

Incomplete Preferences under Uncertainty: Indecisiveness in Beliefs vs. Tastes*

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Abstract

We investigate the classical Anscombe-Aumann model of decision-making under uncertainty without assuming the completeness axiom. We deduce the exact nature of preferences in this setup, and give a simple example that shows why one cannot use state-independent utility functions to represent such preferences (even for those that are separable and monotonic). In turn, we distinguish between the dual traits of “indecisiveness in beliefs” and “indecisiveness in tastes.” The former is captured by the Knightian Uncertainty model while the latter by the single-prior expected multi-utility model. We show how these dual models would jointly obtain on the basis of an ambiguity aversion property, and then show how each model may be deduced from this result in isolation. As an application, we provide a new characterization of the Anscombe-Aumann representation of a preference relation in which we obtain the completeness and monotonicity hypotheses as consequences of a standard ambiguity aversion axiom and completeness of risk preferences alone.

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1. Introduction

Recently, there has been a surge of interest in the role of incomplete preferences within the broad spectrum of decision theory. Many authors have studied such preferences from both the ordinal and cardinal viewpoints, often yielding elegant generalizations of some of the classical findings of individual choice theory. Furthermore, the related results have found applications in several branches of decision theory – for instance, in the analysis of reference-dependent choice and comparative ambiguity aversion – as well as within game theory, political science and financial economics at large.¹

However, despite the abundance of recent work in this area, much remains to be learned about incomplete preferences. In particular, such preferences are studied in the context of uncertainty only in special cases. Indeed, the main model in this regard is the Knightian Uncertainty model of Bewley (1986), which, while dealing with an extremely important case, allows one to work with only a particular type of incompleteness. After all, the Knightian Uncertainty model is not what one would get by relaxing the completeness hypothesis within the classical Anscombe-Aumann model. The first main objective of the present paper is to find exactly what kind of a decision making model would obtain as a consequence of this sort of a relaxation.

In the context of uncertainty, there appears to be two main sources for the incompleteness of the preferences of a decision maker. First, one may find it impossible to compare the desirability of two acts due to the variation of the promises of these acts across states, because, for some reason or another, she cannot formulate a “precise guess” about the likelihood of the states of the world. This sort of an incompleteness of preferences, which may be called “indecisiveness in beliefs,” is, of course, what Bewley’s 1986 model is primed to capture. Second, even if one is able to assess, subjectively, the likelihood of each state, she may still be unable to compare two lotteries, perhaps because the lotteries in question are of complicated nature.² In turn, such sort of an incompleteness of preferences, which may be called “indecisiveness in tastes,” was precisely what Aumann (1962) has aimed to capture.

¹The ordinal utility representation of an incomplete preference relation was studied by Peleg (1970), Jaffray (1975), Sondermann (1980), Ok (2002) and Evren and Ok (2008), among many others. In turn, revealed preference theories of decision-making with incomplete preferences were recently explored by Eliaz and Ok (2006) and Mandler (2008). The cardinal utility representation of an incomplete preference relation (over lotteries or horse race lotteries) owes much to the seminal works of Aumann (1962) and Bewley (1982). Recent advances in this line of thought include Dubra, Maccheroni and Ok (2004), Sagi (2006), Nau (2007), Baucells and Shapley (2008), and Kochov (2008). On the other hand, the list of references of recent applications of incomplete preferences in economics is too long to cite here.

²Notice that this does not mean that the agent is indifferent between the two lotteries, for improving one of these lotteries marginally need not lead the agent to abandon her indecisiveness. How one would make her choice in this sort of a situation, or put differently, how a decision theorist may distinguish between one’s indecisiveness and indifference, is a well-known problem of revealed preference theory. (The work of Eliaz and Ok (2006) is devoted to the solution of this problem.)

Surprisingly, the literature has largely ignored models that would allow one to work with both of these types of indecisiveness in tandem. Abstract, in particular, is a “dual” to Bewley’s model in which a decision maker is decisive with respect to her beliefs but not with respect to her tastes. Our second main objective here is to understand how we may be able to distinguish between one’s indecisiveness in her beliefs and tastes from a behavioral perspective.³ This, in turn, will lead us to characterize the “dual” of the Knightian Uncertainty model.

The findings of the present paper can be summarized as follows. We adopt the framework of the classical Anscombe-Aumann model with a finite state space Ω and a compact (metric) prize space X . Our interest in this paper is confined to those (possibly incomplete) preference relations (on the space of horse race lotteries, or simply, acts) that satisfy the Independence and Continuity Axioms. Indeed, as a first order of business here, we provide a characterization of such preference relations. Not surprisingly, our characterization is a multi-utility version of the classical state-dependent expected utility representation. It generalizes that obtained by Nau (2006) to the case of infinite prize spaces, but its main purpose here is to position us for examining the effects of various state-independence axioms that one may impose in this framework.

By contrast to the case of complete preferences, we find that the classical separability and monotonicity properties are logically distinct for incomplete preference relations. We thus treat these properties separately, and show exactly how the Independence and Continuity Axioms interact with these properties in isolation. These results achieve our first goal, that is, they show precisely which type of a utility model would obtain as a result of the relaxation of the Completeness Axiom within the Anscombe-Aumann setup.

A natural candidate in this regard is, of course, the *multi-prior expected multi-utility* model according to which an arbitrary act f is preferred to another act g iff

$$\sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)) \geq \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, g(\omega)) \quad \text{for every } (\mu, u) \in \mathcal{H}, \quad (1)$$

where \mathcal{H} is a nonempty set of pairs of beliefs on Ω and (state-independent) continuous utility functions on X . (Here, of course, $\mathbf{E}(u, p)$ stands for the expectation of u with respect to the Borel probability measure p on X .) However, it was noted by Seidenfeld, Schervish and Kadane (1995), and reemphasized by Nau (2006), that relaxing the Completeness Axiom in the Anscombe-Aumann setup does not lead one to the multi-prior expected multi-utility model. As this is an important point that shows how big of a role is played by the completeness hypothesis in this framework, and as the examples provided by the previous papers

³We should make note of the recent paper by Nau (2006) at this junction, where a model of incomplete preferences under uncertainty which incorporates both types of indecisiveness, namely the multi-prior expected multi-utility model, is axiomatically characterized. Nau’s characterization, while novel, is based on a complicated axiom whose behavioral basis is difficult to interpret, and is valid only when the involved prize space is finite. Furthermore, it does not tell us the exact consequences of dropping the Completeness Axiom from the Anscombe-Aumann setup, nor it helps to distinguish, behaviorally, between different types of indecisiveness.

are quite complicated, we also provide here a simple illustration that highlights why the separability and monotonicity properties do not yield state-independent utility functions in the broader context of incomplete preferences. In turn, our first set of results clarifies the situation entirely, by showing exactly how far off one falls from the multi-prior expected multi-utility model by postulating only the Anscombe-Aumann axioms (except the completeness hypothesis).

Clearly, the multi-prior expected multi-utility model allows for one to be indecisive both in her beliefs and tastes. The main body of the present paper uses our general theory of incomplete preferences to learn more about the special cases of this model that corresponds to these traits in isolation. The first of these is the *multi-prior expected single-utility* model – this is, of course, none other than Bewley’s Knightian Uncertainty model – in which \mathcal{H} in (1) can be written as

$$\mathcal{H} = \mathcal{M} \times \{u\},$$

where \mathcal{M} is a nonempty set of beliefs on Ω and u is a (state-independent) continuous utility function on X . Obviously, this model captures “indecisiveness in beliefs” alone, for it maintains that one’s risk preferences (that is, her preferences over constant acts) are complete. The natural “dual” to this model is the *single-prior expected multi-utility* model in which \mathcal{H} in (1) can be written as

$$\mathcal{H} = \{\mu\} \times \mathcal{U},$$

where μ is a belief on Ω and \mathcal{U} is a nonempty set of (state-independent) continuous utility functions on X . Clearly, this model captures “indecisiveness in tastes” while it allows one to form beliefs about the states of the world in a precise manner.

In what follows, we shall show how easily one can obtain the first of these two models from our general results, much the same way the Anscombe-Aumann representation obtains from the state-dependent expected utility representation. Among other things, this streamlines the treatment of the Knightian Uncertainty model. Second, and far more important, we shall find here a rather curious connection between the above mentioned dual models in terms of the trait of ambiguity aversion. Notice that both of these models allow one to reduce any given act into a constant one by means of mixing the outcomes of that act suitably. Put differently, either of these models permits the reduction of subjective uncertainty into objective uncertainty in a desirable manner. As it allows one to use different sorts of mixtures for different acts, this property, which we propose to call the Weak Reduction Axiom, appears to be a fairly weak reflection of the notion of ambiguity aversion. (Indeed, it is satisfied by any complete and monotonic preference relation. Further, the definitions of ambiguity aversion by Epstein (1999) and Ghirardato and Marinacci (2002) entail a much stronger form of this property.)

Surprisingly, we find here that, at least for standard preferences, the Weak Reduction is what sets the dual models we mentioned above apart from the rest. Our main result shows that a preference relation satisfies the Independence, Continuity and Weak Reduction Axioms if, and only if, it can be represented as in one of these two models. Thus, at a

basic level, the principle of ambiguity aversion restricts not only the extent – this is trivial – but the nature of the incompleteness of a standard preference relation. It turns out that a preference relation that respects (our formulation of) this principle is bound to be decisive either in beliefs or in tastes (or both, in which case that preference relation is complete).

Apart from its conceptual teaching, it seems to us that this theorem may serve as a useful tool for decision theory as well. We conclude the present paper by considering two applications that illustrate this possibility. First, we show here how this result leads one to a straightforward characterization of the single-prior expected multi-utility model, thereby providing an axiomatic treatment of a “dual” to Bewley’s model in which a decision maker is decisive with respect to her beliefs but not with respect to her tastes. Second, we use this result to derive an interesting characterization of the Anscombe-Aumann model where both the Completeness and Monotonicity Axioms are replaced by completeness-over-constant-acts and ambiguity aversion (the way this notion is defined by either Epstein (1999) and Ghirardato and Marinacci (2002)). Thus, we arrive at the completeness of the Anscombe-Aumann preferences by deriving it from ambiguity aversion, instead of assuming it. This is a case in point that shows how the theory of incomplete preferences may be useful in learning about complete preferences.

The paper is organized as follows. Section 2 introduces the basic setup. In Section 3, we generalize a well-known result of decision theory on the state-dependent expected utility representation of a complete preference relation to the broader context of incomplete preferences. Section 4 gives characterizations of preferences that satisfy all of the postulates of the Anscombe-Aumann model, except the completeness hypothesis. Section 5 provides a simple example that illustrates why the multi-prior expected multi-utility model demands more than these postulates. Section 6 introduces the notions of “indecisiveness in beliefs” and “indecisiveness in tastes,” and Section 7 provides a characterization of the corresponding utility representation models jointly. In Section 8, we deduce a characterization of the “dual” to the Knightian Uncertainty model. In turn, Section 9 provides an application to the classical expected utility theory in which we obtain the Anscombe-Aumann representation without using either the completeness or monotonicity properties. While Section 10 concludes, the Appendix contains the proofs of our main results.

2. The Anscombe-Aumann Model without Completeness

We work within the classical Anscombe-Aumann setup with a finite state space. In what follows, therefore, Ω stands for a finite (nonempty) set of states of the world, X for a compact metric space of alternatives, and $\Delta(X)$ for the set of all Borel probability measures (lotteries) on X . As usual, we metrize $\Delta(X)$ in such a way that metric convergence on it coincides with weak convergence of Borel probability measures. (This, in particular, ensures that $\Delta(X)$ is a compact metric space.) The expectation of any continuous real map u on X with respect

to probability measure p in $\Delta(X)$ is denoted by $\mathbf{E}(u, p)$, that is,

$$\mathbf{E}(u, p) := \int_X u dp.$$

An **act** (or **horse race lottery**) in this setup is a function that maps the state space Ω into the space $\Delta(X)$ of lotteries. The set of all acts, denoted by \mathcal{F} , is, therefore, $\Delta(X)^\Omega$. In what follows, for any nonempty subset S of Ω , we denote by $\mathbf{1}_S$ the indicator function of S on Ω . Thus, for any lottery $p \in \Delta(X)$ and act $f \in \mathcal{F}$, the act that yields p if any one of the states in S occurs, and $f(\omega)$ at any other state ω , is written as

$$p\mathbf{1}_S + f\mathbf{1}_{\Omega \setminus S}.$$

In particular, any act f is decomposed as

$$f = \sum_{\omega \in \Omega} f(\omega)\mathbf{1}_{\{\omega\}}.$$

An act f in \mathcal{F} is said to be **constant** if $|f(\Omega)| = 1$, that is, if it yields the same lottery at every state of the world. Obviously, any such act can be written as $p\mathbf{1}_\Omega$ for some $p \in \Delta(X)$. We denote the set of all constant acts by \mathcal{F}_c .⁴ Naturally, we metrize \mathcal{F} by means of the product metric – this makes \mathcal{F} also a compact metric space. In turn, \mathcal{F}_c is a closed (and hence compact) subset of \mathcal{F} relative to the subspace topology. (Evidently, \mathcal{F}_c and $\Delta(X)$ are isometric.)

A **preference relation** \succsim in the present framework is a preorder (that is, a reflexive and transitive binary relation) on \mathcal{F} . (As usual, the asymmetric part of this preorder is denoted by \succ , and its symmetric part by \sim .) Such a preference relation tells us how an agent makes ordinal comparisons between various acts, and hence summarizes the attitudes of the agent toward both uncertainty and risk. The restriction of \succsim on \mathcal{F}_c , in turn, corresponds to the preferences of the agent over risky prospects, that is, lotteries on X , alone.

The axioms of the Anscombe-Aumann model are imposed on a fixed, but arbitrary, preference relation \succsim on \mathcal{F} . The following are two of the most basic of those postulates.

The Independence Axiom. For any acts f, g and h in \mathcal{F} , and $0 < \alpha \leq 1$,

$$f \succsim g \quad \text{implies} \quad \lambda f + (1 - \lambda)h \succsim \lambda g + (1 - \lambda)h.$$

The Continuity Axiom. \succsim is a closed subset of $\mathcal{F} \times \mathcal{F}$.⁵

⁴Obviously, both \mathcal{F} and \mathcal{F}_c depend on the primitives Ω and X . We do not use a notation that makes this dependence explicit to simplify our exposition.

⁵That is, for any convergent sequences (f_m) and (g_m) in \mathcal{F} with $f_m \succsim g_m$ for each m , we have $\lim f_m \succsim \lim g_m$.

As these axioms are entirely standard, we shall not discuss their content here. But we should note that the other standard axiom of the Anscombe-Aumann model, namely, the completeness of \succsim , is not assumed in the present setup. As we shall see, this allows for richer types of representations for \succsim . Identification of these, and more generally, determining the outline of the Anscombe-Aumann model without the completeness axiom, is the primary objective of the present paper.

3. Additively Separable Expected Multi-Utility Representation

If a complete preference relation \succsim on \mathcal{F} satisfies the Independence and Continuity Axioms, then it can be represented by an aggregate utility function of the form

$$f \mapsto \sum_{\omega \in \Omega} \int_X U(\cdot, \omega) df(\omega),$$

where U is a continuous real map on $X \times \Omega$. This is indeed a standard result of decision theory. As a first order of business in this paper, we shall determine how this representation result alters when \succsim is not known to be complete. The Expected Multi-Utility Theorem of Dubra, Maccheroni and Ok (2004) shows that, if there was only one state of the world – that is, if we were in the context of risk, but not of uncertainty – then the above representation would hold, albeit by means of a *set* of utility functions. Therefore, a natural conjecture here is that, in the general case, we would also have an additively separable representation in which a multitude of (state-dependent) utility functions are used instead of a single (state-dependent) utility function.

Let us first define the notion of utility representation we have in mind in formal terms.

Definition. A preference relation \succsim on \mathcal{F} is said to admit an **additively separable expected multi-utility representation** if there exists a nonempty subset \mathcal{U} of $\mathbf{C}(X \times \Omega)$ such that, for any acts f and g in \mathcal{F} , we have $f \succsim g$ iff

$$\sum_{\omega \in \Omega} \int_X U(\cdot, \omega) df(\omega) \geq \sum_{\omega \in \Omega} \int_X U(\cdot, \omega) dg(\omega) \quad \text{for every } U \in \mathcal{U}. \quad (2)$$

In what follows, we refer to any such subset \mathcal{U} of $\mathbf{C}(X \times \Omega)$ as an **additively separable expected multi-utility** for \succsim .

Our first result shows that this sort of a representation is warranted solely on the basis of the properties of independence and continuity.

Theorem 1. A preference relation \succsim on \mathcal{F} satisfies the Independence and Continuity Axioms if, and only if, it admits an additively separable expected multi-utility representation.

While its proof is somewhat involved, we should note that the basic content of this result is not new. Indeed, it reduces to Theorem 2 of Nau (2006) when X is finite (and to the Expected Multi-Utility Theorem of Dubra, Maccheroni and Ok (2002) when $|\Omega| = 1$).⁶ Our contribution here is mainly to show that Nau’s said theorem extends to the case where the outcome space X is an arbitrary compact metric space.⁷

In what follows, we shall examine various ways of refining Theorem 1 to derive more structured representation results. While it is interesting in its own right, this result thus mainly serves for us here as a starting point for the theory of incomplete preferences under uncertainty.

4. State Independence

The representation obtained in Theorem 1 uses state-dependent utility functions (over outcomes). There is nothing surprising about this, for none of the axioms used in Theorem 1 relates the conditional preferences of the agent across states of the world. A standard way of building such a connection is by imposing on \succsim a separability requirement of the following form:

The Separability Axiom. For any lotteries p and q in $\Delta(X)$, act f in \mathcal{F} , and states ω and τ in Ω ,

$$p\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{\Omega \setminus \{\omega\}} \succsim q\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{\Omega \setminus \{\omega\}} \quad \text{iff} \quad p\mathbf{1}_{\{\tau\}} + f\mathbf{1}_{\Omega \setminus \{\tau\}} \succsim q\mathbf{1}_{\{\tau\}} + f\mathbf{1}_{\Omega \setminus \{\tau\}}.$$

In words, \succsim satisfies the Separability Axiom iff the induced preferences of the agent conditional on any two states of the world are identical. This is a crucial property, for it is well-known that a complete preference relation \succsim on \mathcal{F} that satisfies the Independence, Continuity and Separability Axioms can be represented by an aggregate utility function of the form

$$f \mapsto \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)),$$

where μ is a probability distribution on Ω – interpreted as one’s prior beliefs about the states of the world – and u is a continuous real function on X – interpreted as one’s von Neumann-Morgenstern utility function over (riskless) alternatives. This is, of course, the *Anscombe-Aumann Expected Utility Theorem*. It is also well-known that one can replace the Separability Axiom with the following property in this theorem:

⁶See García del Amo and Ríos Insua (2002) for a related result (in which Ω and X are assumed to be subsets of a finite-dimensional Euclidean space).

⁷As a matter of fact, compactness of X is essential here. Theorem 1 does not generalize even to the case where X is a Polish space, and this, even if $|\Omega| = 1$. See Evren (2008) for an elaborate example.

The Monotonicity Axiom. For any acts f and g in \mathcal{F} , if $f(\omega)\mathbf{1}_\Omega \succsim g(\omega)\mathbf{1}_\Omega$ for every state ω in Ω , then $f \succsim g$.

In words, \succsim satisfies the Monotonicity Axiom iff \succsim would rank an act over another whenever the (risky) outcome of the former act is deemed better than that of the latter (according to the induced risk preferences, that is, $\succsim \cap (\mathcal{F}_c \times \mathcal{F}_c)$) for *every* state of the world. It is well-known that, for a complete preference relation \succsim on \mathcal{F} that satisfies the Independence and Continuity Axioms, the properties of separability and monotonicity are equivalent (provided that no state is deemed null by \succsim). However, it turns out that this equivalence ceases to hold when we drop the completeness hypothesis. As we shall see, in general, these two properties refine the representation we obtained in Theorem 1 in different ways.

Let us first introduce the following bits of notation.

Notation. Let Z be a compact metric space and \mathcal{U} a nonempty subset of $\mathbf{C}(Z)$. We write $\langle \mathcal{U} \rangle$ for the closed convex hull of \mathcal{U} and all constant functions on Z . That is, $v \in \langle \mathcal{U} \rangle$ iff either v can be written as a positive linear combination of finitely many maps in \mathcal{U} plus a constant, or it is equal to the (uniform) limit of a sequence of maps in $\mathbf{C}(Z)$ each of which can be written this way.

Notation. For any nonempty subset \mathcal{U} of $\mathbf{C}(X \times \Omega)$, and any ω in Ω , we write

$$\mathcal{U}_\omega := \{U(\cdot, \omega) : U \in \mathcal{U}\},$$

and

$$\mathcal{U}_\Omega := \left\{ \sum_{\omega \in \Omega} U(\cdot, \omega) : U \in \mathcal{U} \right\}.$$

Clearly, both of these sets are subsets of $\mathbf{C}(X)$. If \mathcal{U} is an additively separable expected multi-utility for an agent's preferences, then \mathcal{U}_ω is the set of utility functions that represents this agent's (incomplete) preferences over lotteries *given state* ω , while \mathcal{U}_Ω is the set of utility functions that represents her risk preferences (that is, her preferences over constant acts).

We are now ready to show how Theorem 1 modifies when we add to its statement the Separability Axiom.

Theorem 2.a. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity and Separability Axioms if, and only if, there exists an additively separable expected multi-utility \mathcal{U} for \succsim such that

$$\langle \mathcal{U}_\omega \rangle = \langle \mathcal{U}_\tau \rangle \quad \text{for every } \omega, \tau \in \Omega. \quad (3)$$

The Ancombe-Aumann Expected Utility Theorem shows that, when \succsim is complete, the Separability Axiom forces the state-dependent (von Neumann-Morgenstern) utility functions to represent identical risk preferences. In turn, Theorem 2.a shows that the gist of this fact remains true when we allow \succsim to be incomplete. Put precisely, in this case, the Separability Axiom forces the state-dependent *sets* of utility functions to represent identical (but possibly incomplete) risk preferences. Indeed, it can be shown that (3) is equivalent to say that, for any lotteries p and q in $\Delta(X)$,

$$\int_X u dp \geq \int_X u dq \text{ for each } u \in \mathcal{U}_\omega \quad \text{iff} \quad \int_X v dp \geq \int_X v dq \text{ for each } v \in \mathcal{U}_\tau$$

for every states ω and τ in Ω . (See Lemma 1 of the Appendix.) Thus, (3) means simply that \mathcal{U}_ω and \mathcal{U}_τ represent the same (incomplete) preferences over $\Delta(X)$.

Theorem 2.a does not tell us how Theorem 1 would modify if we assumed that \succsim was monotonic. This is because, in the broader context of incomplete preferences, the Separability Axiom does not imply the Monotonicity Axiom even in the presence of the Independence and Continuity Axioms (and the absence of any null states). The following example illustrates this point.

Example 1. We consider a two-state two-outcome environment, so assume that $\Omega := \{\omega_1, \omega_2\}$ and $X := \{x, y\}$. Consider the following state-dependent utility functions:

$$\begin{array}{c|cc} U & \omega_1 & \omega_2 \\ \hline x & 3 & 0 \\ y & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} V & \omega_1 & \omega_2 \\ \hline x & 0 & 3 \\ y & 1 & 0 \end{array},$$

and let $\mathcal{U} := \{U, V\}$. Finally, let \succsim be the preference relation on $\mathcal{F} := \Delta(X)^\Omega$ such that $f \succsim g$ iff (2) holds in this setup. It is obvious that \succsim satisfies the Independence and Continuity Axioms (and that no state is deemed null by \succsim). It is also easy to see that, for any $\omega \in \Omega$ and any two lotteries p and q on X , we have $p\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{\Omega \setminus \{\omega\}} \succsim q\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{\Omega \setminus \{\omega\}}$ iff $p = q$.⁸ Consequently, \succsim satisfies the Separability Axiom trivially. (This fact is also evident from Theorem 2.a, for here we obviously have $\langle \mathcal{U}_{\omega_1} \rangle = \mathbb{R}^X = \langle \mathcal{U}_{\omega_2} \rangle$.) Yet, \succsim does not satisfy the Monotonicity Axiom. Indeed, if p is the lottery that yields x with probability one, and q is the lottery that yields y with probability one, then, $p\mathbf{1}_\Omega \succ q\mathbf{1}_\Omega$ while, as we have just seen, $p\mathbf{1}_{\{\omega_1\}} + q\mathbf{1}_{\{\omega_2\}}$ and $q\mathbf{1}_{\{\omega_1\}} + p\mathbf{1}_{\{\omega_2\}}$ are not comparable by \succsim . \square

We show next how Theorem 1 modifies if we add to its statement, instead, the Monotonicity Axiom.

⁸By definition of \succsim , for instance, we have $p\mathbf{1}_{\{\omega_1\}} + f\mathbf{1}_{\Omega \setminus \{\omega_1\}} \succsim q\mathbf{1}_{\{\omega_1\}} + f\mathbf{1}_{\Omega \setminus \{\omega_1\}}$ iff $3p(x) \geq 3q(x)$ and $p(y) \geq q(y)$.

Theorem 2.b. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity and Monotonicity Axioms if, and only if, there exists an additively separable expected multi-utility \mathcal{U} for \succsim such that

$$\mathcal{U}_\omega \subseteq \langle \mathcal{U}_\Omega \rangle \quad \text{for every } \omega \in \Omega. \quad (4)$$

Consequently, the only effect of the Monotonicity Axiom in the present framework is to force the state-dependent *sets* of utility functions (found in Theorem 1) to be compatible with the preferences of the agent over constant acts. In particular, unless \succsim is complete, this axiom does not guarantee that the state-dependent sets of utility functions of the agent represent identical risk preferences. As this is hardly an obvious point, we give next a concrete illustration of it.

Example 2. We consider again a two-state two-outcome environment, so assume that $\Omega := \{\omega_1, \omega_2\}$ and $X := \{x, y\}$. Consider the following state-dependent utility functions:

$$\begin{array}{c|cc} U & \omega_1 & \omega_2 \\ \hline x & 3 & 0 \\ y & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} V & \omega_1 & \omega_2 \\ \hline x & 3 & 0 \\ y & 0 & 4 \end{array} .$$

Set $\mathcal{U} := \{U, V\}$, and let \succsim be the preference relation on $\mathcal{F} := \Delta(X)^\Omega$ such that $f \succsim g$ iff (2) holds in this setup. It is obvious that \succsim satisfies the Independence and Continuity Axioms. It is also easy to see that, for any two lotteries p and q on X , we have $p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega$ iff $p = q$, that is, \succsim is unable to compare any two distinct constant acts.⁹ Consequently, \succsim satisfies the Monotonicity axiom trivially. Yet, one can easily show that $\langle \mathcal{U}_{\omega_1} \rangle = \{u \in \mathbb{R}^X : u(x) \geq u(y)\}$ while $\langle \mathcal{U}_{\omega_2} \rangle = \{u \in \mathbb{R}^X : u(y) \geq u(x)\}$, that is, the risk preferences of the agent conditional on the two states are distinct. Further, we have $\langle \mathcal{U}_\Omega \rangle = \mathbb{R}^X$ here, so we see that the containment in (4) may hold strictly. \square

In passing, let us note that using both the Separability and Monotonicity Axioms allows us to obtain a greater degree of compatibility for the attitudes of the agent towards risk in different states of the world:

Theorem 2.c. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity, Separability, and Monotonicity Axioms if, and only if, there exists an additively separable expected multi-utility \mathcal{U} for \succsim such that

$$\langle \mathcal{U}_\omega \rangle = \langle \mathcal{U}_\Omega \rangle \quad \text{for every } \omega \in \Omega.$$

⁹By definition of \succsim , we have $p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega$ iff $3p(x) + p(y) \geq 3q(x) + q(y)$ and $3p(x) + 4p(y) \geq 3q(x) + 4q(y)$. The latter two inequalities hold iff $p(x) = q(x)$.

Put differently, when we assume that \succsim (of Theorem 1) is both separable and monotonic, we are guaranteed that the state-dependent sets of utility functions of the agent represent identical risk preferences and that these risk preferences coincide with the preferences of the agent over constant acts.

Remark 1. The implications of other (stronger) separability conditions can be found similarly. For instance, the original “separability” axiom of Anscombe and Aumann (1963) is the following: For any lotteries p and q in $\Delta(X)$, acts f and g in \mathcal{F} , and nonempty subsets S and T of Ω ,

$$p\mathbf{1}_S + f\mathbf{1}_{\Omega \setminus S} \succsim q\mathbf{1}_S + f\mathbf{1}_{\Omega \setminus S} \quad \text{iff} \quad p\mathbf{1}_T + g\mathbf{1}_{\Omega \setminus T} \succsim q\mathbf{1}_T + g\mathbf{1}_{\Omega \setminus T}.$$

This axiom is easily seen to be stronger than both the Separability and Monotonicity Axioms. Indeed, replacing Separability and Monotonicity with this axiom in Theorem 2.c would mean replacing the final condition of that result with the following: $\langle \mathcal{U}_S \rangle = \langle \mathcal{U}_T \rangle$ for every nonempty subsets S and T of Ω , where $\mathcal{U}_S := \{\sum_{\omega \in S} u(\cdot, \omega) : u \in \mathcal{U}\}$ and similarly for \mathcal{U}_T . \square

5. Multi-Prior Expected Multi-Utility Representation

It is a general theme that an incomplete preference relation that satisfies a certain property is often realized as a multi-person preference relation in the sense that such a relation can be expressed as the intersection of a set of complete preference relations that satisfy that property.¹⁰ Given the Anscombe-Aumann Expected Utility Theorem, this theme suggests the following utility representation notion in the present context:

Definition. A preference relation \succsim on \mathcal{F} is said to admit a **multi-prior expected multi-utility representation** if there exists a nonempty subset \mathcal{H} of $\Delta(\Omega) \times \mathbf{C}(X)$ such that, for any acts f and g in \mathcal{F} , we have $f \succsim g$ iff

$$\sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)) \geq \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, g(\omega)) \quad \text{for every } (\mu, u) \in \mathcal{H}. \quad (5)$$

In what follows, we refer to any such subset \mathcal{H} of $\Delta(\Omega) \times \mathbf{C}(X)$ as a **multi-prior expected multi-utility** for \succsim .

This is very much in the spirit of the “multi-person” way of looking at an incomplete preference relation \succsim on \mathcal{F} . After all, it maintains that \succsim be viewed as the common part of a

¹⁰See, for instance, Ok (2002), Dubra, Maccheroni and Ok (2004), Evren and Ok (2007), and Baucells and Shapley (2008).

set of preference relations on \mathcal{F} every one of which admits a standard expected utility representation – it thus abides fully with the viewpoint of multi-person Savagean decision-making. Unfortunately, and by stark contrast to the case of complete preferences, this representation notion is a good deal more demanding than that obtained in any one of the theorems we reported in the previous section. Put more precisely, a preference relation that satisfies the Independence, Continuity, Separability and Monotonicity Axioms need not admit a multi-prior expected multi-utility representation. This surprising fact is put on record both by Seidenfeld, Schervish and Kadane (1995) and Nau (2006), but the examples considered by these authors are extremely complex (and, indeed, the computations regarding them are incomplete). It thus seems like there is room for providing a simple example that points to the root of the difficulty of obtaining multi-prior expected multi-utility representations in general. We do this next.

Example 3. We again consider a two-state two-outcome environment, so assume that $\Omega := \{\omega_1, \omega_2\}$ and $X := \{x, y\}$. Consider the following state-dependent utility functions:

$$\begin{array}{c|cc} U & \omega_1 & \omega_2 \\ \hline x & 3 & 0 \\ y & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} V & \omega_1 & \omega_2 \\ \hline x & 0 & 1 \\ y & 2 & 0 \end{array},$$

and let $\mathcal{U} := \{U, V\}$. Finally, let \succsim be the preference relation on $\mathcal{F} := \Delta(X)^\Omega$ such that $f \succsim g$ iff (2) holds in this setup. It is obvious that \succsim satisfies the Independence and Continuity Axioms. Besides, for any two lotteries p and q on X , we have $p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega$ iff $p = q$, that is, \succsim is unable to compare any two distinct constant acts.¹¹ It follows that \succsim satisfies the Monotonicity Axiom trivially. That \succsim satisfies the Separability Axiom is similarly verified.¹²

We wish to show that \succsim does not admit a multi-prior expected multi-utility representation. To derive a contradiction, then, suppose we can find a multi-prior expected multi-utility \mathcal{H} for \succsim in this setup. Since \succsim does not compare any two distinct constant acts, there must exist two pairs (μ, u) and (ν, v) in \mathcal{H} such that $u(x) > u(y)$ and $v(y) > v(x)$. Now let f be the constant act that gives x for sure at every state, and let g be the act that gives x with probability .99 at state ω_1 , and that with probability .98 at state ω_2 . It is readily checked that $f \succ g$ by definition of \succsim . But this is not supported by (ν, v) , for we have

$$v(x) < \nu\{\omega_1\}(.99v(x) + .01v(y)) + \nu\{\omega_2\}(.98v(x) + .02v(y)).$$

Since $(\nu, v) \in \mathcal{H}$ and \mathcal{H} is a multi-prior expected multi-utility for \succsim , this means that $f \succ g$ is false, a contradiction. \square

¹¹By definition of \succsim , we have $p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega$ iff $3p(x) + p(y) \geq 3q(x) + q(y)$ and $2p(y) + p(x) \geq 2q(y) + q(x)$. The latter two inequalities hold iff $p(x) = q(x)$.

¹²Alternatively, we may apply Theorem 2.c here. After all, any one of the convex cones $\langle \mathcal{U}_{\omega_1} \rangle$, $\langle \mathcal{U}_{\omega_2} \rangle$ and $\langle \mathcal{U}_\Omega \rangle$ is equal to \mathbb{R}^X in this example.

In passing, we note that it is not known how, in the present setting, one can characterize a preference relation on \mathcal{F} that admits a multi-prior expected multi-utility representation. Unfortunately, it is likely that such a representation will not be simple, for, as Nau (2006) shows, the needed strengthening of the separability axiom is bound to be rather difficult to interpret from a behavioral angle, even when X is a finite set.

6. Types of Indecisiveness

There are two principal types of indecisiveness that an agent may display when comparing the acts in \mathcal{F} . First, the agent's *tastes* may be incomplete, in the sense that she may be unable to rank certain alternatives in X , perhaps because, at the time choice, she cannot be certain about how she will feel about these alternatives at the time of consumption. Alternatively, even if she is able to compare any two alternatives in X , she may get confused when comparing certain lotteries in $\Delta(X)$, perhaps because the nature of such lotteries are often quite complicated. Second, even if the agent may have no problem whatsoever in ranking any two risky prospects, her *beliefs* may be incomplete, in the sense that she may remain indecisive about comparing two acts in \mathcal{F} due to her inability of forming a “precise guess” about which states of the world would ensue at the time of consumption.

A preference relation on \mathcal{F} that admits a multi-prior expected multi-utility representation allows for the presence of both of these types of indecisiveness. In fact, the “multi-prior” part of this representation corresponds to the (potential) indecisiveness of the agent with respect to her beliefs, and its “multi-utility” part corresponds to her (potential) indecisiveness with respect to her tastes. In this section and the next, we shall concentrate on these two types of indecisiveness separately.

6.1 Indecisiveness in Beliefs

The most well-known model of incomplete preferences under uncertainty is, of course, the so-called *Knighian Uncertainty* model developed by the seminal contribution of Bewley (1986). This model is fully axiomatized, and have proved to possess an admirable scope for applications. It corresponds to a special case of the multi-prior expected multi-utility representation which allows one to work with an individual's indecisiveness only when this indecisiveness arises due to the presence of uncertainty, and not due to the presence of risk. Put differently, this is a model of indecisiveness in beliefs. After all, one of the main features of Bewley's model is its restriction of the incompleteness of \succsim outside the projection of this preference relation onto risky (but not uncertain) prospects. More precisely, Bewley's model maintains that $\succsim \cap (\mathcal{F}_c \times \mathcal{F}_c)$ is a complete preorder, or, to say this still in another way, it assumes:

The C-Completeness Axiom. For any lotteries p and q in $\Delta(X)$,

$$\text{either } p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega \quad \text{or} \quad q\mathbf{1}_\Omega \succsim p\mathbf{1}_\Omega.$$

When combined with the axioms used in Section 4, the C-Completeness Axiom yields the following special case of the multi-prior expected multi-utility representation:

Definition. A preference relation \succsim on \mathcal{F} is said to admit a **multi-prior expected utility representation** if there exists a nonempty subset \mathcal{M} of $\Delta(\Omega)$ and a function $u \in \mathbf{C}(X)$ such that, for any acts f and g in \mathcal{F} , we have $f \succsim g$ iff

$$\sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)) \geq \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, g(\omega)) \quad \text{for every } \mu \in \mathcal{M}. \quad (6)$$

Using this terminology, we can rephrase Bewley’s characterization as follows:

Bewley’s Expected Utility Theorem. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity, Monotonicity and C-Completeness Axioms if, and only if, it admits a multi-prior expected utility representation.

To reiterate, within the contexts of either Theorems 2a or 2b, completeness of the preferences of an agent over lotteries (but not on acts) guarantees that her indecisiveness pertains to her *beliefs*, and not to her tastes. Put simply, Bewley’s representation captures this formally by taking the “multi-” out of the “multi-utility” in the notion of multi-prior expected multi-utility representation.

Before turning to modeling incompleteness of one’s tastes, we wish to make a passing observation about the proof of Bewley’s Theorem. Recall that, for a complete preference relation \succsim on \mathcal{F} , it is routine to deduce the Anscombe-Aumann Expected Utility Theorem from the additively separable expected utility representation of \succsim by using the uniqueness of expected utility representation of a complete preference relation on $\Delta(X)$.¹³ By contrast, the proofs of Bewley’s Theorem that appeared in the literature are based on rather direct (convex-analytic) arguments, and hence they do not make clear the specific contributions of the axioms used in the characterization.

In fact, the proof of Bewley’s Theorem can be streamlined in a way analogous to the situation that ensues in the case of complete preferences. That is, we may deduce this result from Theorem 1 quite easily by using the uniqueness of the expected multi-utility representation of a preference relation on $\Delta(X)$. Not only that this method illustrates the strength of the general theory of incomplete preferences we have outlined so far, but it also makes transparent the individual contributions of the Monotonicity and C-Completeness Axioms in Bewley’s framework. Consequently, as an application of Theorem 2.b, we sketch a short (new) proof for Bewley’s Theorem here.

¹³Indeed, the uniqueness part of the von Neumann-Morgenstern Expected Utility Theorem tells us that, in this case, each of the involved state-dependent utility functions are positive affine transformations of each other, and hence, of a fixed state-independent utility function. See, for instance, Kreps (1988).

Proof of Bewley’s Expected Utility Theorem. We only need to prove the “only if” part of the assertion. To this end, let \succsim be a preference relation on \mathcal{F} that satisfies the Independence, Continuity, Monotonicity and C-Completeness Axioms. We assume in what follows that $\succsim \neq \mathcal{F} \times \mathcal{F}$, for otherwise the assertion is trivially true.

We apply Theorem 2.b to find an additively separable expected multi-utility \mathcal{U} such that $\langle \mathcal{U}_\omega \rangle \subseteq \langle \mathcal{U}_\Omega \rangle$ for every $\omega \in \Omega$. Now consider the binary relation \succsim_c on $\Delta(X)$, defined by $p \succsim_c q$ iff $p\mathbf{1}_\Omega \succsim q\mathbf{1}_\Omega$. Clearly, for any lotteries p and q in $\Delta(X)$, we have

$$p \succsim_c q \quad \text{iff} \quad \int_X v dp \geq \int_X v dq \quad \text{for every } v \in \mathcal{U}_\Omega.$$

On the other hand, thanks to the C-Completeness, Independence and Continuity Axioms, we may apply the von Neumann-Morgenstern Expected Utility Theorem to \succsim_c to find a map $u \in \mathbf{C}(X)$ such that

$$p \succsim_c q \quad \text{iff} \quad \int_X u dp \geq \int_X u dq$$

for any lotteries p and q in $\Delta(X)$. Therefore, by the uniqueness of expected multi-utility representations – see Lemma 1 in the Appendix – we find $\langle \{u\} \rangle = \langle \mathcal{U}_\Omega \rangle$. Consequently, $\langle \mathcal{U}_\omega \rangle$ lies within the convex cone generated by u and all constant real functions on X , for every ω in Ω . That is, for every $\omega \in \Omega$ and $U \in \mathcal{U}$, there exist real numbers $\alpha_{\omega,U} \geq 0$ and $\beta_{\omega,U}$ such that

$$U(\cdot, \omega) = \alpha_{\omega,U} u + \beta_{\omega,U}.$$

Notice that, since $\succsim \neq \mathcal{F} \times \mathcal{F}$, there exist a map $U \in \mathcal{U}$ and a state $\omega \in \Omega$ such that $\alpha_{\omega,U} > 0$. Define the set $\mathcal{U}^* := \{U \in \mathcal{U} : \alpha_{\omega,U} > 0 \text{ for some } \omega \in \Omega\}$, and notice that $f \succsim g$ iff

$$\sum_{\omega \in \Omega} \alpha_{\omega,U} \mathbf{E}(u, f(\omega)) \geq \sum_{\omega \in \Omega} \alpha_{\omega,U} \mathbf{E}(u, g(\omega)) \quad \text{for every } U \in \mathcal{U}^*$$

for any $f, g \in \mathcal{F}$. Defining $\mu_U \in \Delta(\Omega)$ by $\mu_U(\omega) := \alpha_{\omega,U} / \sum_{\tau \in \Omega} \alpha_{\tau,U}$, for each $U \in \mathcal{U}^*$, and letting $\mathcal{M} := \{\mu_U : U \in \mathcal{U}^*\}$, therefore, we find that $f \succsim g$ iff (6) holds, for any acts f and g in \mathcal{F} . ■

6.2 Indecisiveness in Tastes

Despite – or perhaps, because – of the influence of the Knightian Uncertainty model in decision theory and its applications, the literature does not provide a “dual” to this model, one that would concentrate on the incompleteness of the tastes of a decision maker, and not on that of her beliefs. Just as Bewley’s model takes the “multi-” out of the “multi-utility” in the notion of multi-prior expected multi-utility representation, such a “dual” model would convert the “multi-prior” into “single-prior” in this representation. Put precisely:

Definition. A preference relation \succsim on \mathcal{F} is said to admit a **single-prior expected multi-utility representation** if there exist a probability distribution $\mu \in \Delta(\Omega)$ and nonempty

subset \mathcal{U} of $\mathbf{C}(X)$ such that, for any acts f and g in \mathcal{F} , we have $f \succsim g$ iff

$$\sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)) \geq \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, g(\omega)) \quad \text{for every } u \in \mathcal{U}.$$

In this case, we say that (μ, \mathcal{U}) is a **single-prior expected multi-utility** for \succsim .

This utility representation notion extends the so-called expected multi-utility representation of preferences over lotteries to the context of preferences over uncertain acts in a straightforward manner. It models an agent who is confident in assessing the likelihood of the states of nature, but who is unable to compare certain lotteries with assurance. However, while its interpretation parallels, and complements, that of the Knightian Uncertainty model, this model has not received much attention in the literature. In particular, perhaps because it is not clear how one may formulate “completeness in beliefs” as a behavioral postulate – the “completeness of tastes” is, by contrast, captured trivially by the C-Completeness Axiom – no axiomatization has yet been given for this model. We shall solve this problem in Section 8, but first, we need to tackle the important issue of the ability of a decision maker, or lack of it thereof, to convert uncertainty to risk.

7. Reduction of Uncertainty to Risk

7.1 The Weak Reduction Axiom

An essential property of the classical Anscombe-Aumann model is its universal allowance for transforming subjective uncertainty to objective uncertainty. For instance, in an environment with two states, say, ω_1 and ω_2 , this model maintains that any act f that maps $\{\omega_1, \omega_2\}$ into $\Delta(X)$ is indifferent to a constant act that yields the lottery $\alpha f(\omega_1) + (1 - \alpha)f(\omega_2)$ at both of the states for some number α in $[0, 1]$. (Of course, α is the prior probability that the agent assigns to the event $\{\omega_1\}$.) This is an important property that allows one to factor uncertainty out in general, and reduce any choice problem of uncertainty to one of risk. Furthermore, this feature of the Anscombe-Aumann framework is shared by many of its extensions (such as the maxmin expected utility theory, multiplier preferences and α -maxmin expected utility model).

Given the sizable literature on uncertainty aversion, it is tempting to generalize this reduction property and ask for the possibility of reducing subjective uncertainty to objective uncertainty (that is, risk) in a manner that would be deemed *desirable* by the decision maker. To formulate this precisely, we need to introduce one final bit of notation.

Notation. For any act f in \mathcal{F} and any probability distribution α in $\Delta(\Omega)$, we write f^α to denote the constant act that yields the lottery $\sum_{\omega \in \Omega} \alpha\{\omega\}f(\omega)$ at every state, that is,

$$f^\alpha := \left(\sum_{\omega \in \Omega} \alpha\{\omega\} f(\omega) \right) \mathbf{1}_\Omega.$$

The axiom we have in mind can now be stated as follows:

The Weak Reduction Axiom. For any act f in \mathcal{F} , there is an $\alpha \in \Delta(\Omega)$ such that

$$f^\alpha \succsim f.$$

Given an act f in \mathcal{F} and a probability distribution α on Ω , the *constant act* f^α is obtained from f by mixing, in a manner indicated by α , the lotteries that f may potentially lead to, and setting the resulting lottery as the consequence at every state of the world. It thus makes sense to view this new act as one that is induced by f upon factoring out the underlying uncertainty in a specific manner. The Weak Reduction Axiom asks for the possibility of finding a mixture α that would make such a “reduction of uncertainty to risk” desirable for the involved agent. It is important to note that this is a *local* formulation of the basic premise of ambiguity aversion. For, and this is quite important, the needed type of mixing (that is, α) is allowed to depend on f here.

Of course, a preference relation \succsim on \mathcal{F} that admits a representation as in the Anscombe-Aumann model satisfies the Weak Reduction Axiom trivially. More generally, the same is true for any \succsim that admits a single-prior expected multi-utility representation. Indeed, if such representation holds for a prior μ in $\Delta(\Omega)$ and a nonempty set $\mathcal{U} \subseteq \mathbf{C}(X)$ of (state-independent) utility functions, we are sure to have $f^\mu \sim f$ for every $f \in \mathcal{F}$. However, the situation is much less clear for a general preference relation that admits a multi-prior expected multi-utility representation, as the following two examples illustrate.

Example 4. Let \succsim be a preference relation on \mathcal{F} that satisfies the Monotonicity and C-Completeness Axioms. Take any act f in \mathcal{F} , and notice that, by the C-Completeness Axiom and finiteness of Ω , we can find a state ω_f such that $f(\omega_f)\mathbf{1}_\Omega \succsim f(\omega)\mathbf{1}_\Omega$ for every $\omega \in \Omega$. Defining $\alpha \in \Delta(\Omega)$ by $\alpha\{\omega_f\} := 1$, then, we have $f^\alpha(\omega)\mathbf{1}_\Omega = f(\omega_f)\mathbf{1}_\Omega \succsim f(\omega)\mathbf{1}_\Omega$ for each ω in Ω , and hence, by the Monotonicity Axiom, we find $f^\alpha \succsim f$. Conclusion: The Monotonicity and C-Completeness Axioms imply the Weak Reduction Axiom. In particular: The Knightian Uncertainty model satisfies the Weak Reduction Axiom.

We note, however, that it may be impossible to have $f^\alpha \sim f$ within the Knightian Uncertainty model. To illustrate this, assume that $\Omega := \{\omega_1, \omega_2\}$ and $X := \{x, y\}$, and let \succsim admit a multi-prior expected utility representation where the set of priors consists of the two degenerate lotteries, one that assigns probability one to the state ω_1 and the other to ω_2 . Let f be the act that yields x with probability one in state ω_1 and y with probability one in state ω_2 . Then, we have $f^\alpha \sim f$ for some $\alpha \in \Delta(\Omega)$ iff the number

$\alpha\{\omega_1\}u(x) + (1 - \alpha\{\omega_1\})u(y)$ equals to both $u(x)$ and $u(y)$, where $u : X \rightarrow \mathbb{R}$ is the utility function used in the representation of \succsim . Thus, if $u(x)$ and $u(y)$ are distinct numbers – that is, if the agent prefers x to y strictly, or the converse of this is true – then $f^\alpha \sim f$ cannot hold for any $\alpha \in \Delta(\Omega)$. \square

The “local” nature of the Weak Reduction Axiom, and Example 4, may suggest that this property is unduly weak. The following example casts shadow on this point of view.

Example 5. We again consider a two-state two-outcome environment, so assume that $\Omega := \{\omega_1, \omega_2\}$ and $X := \{x, y\}$. Consider the following state-independent utility functions:

$$\begin{array}{c|cc} u & \omega_1 & \omega_2 \\ \hline x & 2 & 2 \\ y & 0 & 0 \end{array} \quad \text{and} \quad \begin{array}{c|cc} v & \omega_1 & \omega_2 \\ \hline x & 0 & 0 \\ y & 1 & 1 \end{array},$$

and let $\mathcal{H} := \{(\mu, u), (\nu, v)\}$, where μ and ν are the probability distributions on Ω such that $\mu\{\omega_1\} = 1 = \nu\{\omega_2\}$. Finally, let \succsim be the preference relation on $\mathcal{F} := \Delta(X)^\Omega$ such that \mathcal{H} is a multi-prior expected multi-utility for \succsim in this setup. Consider the act f that gives x with probability $\frac{3}{4}$ in state ω_1 , and that with $\frac{1}{4}$ in state ω_2 . Then, for any real number α in $[0, 1]$, one can check that the constant act that yields $\alpha f(\omega_1) + (1 - \alpha)f(\omega_2)$ at both of the states is preferred to f iff $\alpha \geq 1$ and $\alpha \leq 0$ hold simultaneously. Conclusion: A preference relation \succsim on \mathcal{F} that satisfies the multi-prior expected multi-utility representation need not satisfy the Weak Reduction Axiom. \square

7.2 Relation to Ambiguity Aversion

The following remarks summarize the relation of the Weak Reduction Axiom with the various formulations of the principle of uncertainty/ambiguity aversion present in the literature.

Remark 2. The Weak Reduction Axiom is logically independent of the famous Uncertainty Aversion Axiom of Gilboa-Schmeidler (1989) and Schmeidler (1989), which asks for the quasi-concavity of a preference relation on \mathcal{F} . However, it should be noted that the latter axiom is not very suitable for studying the ambiguity aversion that may be present in an incomplete preference relation. Indeed, “defining” uncertainty aversion through this axiom would force one to declare the Knightian Uncertainty model as *uncertainty neutral*, which seems, at least intuitively, an awkward conclusion.

By contrast, the Weak Reduction Axiom works well with incomplete preferences that may or may not satisfy the Independence Axiom. Indeed, as we have seen in Example 4, for a preference relation \succsim on \mathcal{F} that satisfies the Independence Axiom, we may have $f^\alpha \succ f$ for some α . This is true, in particular, for the Knightian Uncertainty model. \square

Remark 3. A useful method of defining the notion of ambiguity aversion is by using the comparative ambiguity aversion theory developed by Epstein (1999) and Ghirardato and Marinacci (2002). We adopt here a reflection of the definition deduced by the latter paper, and call a preference relation \succsim on \mathcal{F} **ambiguity averse** if there exists a complete preference relation \succsim^* on \mathcal{F} that admits a single-prior expected multi-utility representation, that coincides with \succsim on constant acts, and that satisfies the condition:

$$p\mathbf{1}_\Omega \succsim^* f \quad \text{implies} \quad p\mathbf{1}_\Omega \succsim f \quad (7)$$

for any $p \in \Delta(X)$ and $f \in \mathcal{F}$. The idea here is that any preference relation \succsim^* on \mathcal{F} that admits a single-prior expected multi-utility representation can reasonably be viewed as *ambiguity neutral*. After all, a decision-maker with such a preference relation assigns precise subjective probabilities to every state of the world. (It is also checked readily that any such preference relation is probabilistically sophisticated in the sense of Machina and Schmeidler (1992).)¹⁴

Consequently, it makes sense to deem \succsim as *ambiguity averse* if (i) the risk preferences induced by \succsim and \succsim^* are the same (so that the difference between the two relations stem from differences in attitudes toward uncertainty, not risk) and (ii) if \succsim decides in favor of a constant (hence unambiguous) act whenever so does \succsim^* (so that \succsim never prefers uncertainty over risk unless so does the ambiguity neutral relation \succsim^*).¹⁵

We note that any ambiguity averse preference relation \succsim on \mathcal{F} , whether complete or incomplete, satisfies the Weak Reduction Axiom, giving credence to viewing the latter property as a fairly weak (local) uncertainty aversion axiom. In fact, if \succsim is ambiguity averse, then it satisfies the *global* version of the latter axiom, that is, in that case, we can find one mixture α in $\Delta(\Omega)$ such that $f^\alpha \succsim f$ holds for every f in \mathcal{F} . For, if \succsim^* is a preference relation \succsim^* on \mathcal{F} that admits a single-prior expected multi-utility representation, then there exists a belief μ in $\Delta(\Omega)$ such that $f^\mu \sim^* f$ for every $f \in \mathcal{F}$. But then, if (7) holds for every $p \in \Delta(X)$ and $f \in \mathcal{F}$, it follows that $f^\mu \succsim f$ for every $f \in \mathcal{F}$, as we sought. \square

¹⁴One may wish to choose another benchmark for ambiguity neutral preferences here. For instance, one may ask \succsim^* to be complete, and hence, admit a representation in the sense of Anscombe-Aumann (as in Ghirardato and Marinacci (2002)). At the other extreme, one can ask \succsim^* to only be probabilistically sophisticated (as in Epstein (1999)) or satisfy the (slightly stronger) property that there exists a probability distribution μ on Ω such that $f^\mu \sim f$ for every act f in \mathcal{F} . For the purposes of this paper, it is irrelevant which of these benchmarks are used to define ambiguity neutrality, and hence, ambiguity aversion. The resulting notion of ambiguity aversion would always imply the Weak Reduction Axiom, and Theorem 5 below would remain intact with any one of these notions.

¹⁵It may be worth noting that our definition of ambiguity aversion reduces to that given by Ghirardato and Marinacci (2002) for complete preference relations. Indeed, if \succsim is complete and ambiguity averse (in the sense just described), then \succsim^* must be C-complete (because \succsim and \succsim^* behave identically when comparing constant acts). But it is easy to see that any C-complete preference relation on \mathcal{F} that admits a single-prior expected multi-utility representation must be representable in the sense of Anscombe-Aumann.

7.3 The Main Theorem

The discussion above suggests that the Weak Reduction Axiom is a curious property that is based on some “local” form of the principle of ambiguity aversion. This property may or may not be satisfied by preference relations that admit a multi-prior expected multi-utility representation. In particular, both Bewley’s model and its dual, the single-prior expected multi-utility model, are compatible with this property. Rather surprisingly, we find here that the converse of this is also true. Among the preferences that admit multi-prior expected multi-utility representations, these two models are the only ones that satisfy the Weak Reduction Axiom. Even more surprisingly, it turns out that these two models are the only ones with this property within the entire class of preference relations on \mathcal{F} that admit additively separable expected multi-utility representations.

Theorem 3. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity, and Weak Reduction Axioms if, and only if, \succsim admits either

- (i) a multi-prior expected utility representation; or
- (ii) a single-prior expected multi-utility representation.

This result is the main finding of the present paper. It shows that, among the vast set of preference relations that satisfy the Independence and Continuity Axioms – recall that it is often not possible to represent such a preference relation by means of state-independent utility functions (cf. Theorem 1) – the only ones that satisfy the Weak Reduction Axiom are those that arise in the Knightian Uncertainty model and its dual. Behaviorally speaking, this means that, among such preferences, compatibility with the Weak Reduction Axiom brings a considerable restraint on the type of indecisiveness an agent may display. Such a preference relation may exhibit a (potential) indecisiveness in beliefs but this must be coupled with decisiveness in tastes. Or, alternatively, such a preference relation may display a (potential) indecisiveness in tastes but then this must be coupled with decisiveness in beliefs. Of course, the model allows for both of these two possibilities to hold simultaneously, in which case it reduces to the classical Anscombe-Aumann model.

The following two sections are devoted to some applications of Theorem 3.

8. Application: Characterization of Indecisiveness in Tastes

We have noted earlier that there is no known axiomatization of the single-prior expected multi-utility model. Theorem 3 provides an easy solution to this problem. After all, we know that this model satisfies the Weak Reduction Axiom trivially, while, for some acts f in \mathcal{F} , we may have to have $f^\alpha \succ f$ for every α in $\Delta(\Omega)$ within Bewley’s model. Consequently, the following property is bound to yield the model we are after when appended to Theorem 3.

The Reduction Axiom. For any act f in \mathcal{F} , there is an $\alpha \in \Delta(\Omega)$ such that

$$f^\alpha \sim f.$$

The interpretation of this axiom is not different than that of the Weak Reduction Axiom. It says that when the preferences of an agent satisfy this property, then that person can always reduce subjective uncertainty to objective uncertainty by mixing the outcomes of a given act suitably. (Note that, again, this is a “local” property in the sense that the nature of the “mixing” needed here may very well depend on the act that one is interested in.)

We may now easily characterize the aforementioned “dual” of the Knightian Uncertainty model:

Theorem 4. A preference relation \succsim on \mathcal{F} satisfies the Independence, Continuity, and Reduction Axioms if, and only if, \succsim admits a single-prior expected multi-utility representation. Furthermore, provided that $\succsim \neq \mathcal{F} \times \mathcal{F}$, for any two single-prior expected multi-utilities (μ, \mathcal{U}) and (ν, \mathcal{V}) for \succsim , we have $\mu = \nu$ and $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$.

Proof. We only need to prove the “only if” part of our assertion. To this end, let \succsim be a preference relation on \mathcal{F} that satisfies the Independence, Continuity, and Reduction Axioms. By Theorem 3, \succsim satisfies the Monotonicity Axiom, for it admits either a multi-prior expected utility representation or a single-prior expected multi-utility representation. Of course, in the former case, \succsim satisfies the C-Completeness Axiom as well. But any preference relation \succsim on \mathcal{F} that satisfies both the C-Completeness and Reduction Axioms is, obviously, complete. Therefore, either \succsim is complete (and the Anscombe-Aumann Expected Utility Theorem applies) or \succsim admits a single-prior expected multi-utility representation.

To prove the uniqueness part of the assertion, let (μ, \mathcal{U}) and (ν, \mathcal{V}) be two single-prior expected multi-utilities for \succsim , and assume $\succsim \neq \mathcal{F} \times \mathcal{F}$. By the uniqueness of expected multi-utility representations – see Lemma 1 in the Appendix – we readily have $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$. Since $\succsim \neq \mathcal{F} \times \mathcal{F}$, we can find two lotteries p and q in $\Delta(X)$ and a map v in \mathcal{V} such that $\mathbf{E}(v, p) > \mathbf{E}(v, q)$. Now take an arbitrary ω in Ω , and consider the acts

$$f := p\mathbf{1}_{\{\omega\}} + q\mathbf{1}_{\Omega \setminus \{\omega\}} \quad \text{and} \quad f^\mu := (\mu\{\omega\}p + (1 - \mu\{\omega\})q)\mathbf{1}_\Omega.$$

Then, for any u in \mathcal{U} , we have

$$\sum_{\tau \in \Omega} \mu\{\tau\} \mathbf{E}(u, f(\tau)) = \sum_{\tau \in \Omega} \mu\{\tau\} \mathbf{E}(u, f^\mu(\tau)),$$

whence $f \sim f^\mu$. But this implies that

$$\sum_{\tau \in \Omega} \nu\{\tau\} \mathbf{E}(v, f(\tau)) = \sum_{\tau \in \Omega} \nu\{\tau\} \mathbf{E}(v, f^\mu(\tau)),$$

which can be true only if $\nu\{\omega\} = \mu\{\omega\}$. Since ω was chosen arbitrarily in Ω , therefore, we may conclude that $\mu = \nu$. ■

Apparently, then, the Independence Axiom is powerful enough to extend the local nature of the Reduction Axiom to a global one. Informally speaking, this is the reason behind guaranteeing the presence of a unique prior in the representation.

9. Application: A Characterization of the Anscombe-Aumann Model

As Theorem 4 illustrates, Theorem 3 is likely to be a useful tool in decision theory. As a final order of business in this paper, we shall provide another illustration of this, and use Theorem 3 to build an unexpected link between the completeness and ambiguity aversion of a preference relation. Put precisely, we wish to drop both the Completeness and Monotonicity Axioms from the axiomatic framework of the Anscombe-Aumann model, and instead, posit the C-completeness and ambiguity aversion of the preference relation in question (in the sense of Ghirardato and Marinacci (2002)) to arrive at the very same decision-making model. In other words:

Theorem 5. A preference relation \succsim on \mathcal{F} satisfies the Independence, C-Completeness, and Continuity Axioms, and it is ambiguity averse, if, and only if, \succsim admits a representation in the sense of Anscombe-Aumann.

Proof. We only need to prove the “only if” part of our assertion. To this end, let \succsim be an ambiguity averse preference relation on \mathcal{F} that satisfies the Independence, C-Completeness, and Continuity Axioms. As we have noted in Remark 3, the ambiguity aversion of \succsim implies that there exists a probability distribution μ on $\Delta(\Omega)$ such that

$$f^\mu \succsim f \quad \text{for every } f \in \mathcal{F}. \quad (8)$$

In particular, \succsim satisfies the Weak Reduction Axiom, so we may apply Theorem 3 to conclude that \succsim admits either a multi-prior expected utility representation or a single-prior expected multi-utility representation. As \succsim satisfies the C-Completeness Axiom, therefore, \succsim admits a multi-prior expected utility representation.

Let \mathcal{M} be a nonempty subset of $\Delta(\Omega)$ and u a map in $\mathbf{C}(X)$ such that, for every acts f and g in \mathcal{F} , we have $f \succsim g$ iff (6) holds. If u is constant, then \succsim is complete, so there is nothing to prove. We assume, then, that $u(x) > u(y)$ for some $x, y \in X$. To complete the proof, we need to show that \mathcal{M} is a singleton. We will in fact show that $\mathcal{M} = \{\mu\}$. To derive a contradiction, suppose this is not the case. Then, we can find a probability distribution

$\sigma \in \mathcal{M}$ and a state $\tau \in \Omega$ such that $\sigma\{\tau\} > \mu\{\tau\}$. Let f be the act that yields x with probability one at state τ , and y with probability one in any other state. Then,

$$\begin{aligned} \sum_{\omega \in \Omega} \sigma\{\omega\} \mathbf{E}(u, f^\mu(\omega)) &= \sum_{\omega \in \Omega} \mu\{\omega\} \mathbf{E}(u, f(\omega)) \\ &= \mu\{\tau\}u(x) + (1 - \mu\{\tau\})u(y) \\ &< \sigma\{\tau\}u(x) + (1 - \sigma\{\tau\})u(y) \\ &= \sum_{\omega \in \Omega} \sigma\{\omega\} \mathbf{E}(u, f(\omega)), \end{aligned}$$

which, in view of the representation of \succsim , means that $f^\mu \succsim f$ is false. This contradicts (8), and completes the proof. ■

While we report this result mainly to highlight how our main finding, Theorem 3, can be utilized in practice, Theorem 5 is of interest in its own right. After all, this result shows that, in the presence of the Independence and Continuity Axioms, the properties of C-completeness and ambiguity aversion imply both the monotonicity and completeness of a preference relation, thereby establishing a different perspective about the completeness of one's preferences in the context of uncertainty. In turn, this result shows that among all ambiguity averse preference relations, complete or incomplete, the only ones that satisfy the C-Completeness, Independence and Continuity Axioms are the ones that are ambiguity neutral and complete. This appears to be highly non-trivial, especially in our broader context of incomplete preferences. In particular, it entails that no preference relation that admits a representation as in the Knightian Uncertainty model can be ambiguity averse (unless it is complete to begin with).

10. Concluding Comments

This paper was devoted to the investigation of the classical Anscombe-Aumann model of decision-making under uncertainty without assuming the Completeness Axiom. We determined the exact nature of preference relations that would arise in this setup, and showed how the Weak Reduction Axiom, a weak formulation of the principle of ambiguity aversion, identifies in this setup two “dual” models, one corresponding to decisiveness in tastes and the other to decisiveness in beliefs.

The applications we considered – such as the alternative proof of Bewley's Expected Utility Theorem, or the characterization of the Anscombe-Aumann representation by using ambiguity aversion, as opposed to completeness and monotonicity – suggest that our main results, namely, Theorems 1, 2 and 3, may prove useful in decision theory. Much remains to be learned about incomplete preferences under uncertainty, however. In particular, little

is known about the “completions” of such preference relations.¹⁶ And indeed, next to nothing is known about such preferences, and their updating thereof, in the classical Savagean framework. In addition, we do not know how the theory laid out here would modify when we replace the Independence Axiom with its weaker formulations, as in the theory of variational preferences. Given the connections between ambiguity aversion and incompleteness of preferences we found here, these are among the intriguing avenues of research that should be explored in future work.

Appendix: Proofs of Main Results

Proof of Theorem 1

The “if” part of the assertion is straightforward, so we focus on its “only if” part alone. The argument for this part is a modification of the one given by Dubra, Maccheroni and Ok (2004) for their Expected Multi-Utility Theorem.

Let \succsim be a preference relation on \mathcal{F} that satisfies the Independence and Continuity Axioms. For any act h in \mathcal{F} , define the Borel probability measure \mathbf{h} on $X \times \Omega$ by

$$\mathbf{h}(S \times \{\omega\}) := \frac{1}{|\Omega|} h(\omega)(S)$$

for any $\omega \in \Omega$ and Borel subset S of X . Let

$$\mathcal{S} := \{\mathbf{h} : h \in \mathcal{F}\},$$

which is a closed and convex subset of $\Delta(X \times \Omega)$. Next, we define the binary relation \trianglerighteq on \mathcal{S} as

$$\mathbf{f} \trianglerighteq \mathbf{g} \quad \text{iff} \quad f \succsim g.$$

It is easy to see that \trianglerighteq is affine. Furthermore, \trianglerighteq is a closed subset of $\Delta(X \times \Omega)^2$. These properties of \trianglerighteq allows us to borrow Lemma 2 of Dubra, Maccheroni and Ok (2004):

Claim 1.1. The set

$$C(\trianglerighteq) := \{\lambda(\mathbf{r} - \mathbf{s}) : \lambda > 0 \text{ and } \mathbf{r} \trianglerighteq \mathbf{s}\}$$

is a convex cone in the linear space spanned by \mathcal{S} . Furthermore,

$$\mathbf{f} \trianglerighteq \mathbf{g} \quad \text{iff} \quad \mathbf{f} - \mathbf{g} \in C(\trianglerighteq)$$

for any \mathbf{f} and \mathbf{g} in \mathcal{S} .

Now let $\text{ca}(X \times \Omega)$ denote the normed linear space of signed finite Borel measures on $X \times \Omega$. While this space is normed by the total variation norm, it is isometrically isomorphic to $\mathbf{C}(X \times \Omega)^*$, because $X \times \Omega$ is compact. We use this duality to topologize $\text{ca}(X \times \Omega)$ with the weak*-topology. For concreteness, let us denote the resulting topological space by $\text{ca}(X \times \Omega)_*$. (The notation $\text{ca}(X)_*$ is similarly interpreted.) We note that, by definition, a net (μ_α) in $\text{ca}(X \times \Omega)_*$ converges to a signed finite Borel measure μ on $X \times \Omega$ iff

$$\int_{X \times \Omega} U d\mu_\alpha \rightarrow \int_{X \times \Omega} U d\mu \quad \text{for all } U \in \mathbf{C}(X \times \Omega).$$

¹⁶But, in the context of “indecisiveness in beliefs,” this issue is explored recently by Gilboa, Schmeidler, Maccheroni and Marinacci (2008) where a curious bridge between the Knightian Uncertainty and the maxmin expected utility models is founded.

We shall show next that the cone $C(\supseteq)$ is closed in $\text{ca}(X \times \Omega)_*$.

Claim 1.2. $C(\supseteq)$ is a closed subset of $\text{ca}(X \times \Omega)_*$.

Proof of Claim 1.2. The Krein-Smulian Theorem says that every sequentially weak*-closed convex set in the dual of a separable normed linear space is weak*-closed. But, since $X \times \Omega$ is compact, $\mathbf{C}(X \times \Omega)$ is separable, and $\text{ca}(X \times \Omega)$ is isometrically isomorphic to $\mathbf{C}(X \times \Omega)^*$. It follows that it is enough to show that $C(\supseteq)$ is sequentially closed in $\text{ca}(X \times \Omega)_*$ to establish Claim 1.2.

Take an arbitrary sequence (μ_m) in $C(\supseteq)$. By definition, there exist a sequence (λ_m) of positive real numbers, and sequences (\mathbf{f}^m) and (\mathbf{g}^m) in \mathcal{S} such that

$$\mu_m = \lambda_m(\mathbf{f}^m - \mathbf{g}^m) \quad \text{and} \quad \mathbf{f}^m \supseteq \mathbf{g}^m$$

for each positive integer m . Assume that (μ_m) converges in $\text{ca}(X \times \Omega)_*$. Clearly, if $\mathbf{f}^m = \mathbf{g}^m$ for infinitely many m , then (μ_m) has a subsequence that converges to the origin of $\text{ca}(X \times \Omega)$. As $C(\supseteq)$ contains the origin, we are done in this case. Let assume then that \mathbf{f}^m is distinct from \mathbf{g}^m for all but finitely many m . Without loss of generality, we assume that $\mathbf{f}^m \neq \mathbf{g}^m$ for each m .

Note that, by definition,

$$\left(\int_{X \times \Omega} U d\mu_1, \int_{X \times \Omega} U d\mu_2, \dots \right)$$

is a convergent real sequence for every U in $\mathbf{C}(X \times \Omega)$. It follows that

$$\sup \left\{ \int_{X \times \Omega} U d\mu_1, \int_{X \times \Omega} U d\mu_2, \dots \right\} < \infty \quad \text{for every } U \in \mathbf{C}(X \times \Omega).$$

By the Uniform Boundedness Principle, therefore, $\sup\{\|\mu_1\|, \|\mu_2\|, \dots\} < \infty$, that is, there exists a real number K such that

$$\|\lambda_m(\mathbf{f}^m - \mathbf{g}^m)\| \leq K \tag{9}$$

for every m , where $\|\cdot\|$ is the total variation norm. On the other hand, by the Jordan Decomposition Theorem, for each positive integer m and state ω in Ω , there exist two mutually singular Borel probability measures $t^m(\omega)$ and $s^m(\omega)$ and a positive real number $\alpha_m(\omega)$ such that

$$f^m(\omega) - g^m(\omega) = \alpha_m(\omega)(t^m(\omega) - s^m(\omega)).$$

Letting

$$\alpha_m := \max\{\alpha_m(\omega) : \omega \in \Omega\}$$

and

$$r^m(\omega) := \frac{\alpha_m(\omega)}{\alpha_m} t^m(\omega) + \left(1 - \frac{\alpha_m(\omega)}{\alpha_m}\right) s^m(\omega),$$

we have

$$f^m(\omega) - g^m(\omega) = \alpha_m(r^m(\omega) - s^m(\omega))$$

for each m and ω . Furthermore, for at least one state ω in Ω , the probability measures $r^m(\omega)$ and $s^m(\omega)$ are mutually singular. Therefore, we have

$$\mathbf{f}^m - \mathbf{g}^m = \alpha_m(\mathbf{r}^m - \mathbf{s}^m) \quad \text{and} \quad \|\mathbf{r}^m - \mathbf{s}^m\| \geq \frac{2}{|\Omega|},$$

and hence,

$$\begin{aligned} \|\lambda_m(\mathbf{f}^m - \mathbf{g}^m)\| &= \|\lambda_m \alpha_m(\mathbf{r}^m - \mathbf{s}^m)\| \\ &= \lambda_m \alpha_m \|\mathbf{r}^m - \mathbf{s}^m\| \\ &\geq \frac{2\lambda_m \alpha_m}{|\Omega|} \end{aligned}$$

for every m . Combining this observation with (9) shows that $(\lambda_m \alpha_m)$ is a bounded real sequence. Thus, there exist a strictly increasing sequence (m_k) of positive integers such that $(\lambda_{m_k} \alpha_{m_k})$ converges. Since X is compact, Prokhorov's Theorem entails that $\Delta(X)$ is a compact subset of $\text{ca}(X)$ relative to the topology of weak convergence, and hence, for every state ω , both $r^{m_k}(\omega)$ and $s^{m_k}(\omega)$ must have convergent subsequences in $\text{ca}(X)_*$. Passing to these subsequences consecutively, we find a strictly increasing sequence (n_k) of positive integers such that, for each state ω , $(\lambda_{n_k} \alpha_{n_k})$, $(r^{n_k}(\omega))$ and $(s^{n_k}(\omega))$ converge, the latter two in $\text{ca}(X)_*$, to λ , $r(\omega)$ and $s(\omega)$, respectively. Obviously, $\lambda \geq 0$ and r and s are Borel probability measures on X . Moreover, for any $U \in \mathbf{C}(X \times \Omega)$, we have

$$\begin{aligned} \int_{X \times \Omega} U d\mathbf{r}^{n_k} &= \sum_{\omega \in \Omega} \int_{X \times \{\omega\}} U d\mathbf{r}^{n_k} \\ &= \sum_{\omega \in \Omega} \frac{1}{|\Omega|} \int_X U(\cdot, \omega) dr^{n_k}(\omega) \\ &\rightarrow \sum_{\omega \in \Omega} \frac{1}{|\Omega|} \int_X U(\cdot, \omega) dr(\omega) \\ &= \int_{X \times \Omega} U d\mathbf{r}, \end{aligned}$$

and hence $\mathbf{r}^{n_k} \rightarrow \mathbf{r}$ in $\text{ca}(X)_*$, and similarly, $\mathbf{s}^{n_k} \rightarrow \mathbf{s}$ in $\text{ca}(X)_*$. Consequently,

$$\mu_{m_k} = \lambda_{n_k} (\mathbf{f}^{n_k} - \mathbf{g}^{n_k}) = \lambda_{n_k} \alpha_{n_k} (\mathbf{r}^{n_k} - \mathbf{s}^{n_k}) \rightarrow \lambda(\mathbf{r} - \mathbf{s})$$

as $k \rightarrow \infty$. But since $\mathbf{f}^{n_k} \succeq \mathbf{g}^{n_k}$ and $\mathbf{r}^{n_k} - \mathbf{s}^{n_k} = \frac{1}{\alpha_{n_k}} (\mathbf{f}^{n_k} - \mathbf{g}^{n_k})$, Claim 1.1 implies that $\mathbf{r}^{n_k} \succeq \mathbf{s}^{n_k}$ for each k . As \succeq is a closed subset of $\Delta(X \times \Omega)^2$, therefore, we have $\mathbf{r} \succeq \mathbf{s}$. Thus $\lambda(\mathbf{r} - \mathbf{s}) \in C(\succeq)$, that is, (μ_{n_k}) converges to a point in $C(\succeq)$ relative to $\text{ca}(X \times \Omega)_*$. Since every subsequence of a convergent subsequence converges to the limit of the mother sequence, we are done. \parallel

Given that $C(\succeq)$ is a closed convex cone in the locally convex topological linear space $\text{ca}(X \times \Omega)_*$, we may apply the Separating Hyperplane Theorem to conclude that $C(\succeq)$ equals the intersection of all closed halfspaces in $\text{ca}(X \times \Omega)_*$ that contains $C(\succeq)$ and that goes through the origin. Consequently, there exist a set \mathcal{L} of continuous linear functionals on $\text{ca}(X \times \Omega)_*$ and a real map σ on \mathcal{L} such that, for any \mathbf{f} and \mathbf{g} in \mathcal{S} , we have

$$\mathbf{f} \succeq \mathbf{g} \quad \text{iff} \quad L(\mathbf{f} - \mathbf{g}) \geq 0 \text{ for every } L \in \mathcal{L}$$

that is,

$$\mathbf{f} \succeq \mathbf{g} \quad \text{iff} \quad L(\mathbf{f}) \geq L(\mathbf{g}) \text{ for every } L \in \mathcal{L}. \quad (10)$$

Now, fix an L in \mathcal{L} arbitrarily, and define the binary relation \succsim_L on \mathcal{F} as follows:

$$f \succsim_L g \quad \text{iff} \quad L(\mathbf{f}) \geq L(\mathbf{g}). \quad (11)$$

It is easy to see that the linearity and continuity of L ensures that \succsim_L satisfies the Independence and Continuity Axioms. Since \succsim_L is complete, therefore, we may apply the classical State-Dependent Expected Utility Theorem to conclude that there exists a map $U_L \in \mathbf{C}(X \times \Omega)$ such that, for every f and g in \mathcal{F} ,

$$f \succsim_L g \quad \text{iff} \quad \sum_{\omega \in \Omega} \int_X U_L(\cdot, \omega) df \geq \sum_{\omega \in \Omega} \int_X U_L(\cdot, \omega) dg.$$

Combining this representation with (10) and (11), invoking Claim 1.1, and letting $\mathcal{U} := \{U_L : L \in \mathcal{L}\}$, completes our proof.

Proof of Theorem 2

We shall use the following notation in what follows.

Notation. Let X be a metric space and \mathcal{U} a nonempty set in $\mathbf{C}(X)$. For any p and q in $\Delta(X)$, by the expression

$$\int_X \mathcal{U} dp \geq \int_X \mathcal{U} dq,$$

we mean

$$\int_X u dp \geq \int_X u dq \quad \text{for every } u \in \mathcal{U}.$$

The argument for all three versions of Theorem 2 relies on the following uniqueness theorem, which is a consequence of the Separating Hyperplane Theorem. A more general version of this result was proved by Dubra, Maccheroni and Ok (2004), so we omit its proof here.

Lemma 1. Let X be a compact metric space. Two nonempty sets \mathcal{U} and \mathcal{V} in $\mathbf{C}(X)$ satisfy, for each p and q in $\Delta(X)$,

$$\int_X \mathcal{U} dp \geq \int_X \mathcal{U} dq \quad \text{iff} \quad \int_X \mathcal{V} dp \geq \int_X \mathcal{V} dq$$

if, and only if, $\langle \mathcal{U} \rangle = \langle \mathcal{V} \rangle$.

We are now ready to prove Theorem 2. Let \succsim be a preference relation on \mathcal{F} , and assume that there exists an additively separable expected multi-utility \mathcal{U} for \succsim .

Proof of Theorem 2.a. Fix any act $f \in \mathcal{F}$, and, for any state ω in Ω , define the binary relation \succsim_ω on $\Delta(X)$ by

$$p \succsim_\omega q \quad \text{iff} \quad p\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{X \setminus \{\omega\}} \succsim q\mathbf{1}_{\{\omega\}} + f\mathbf{1}_{X \setminus \{\omega\}}.$$

Clearly, \succsim satisfies the Separability Axiom iff $\succsim_\omega = \succsim_\tau$ for every $\omega, \tau \in \Omega$. By (2) and Lemma 1, therefore, \succsim satisfies the Separability Axiom iff $\langle \mathcal{U}_\omega \rangle = \langle \mathcal{U}_\tau \rangle$ for every $\omega, \tau \in \Omega$. In view of Theorem 1, this establishes Theorem 2.a.

Proof of Theorem 2.b. Assume first that $\mathcal{U}_\omega \subseteq \langle \mathcal{U}_\Omega \rangle$ for every state ω in Ω . We wish to show that \succsim satisfies the Monotonicity Axiom. To this end, take any two acts f and g in \mathcal{F} with $f(\omega)\mathbf{1}_\Omega \succsim g(\omega)\mathbf{1}_\Omega$ for every $\omega \in \Omega$. By (2), then,

$$\sum_{\tau \in \Omega} \int_X U(\cdot, \tau) df(\omega) \geq \sum_{\tau \in \Omega} \int_X U(\cdot, \tau) dg(\omega) \quad \text{for all } U \in \mathcal{U},$$

and hence,

$$\int_X \sum_{\tau \in \Omega} U(\cdot, \tau) df(\omega) \geq \int_X \sum_{\tau \in \Omega} U(\cdot, \tau) dg(\omega) \quad \text{for all } U \in \mathcal{U},$$

that is, by Lemma 1,

$$\int_X \langle \mathcal{U}_\Omega \rangle df(\omega) \geq \int_X \langle \mathcal{U}_\Omega \rangle dg(\omega),$$

for every $\omega \in \Omega$. Since $\mathcal{U}_\omega \subseteq \langle \mathcal{U}_\Omega \rangle$ for each ω in Ω , then,

$$\int_X \mathcal{U}_\omega df(\omega) \geq \int_X \mathcal{U}_\omega dg(\omega) \quad \text{for every } \omega \in \Omega.$$

Summing over ω , we find

$$\sum_{\omega \in \Omega} \int_X U(\cdot, \omega) df(\omega) \geq \sum_{\omega \in \Omega} \int_X U(\cdot, \omega) dg(\omega) \quad \text{for all } U \in \mathcal{U},$$

that is, $f \succsim g$, as we sought.

Conversely, suppose \succsim satisfies the Monotonicity Axiom. To derive a contradiction, we assume that there exist a state τ in Ω and a map $V \in \mathcal{U}$ such that $V(\cdot, \tau)$ lies outside $\langle \mathcal{U}_\Omega \rangle$. As a closed convex cone in a normed linear space can be strictly separated from a point in its exterior by a closed hyperplane that passes through the origin, there exists a nonzero continuous linear functional L on $\mathbf{C}(X)$ such that

$$L(V(\cdot, \tau)) > 0 \geq L(u) \quad \text{for all } u \in \langle \mathcal{U}_\Omega \rangle.$$

By the Riesz-Radon Representation Theorem, then, there exists a nonzero (finite) signed measure μ in $\text{ca}(X)$ such that

$$\int_X V(\cdot, \tau) d\mu > 0 \geq \int_X u d\mu \quad \text{for all } u \in \langle \mathcal{U}_\Omega \rangle.$$

As $\langle \mathcal{U}_\Omega \rangle$ contains all real constant functions on X , the latter set of inequalities may hold only if $\mu(X) = 0$. Therefore, $\mu = \mu^+ - \mu^-$ for some finite Borel measures μ^+ and μ^- on X with $c := \mu^+(X) = \mu^-(X)$. Since μ is nonzero, we have $c > 0$. Thus, $p := \mu^+/c$ and $q := \mu^-/c$ belong to $\Delta(X)$, while we have

$$\int_X V(\cdot, \tau) dp > \int_X V(\cdot, \tau) dq \tag{12}$$

and

$$\sum_{\omega \in \Omega} \int_X U(\cdot, \omega) dq \geq \sum_{\omega \in \Omega} \int_X U(\cdot, \omega) dp \quad \text{for all } U \in \mathcal{U}. \tag{13}$$

Clearly, (13) implies that $q\mathbf{1}_\Omega \succsim p\mathbf{1}_\Omega$. So, if we pick any act h in \mathcal{F} , and define $f := q\mathbf{1}_{\{\tau\}} + h\mathbf{1}_{\Omega \setminus \{\tau\}}$ and $g := p\mathbf{1}_{\{\tau\}} + h\mathbf{1}_{\Omega \setminus \{\tau\}}$, the Monotonicity Axiom ensures that $f \succsim g$. In particular, as V is in \mathcal{U} ,

$$\sum_{\omega \in \Omega} \int_X V(\cdot, \omega) df(\omega) \geq \sum_{\omega \in \Omega} \int_X V(\cdot, \omega) dg(\omega).$$

But, by definition of f and g , this inequality is equivalent to say that

$$\int_X V(\cdot, \tau) dq \geq \int_X V(\cdot, \tau) dp,$$

contradicting (12).

Proof of Theorem 2.c. Given Theorems 2.a and 2.b, we only need to show that if \succsim satisfies both the Separability and Monotonicity Axioms, then $\langle \mathcal{U}_\Omega \rangle \subseteq \langle \mathcal{U}_\omega \rangle$ for some (and hence all) ω in Ω . Fix, then, any $\omega \in \Omega$. Since, by Theorem 2.a, $\langle \mathcal{U}_\tau \rangle = \langle \mathcal{U}_\omega \rangle$ for every τ and ω in Ω , we have

$$\mathcal{U}_\Omega \subseteq \sum_{\tau \in \Omega} \mathcal{U}_\tau \subseteq \sum_{\tau \in \Omega} \langle \mathcal{U}_\tau \rangle = \langle \mathcal{U}_\omega \rangle,$$

and hence $\langle \mathcal{U}_\Omega \rangle \subseteq \langle \mathcal{U}_\omega \rangle$, as we sought.

Proof of Theorem 3

The “if” part of the assertion was already proved in Section 7, so we concentrate here on its “only if” part alone. To this end, let \succsim be a preference relation on \mathcal{F} that satisfies the Independence, Continuity, and Weak Reduction Axioms. We assume in what follows that $\succsim \neq \mathcal{F} \times \mathcal{F}$, for otherwise the assertion is trivially true.

We begin by applying Theorem 1 to find an additively separable expected multi-utility \mathcal{U} for \succsim . Since the convex hull of \mathcal{U} is also an additively separable expected multi-utility for \succsim , we may assume that \mathcal{U} is a convex subset of $\mathbf{C}(X \times \Omega)$. Moreover, since $\succsim \neq \mathcal{F} \times \mathcal{F}$, there is at least one function U in \mathcal{U} such that $U(\cdot, \omega_0)$ is not constant for some state ω_0 in Ω . The following claim applies to any such function U and state ω_0 (which depends on U).

Claim 3.1. For every ω in Ω , there exist real numbers $\alpha\{\omega\}$ and $\beta(\omega)$ such that $\alpha\{\omega\} \geq 0$ and

$$U(\cdot, \omega) = \alpha\{\omega\}U(\cdot, \omega_0) + \beta(\omega).$$

Proof of Claim 3.1. Suppose this claim is false, that is, there exists a state τ in Ω such that the map $U(\cdot, \tau)$ is distinct from the map $\alpha U(\cdot, \omega_0) + \beta$ for any (α, β) in $\mathbb{R}_+ \times \mathbb{R}$. This means that $U(\cdot, \tau)$ lies outside the cone C generated by $U(\cdot, \omega_0)$ and all constant real functions on X . We may thus apply the Separating Hyperplane Theorem to find a nonzero continuous linear functional L on $\mathbf{C}(X)$ such that

$$L(U(\cdot, \tau)) > 0 \geq L(u) \quad \text{for all } u \in C.$$

Similarly, we can separate $U(\cdot, \omega_0)$ from the set of all real constant functions to find a nonzero continuous linear functional K on $\mathbf{C}(X)$ such that

$$K(U(\cdot, \omega_0)) < 0.$$

Therefore, for any real number λ in $(0, 1)$ close enough to 1, we have

$$(\lambda L + (1 - \lambda)K)(U(\cdot, \tau)) > 0 \geq (\lambda L + (1 - \lambda)K)(U(\cdot, \omega_0)).$$

Using the Riesz-Radon Representation Theorem and proceeding as in the proof of the ‘‘only if’’ part of Theorem 2.b, we then obtain two Borel probability measures on X such that

$$\int_X U(\cdot, \tau) dp > \int_X U(\cdot, \tau) dq \quad \text{and} \quad \int_X U(\cdot, \omega_0) dq > \int_X U(\cdot, \omega_0) dp. \quad (14)$$

Now define $f \in \mathcal{F}$ as

$$f(\omega) := p \quad \text{if} \quad \int_X U(\cdot, \omega) dp \geq \int_X U(\cdot, \omega) dq,$$

$f(\omega) := q$ otherwise. In turn, for an arbitrarily fixed probability distribution $\alpha \in \Delta(\Omega)$, define

$$f^\alpha := \left(\sum_{\omega \in \Omega} \alpha\{\omega\} f(\omega) \right) \mathbf{1}_\Omega.$$

Then, we have

$$\int_X U(\cdot, \omega) df(\omega) \geq \int_X U(\cdot, \omega) df^\alpha(\omega) \quad \text{for each } \omega \in \Omega.$$

Furthermore, by (14), this inequality must be strict for at least one of the states τ and ω_0 . It follows that

$$\sum_{\omega \in \Omega} \int_X U(\cdot, \omega) df(\omega) > \sum_{\omega \in \Omega} \int_X U(\cdot, \omega) df^\alpha(\omega),$$

whence, $f \succ f^\alpha$. But, as α was arbitrarily chosen in $\Delta(\Omega)$, this contradicts the hypothesis that \succsim satisfies the Weak Reduction Axiom.

The following is the main conclusion we wish to derive from Claim 3.1.

Claim 3.2. For every U in \mathcal{U} , there exist a probability distribution $\mu_U \in \Delta(\Omega)$ and maps $u_U \in \mathbf{C}(X)$ and $\beta_U \in \mathbb{R}^\Omega$ such that

$$U(\cdot, \omega) = \mu_U(\omega)u_U + \beta_U(\omega) \quad \text{for each } \omega \in \Omega.$$

Proof of Claim 3.2. Pick any U in \mathcal{U} . If $U(\cdot, \omega)$ is constant for each $\omega \in \Omega$, there is nothing to prove. If $U(\cdot, \omega_0)$ is not constant for some $\omega_0 \in \Omega$, then we use Claim 3.1 to find two maps $\alpha_U \in \mathbb{R}_+^\Omega$ and $\beta_U \in \mathbb{R}^\Omega$ such that

$$U(\cdot, \omega) = \alpha\{\omega\}U(\cdot, \omega_0) + \beta(\omega) \quad \text{for each } \omega \in \Omega.$$

Since $U(\cdot, \omega_0)$ is not constant, $\alpha\{\omega_0\} > 0$, so we may define the probability distribution $\mu_U \in \Delta(\Omega)$ by $\mu_U(\omega) := \alpha\{\omega\} / \sum_{\tau \in \Omega} \alpha\{\tau\}$. Then, setting

$$u_U := \left(\sum_{\tau \in \Omega} \alpha\{\tau\} \right) U(\cdot, \omega_0)$$

completes the proof of Claim 3.2.

We now define the map $\Lambda : \mathcal{U} \rightarrow \Delta(\Omega) \times \mathbf{C}(X)$ by

$$\Lambda(U) := (\mu_U, u_U),$$

where (μ_U, u_U) is an arbitrarily chosen pair from those found in Claim 3.2 for U . By the Axiom of Choice, Λ is well-defined. Furthermore, since \mathcal{U} is an additively separable expected multi-utility for \succsim , Claim 3.2 ensures that $\mathcal{H} := \Lambda(\mathcal{U})$ is a multi-prior expected multi-utility for \succsim .

The following property of \mathcal{H} is essential for our purposes.

Claim 3.3. For any two pairs (μ, u) and (ν, v) in $\Lambda(\mathcal{U})$, and any real number λ in $[0, 1]$, there exist a pair $(\sigma, w) \in \Delta(\Omega) \times \mathbf{C}(X)$ and $\beta \in \mathbb{R}^\Omega$ such that

$$\lambda\mu\{\omega\}u + (1 - \lambda)\nu\{\omega\}v = \sigma\{\omega\}w + \beta(\omega) \quad \text{for each } \omega \in \Omega.$$

Proof of Claim 3.3. Pick any two pairs (μ, u) and (ν, v) in $\Lambda(\mathcal{U})$. Then, by definition of Λ , there exist maps U and V in \mathcal{U} such that

$$U(\cdot, \omega) = \mu\{\omega\}u + \beta_U(\omega) \quad \text{and} \quad V(\cdot, \omega) = \nu\{\omega\}v + \beta_V(\omega)$$

for any state ω in Ω . Fix, arbitrarily, a real number λ between 0 and 1, and consider the real map W on $X \times \Omega$ defined by

$$W(\cdot, \omega) := \lambda\mu\{\omega\}u + (1 - \lambda)\nu\{\omega\}v \quad \text{for each } \omega \in \Omega.$$

Then,

$$W(\cdot, \omega) = \lambda(U(\cdot, \omega) - \beta_U(\omega)) + (1 - \lambda)(V(\cdot, \omega) - \beta_V(\omega))$$

for each $\omega \in \Omega$. Since \mathcal{U} is convex, $\lambda U + (1 - \lambda)V \in \mathcal{U}$, and hence, by Claim 3.2, there exist a probability distribution $\sigma \in \Delta(\Omega)$ and maps $w \in \mathbf{C}(X)$ and $\beta' \in \mathbb{R}^\Omega$ such that

$$\lambda U(\cdot, \omega) + (1 - \lambda)V(\cdot, \omega) = \sigma\{\omega\}w + \beta'(\omega) \quad \text{for each } \omega \in \Omega.$$

It follows that

$$W(\cdot, \omega) = \sigma\{\omega\}w + \beta'(\omega) - \lambda\beta_U(\omega) - (1 - \lambda)\beta_V(\omega)$$

for each $\omega \in \Omega$, so, setting $\beta := \beta' - \lambda\beta_U - (1 - \lambda)\beta_V$ completes the proof of Claim 3.3.

We are now fully prepared to complete the proof of Theorem 3. To derive a contradiction, assume that \succsim admits neither a multi-prior expected utility representation nor a single-prior expected multi-utility representation. Then, there must exist two pairs (μ, u) and (ν, v) in $\Lambda(\mathcal{U})$ such that

- (i) $\mu \neq \nu$;
- (ii) neither u nor v is constant; and

(iii) v is not a positive affine transformation of u .

Pick any two states, say ω_1 and ω_2 , in Ω such that

$$\mu\{\omega_1\} > \nu\{\omega_1\} \quad \text{and} \quad \nu\{\omega_2\} > \mu\{\omega_2\}, \quad (15)$$

and two lotteries, say p and q , in $\Delta(X)$ such that

$$\mathbf{E}(u, p) > \mathbf{E}(u, q) \quad \text{and} \quad \mathbf{E}(v, q) > \mathbf{E}(v, p).$$

Choose λ to be the unique real number in $[0, 1]$ with the property:

$$\mathbf{E}(\lambda u + (1 - \lambda)v, p) = \mathbf{E}(\lambda u + (1 - \lambda)v, q).$$

Obviously, by (15),

$$\mathbf{E}(\lambda\mu\{\omega_1\}u + (1 - \lambda)\nu\{\omega_1\}v, p) > \mathbf{E}(\lambda\mu\{\omega_1\}u + (1 - \lambda)\nu\{\omega_1\}v, q)$$

and

$$\mathbf{E}(\lambda\mu\{\omega_2\}u + (1 - \lambda)\nu\{\omega_2\}v, q) > \mathbf{E}(\lambda\mu\{\omega_2\}u + (1 - \lambda)\nu\{\omega_2\}v, p).$$

On the other hand, by Claim 3.3, there exist a pair $(\sigma, w) \in \Delta(\Omega) \times \mathbf{C}(X)$ and $\beta \in \mathbb{R}^\Omega$ such that

$$\lambda\mu\{\omega\}u + (1 - \lambda)\nu\{\omega\}v = \sigma\{\omega\}w + \beta(\omega) \quad \text{for each } \omega \in \Omega.$$

Setting ω to be ω_1 and ω_2 here, and substituting the resulting expression in the previous two inequalities, we find

$$\sigma\{\omega_1\}\mathbf{E}(w, p) > \sigma\{\omega_1\}\mathbf{E}(w, q)$$

and

$$\sigma\{\omega_2\}\mathbf{E}(w, q) > \sigma\{\omega_2\}\mathbf{E}(w, p)$$

which is impossible.

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