gamma(x + x')). \end{aligned}$$

Since $\arg\max\{f(w) : w \in \Gamma(x + x')\}$ is a singleton (because $\Gamma(x + x')$ is weakly compact and rotund), it follows that $H_f(x+x') = H_f(x)+H_f(x')$ for all $x, x' \in X$, that is, H_f is additive. Then, by Lemma 6, there exists a linear operator $L \in Y^{\text{span}(X)}$ with $L|_X = H_f$, establishing the existence part of the claim.

To prove the uniqueness, let $T \in Y_w^{\text{span}(X)}$ be a linear operator such that $T|_X$ is a continuous selection of Γ and $T(x_0) = y_0$. By the uniqueness part of Lemma 6, it is clear that it is enough to show that $T|_X = H_f$ to conclude that $T = L$. To this end, fix any $x \in X$. Since x_0 belongs to the algebraic interior of X, there exists an $\varepsilon \in (0, 1)$ such that $(1 - \lambda)x_0 + \lambda x \in X$ for all $-\varepsilon \leq \lambda \leq \varepsilon$. Since $T|_X$ is a selection of Γ, we have $T|_X((1 - \lambda)x_0 + \lambda x) \in \Gamma((1 - \lambda)x_0 + \lambda x)$

and hence, by definition of H_f,

$$f(H_f((1-\lambda)x_0+\lambda x)) \geq f(T|_X((1-\lambda)x_0+\lambda x)), \quad -\varepsilon \leq \lambda \leq \varepsilon,$$

that is,

$$\lambda f((L-T)(x)) = f((L-T)((1-\lambda)x_0+\lambda x)) \geq 0, \quad -\varepsilon \leq \lambda \leq \varepsilon.$$

Obviously, this is possible only if $f((L-T)(x)) = 0$, that is, $f(H_f(x)) = f(T|_X(x))$. But $T|_X(x) \in \Gamma(x)$ and $H_f(x)$ is the only member of $\arg\max\{f(w) : w \in \Gamma(x)\}$, so we must have $T|_X(x) = H_f(x)$. Since x is arbitrary in X, it follows that $T|_X = H_f$, and the proof is complete. □

We do not know if the uniqueness part of Proposition 1 remains valid if x_0 is allowed to be arbitrary in X, and/or if y_0 can be arbitrary in $\Gamma(x_0)$. This remains as an interesting open problem.

As another application, we note the following characterization of rotund-valued proper additive multifunctions defined on $[0,\infty)$.

Proposition 2. *Let Y be a Hausdorff locally convex topological linear space.* $\Gamma : [0,\infty) \rightrightarrows Y$ *is a rotund-valued proper additive multifunction if, and only if, there exists a nonempty, weakly compact and strictly convex subset S of Y such that* $\Gamma(t) = tS$ *for all* $t \geq 0$.

Proof. The "if" part is trivial. To see the "only if" part, let $X := [0,\infty)$, and define H_f as in (3.1) for each $f \in Y_0^*$. The argument given in the proof of the previous proposition shows that each H_f is additive. By Lemma 6, therefore, H_f is positively homogeneous so that $H_f(t) = tH_f(1)$ for each $t \geq 0$. By (3.2) and the Krein-Milman theorem, therefore,

$$\Gamma(t) := \overline{\text{co}}\{H_f(t) : f \in Y_0^*\} = t\,(\overline{\text{co}}\{H_f(1) : f \in Y_0^*\}) = t\Gamma(1), \quad t \geq 0.$$

Setting $S := \Gamma(1)$ completes the proof. □

Nikodem (1981) has shown that any additive, continuous, compact and convex-valued multifunction from $[0,\infty)$ to a metric linear space is of the form described in Proposition 1. The present result, in turn, shows that one can take the codomain as any Hausdorff locally convex space and the multifunction Γ only weakly continuous and weakly compact-valued in Nikodem's theorem, provided that Γ is rotund-valued.

Proposition 1 can be extended to the case of multifunctions defined on a finitely generated convex cone in the obvious way. When X is an arbitrary convex cone, however, such a sharp characterization of a rotund-valued proper additive multifunction is not possible. The following result reports what can be said in this general case on the basis of the present approach.

Proposition 3. *Let X be a convex cone (with a closed span) in a linear k-space, and Y a Hausdorff locally convex topological linear space such that Y_w has the C-property. A multifunction* $\Gamma : X \rightrightarrows Y$ *is rotund-valued, proper and additive if, and only if, there exists a convex set $\mathcal{L}$ of (continuous) linear operators from* span(X) *into Y such that* $\{L|_X : L \in \mathcal{L}\} \in \mathbf{c}(\mathbf{C}(X, Y_w))$ *and* $\{L(x) : L \in \mathcal{L}\}$ *is a rotund set that equals* $\Gamma(x)$ *for each* $x \in X$.

Proof. The "if" part of the claim is an obvious consequence of Theorem 3. To prove the "only if" part, let $\mathcal{H}$ be the (point-open) closure of $\{H_f : f \in Y_0^*\}$ and $\mathcal{G} := \overline{\text{co}}(\mathcal{H})$. By the arguments given in the proofs of Theorems 2 and 4, $\mathcal{G} \in \mathbf{cc}(\mathbf{C}(X, Y_w))$, and $\mathcal{G}(x)$ is a rotund set that equals $\Gamma(x)$ for each $x \in X$. The argument given in the proof of Proposition 1 shows that each H_f is additive. It follows that each $G \in \mathcal{G}$ is additive. Let L_G denote the unique extension of $G \in \mathcal{G}$ found by Lemma 6. Then $\mathcal{L} := \{L_G : G \in \mathcal{G}\}$ satisfies all the required properties. □

Due to its connection with the Hyers-Ulam stability problem (Gajda and Ger, 1987), it is interesting to inquire into how this result would modify for sublinear (i.e. subadditive and positively homogeneous) multifunctions. Little is known, however, on this matter.

Our final result is the following extension theorem which gives sufficient conditions for the existence of a subadditive extension of an additive multifunction.

Proposition 4. *Let X be a convex cone with a closed span in a Hilbert space W, and Y a Banach space. If $\Gamma : X \rightrightarrows Y$ is a rotund-valued, proper and additive multifunction, then for any $(x_0, y_0) \in X \times Y$ with $y_0 \in \Gamma(x_0)$, there exists a continuous linear operator L from W to Y such that $y_0 = L(x_0)$ and $L(x) \in \Gamma(x)$ for all $x \in X$.*

Proof. This follows from Proposition 3 and the fact that every bounded linear operator defined on a subspace of a Hilbert space W can be extended to a bounded linear operator defined on W (Kakutani, 1939). □

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