

Axioms for Minimax Regret Choice Correspondences

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Abstract

This paper unifies and extends the recent axiomatic literature on minimax regret. It compares several models of minimax regret, shows how to characterize the according choice correspondences in a unified setting, extends them to choice from convex sets, connects them by defining a behavioral notion of perceived ambiguity, and provides an axiomatization of Hannan regret. Substantively, a main idea is to behaviorally identify ambiguity with failures of independence of irrelevant alternatives. Regarding proof technique, the core contribution is to uncover a dualism between choice correspondences and preferences in an environment where this dualism is not obvious. This insight can be used to generate results by importing findings from the existing literature on preference orderings.

Keywords: Minimax regret, Hannan regret, ambiguity, multiple priors, choice correspondences.

JEL classification codes: C44, D81.

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1 Introduction

1.1 Motivation

The minimax regret decision criterion was first suggested in Savage’s (1951) reading of Wald (1950) and has since seen occasional use in statistics (DasGupta and Studden 1991, Droge 1998, 2006, Eldar et al. 2004). Interest in minimax regret has recently increased among econometricians (Brock 2006, Chamberlain 2000, Manski 2004, 2006, 2007, Hansen 2005, Stoye 2007a, 2007b, 2008) as well as economic theorists (Bergemann and Schlag 2007, 2008, Eozenou et al. 2006, Schlag 2003, 2006; substantively, some of these papers overlap with econometrics). Its decision theoretic foundations, the classic reference for which is Milnor (1954), were revisited as well, specifically in recent work by Hayashi (2008) and Stoye (2006).¹

The present paper provides further insight into axiomatic characterizations of minimax regret. It is partly motivated by the observation that, although all of the aforementioned authors talk about something they call “minimax regret,” there are significant differences between their conception of this object. I hope to clarify some discussions of regret by elaborating on these differences. Building on this, I provide a unified characterization of different “minimax regret” objects in a common framework and axiomatic system. I also give a behavioral characterization of perceived ambiguity and show how to vary the benchmark according to which regret is computed. These are this paper’s main substantive contributions. One leitmotif that connects them is the idea to identify perceived ambiguity with violations of independence of irrelevant alternatives (IIA). Insofar as IIA is the standard axiom that will not be imposed in this paper, this identification resembles Ghirardato et al.’s (2004) identification of perceived ambiguity with violations of von Neumann-Morgenstern independence. This similarity will turn out to be much more than semantic. Finally, the paper’s main technical contribution is a proof technique which recovers a dualism between choice correspondences and preference orderings in an environment where this dualism is not obvious. This allows one to adapt existing results on preferences to statements about regret-based choice correspondences.

I unify previous axiomatizations of regret along two dimensions. First, one can think of a minimax regret preference ordering or of the according choice correspondence. All applications are phrased in terms of the preference ordering, and it is this ordering that Stoye (2006) analyzes. However, this ordering is menu dependent, i.e. the ranking of acts can depend on the feasible set within which they are compared. As a result, axiomatizations of minimax regret preferences are at tension with the revealed preference paradigm. To illustrate, consider the statement $f \succ g \succ h$, where the menu in question is $\{f, g, h\}$. Choice from the menu reveals that $f \succ g$ and $f \succ h$, but not that $g \succ h$. If preferences

¹Both also do other things to which this paper is less related – Stoye (2006) by looking at more preference orderings, Hayashi (2008) by formalizing a notion of smooth (non-minimax) regret aversion.

do not depend on menus, an obvious dualism between preferences and choice correspondences resolves the problem – $g \succ h$ can be inferred from choices over $\{g, h\}$. With menu-dependent preferences, this dualism breaks down, and choice from $\{g, h\}$ will not reveal the ranking of those same acts within $\{f, g, h\}$. Indeed, the second part of $f \succ g \succ h$ need not map onto revealed preference for g over h in *any* menu. One could accordingly prefer to not axiomatize it, that is, one could restrict attention to the minimax regret choice correspondence, as is done by Hayashi (2008). The present paper adopts the choice correspondence approach, but on a deeper level, shows that the two perspectives continue to be tightly related because a somewhat different dualism can be uncovered.

Second, minimax regret can be thought of as presuming no priors, endogenous priors, or exogenous priors.

(i) No priors: Milnor (1954) and Stoye (2006) axiomatize the preference ordering represented by (the negative of)

$$MR(f, M) \equiv \max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\},$$

where f and g are acts in a menu M , \mathcal{S} is a state space, and u is an expected utility functional; I will explain notation in detail below. The idea here is that \mathcal{S} reflects the objective ambiguity inherent in a situation.

In applications, prior-less minimax regret was recently used by Bergemann and Schlag (2008), Manski (2004, 2007), Schlag (2003, 2007), and Stoye (2007a, 2007b, 2008a) and also underlies Hannan regret (to be elaborated later). For example, Manski (2004, 2007, 2008) considers the problem of treatment choice – be it assignment to on-the-job training programs or to medical treatment – as a statistical decision problem and compares the risk functions generated by different statistical treatment rules. He advocates the use of minimax regret risk as decision criterion, but certainly not the use of priors.² Indeed, as this version of minimax regret is the only one that can be interpreted without any notion of priors, it is the one that frequentist statisticians must have in mind and that corresponds to Savage’s (1951) original suggestion.

(ii) Endogenous Priors: Hayashi (2008) axiomatizes minimization of

$$MR_1(f, M) \equiv \max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi,$$

²Notation in statistics is typically somewhat different. In short, statisticians postulate a set of conceivable data generating processes and a loss function. A risk function maps any combination of true data generating process and statistical decision rule onto the implied expectation of loss. Noting that data generating processes correspond to states of the world, statistical decision rules to acts, and loss to (the negative of) utility, risk functions are seen to map onto the functional u (i.e., utility acts).

where the “set of priors” $\Gamma \subseteq \Delta\mathcal{S}$ is behavioral or “as if.” Mathematically, MR_1 generalizes MR because Γ could equal $\Delta\mathcal{S}$. This approach is in line with Gilboa and Schmeidler (1989) and the large literature that builds on them. It may be the most interesting one for a theorist since we cannot typically observe people’s (sets of) beliefs; hence, these axiomatizations are most revealing regarding a theory’s observable implications. Endogenous prior minimax regret was recently applied to the modelling of games by Renou and Schlag (2008).

(iii) Exogenous Priors: An intermediate possibility is that the representation is MR_1 but Γ is a feature of the environment. This might appear unusual to decision theorists, but is faithful to many applications. The quintessential example is the robust (multi-prior) Bayesian literature (Berger 1985), which first specifies a set of priors and hence the extent of ambiguity in a decision situation and subsequently thinks about how to make decisions. Indeed, many contributions make the first and not the second step (e.g., Wasserman and Kadane 1992), and the literature “does not as yet contain substantial work on how exactly a specific action should be chosen” (Zen and DasGupta 1993; see also Arias et al. 2003). If specification of set valued beliefs precedes the contemplation of action, a faithful model of the latter requires that the beliefs are part of the decision theoretic environment, and that beliefs revealed by choices are axiomatically linked to them. Hence, a characterization of minimax regret with exogenous priors would provide axiomatic foundations for Γ -minimax regret (Berger 1985; see Bergemann and Schlag 2007 or Chamberlain 2000 for applications in economics). Indeed, I use the symbol Γ for sets of priors to emphasize this link. Exogenous sets of priors were recently considered by Gajdos et al. (2004) and Gajdos et al. (2008), but these papers are not about regret.

To repeat, which of these possibilities appears most interesting depends on the desired application. The typical, namely descriptive and behavioral, application in economic theory will rely on case (ii). On the other hand, although this paper is firmly rooted in the revealed preference paradigm, it is partly motivated by statistical and econometric applications. Any frequentist application avoids priors and consequently, is an example of case (i). When statistical applications do use (sets of) priors, these priors should typically be thought of as exogenous; these cases, most notably Γ -minimax regret, therefore fall under case (iii). I will characterize all three cases.

1.2 Overview and Brief Summary

This paper provides and compares characterizations of minimax regret choice correspondences for all of (i)-(iii) above. The representations can be connected via a behavioral characterization of perceived ambiguity. They rely on a proof technique which puts into sharp focus the role of certain axioms, recovers a duality between choice correspondences and preference orderings, and can be extended to

generate more results. The results are new with the exception of theorem 4, which resembles a core finding in Hayashi (2008). In that theorem’s case, the main contribution lies in the new proof and the embedding of results.

An overview of the paper’s structure and brief summary of results goes as follows. Section 2 begins by describing the decision theoretic environment and stating axioms. I then establish a lemma that generates the aforementioned connection between choice correspondences and preference orderings. Characterizations of prior-less minimax regret (by importing a result of Stoye 2006) and endogenous prior minimax regret (by importing Gilboa and Schmeidler 1989) follow easily. The former result is a choice correspondence analog of Stoye’s (2006) minimax regret representation. The latter result is substantively similar to one in Hayashi (2008), so I essentially derive his representation from Gilboa and Schmeidler (1989). Similarly, one can use a link to previous work on multi-prior Pareto criteria by Bewley (2002) and Ghirardato et al. (2004) to characterize a notion of ambiguity perception. This technique also leads to an axiomatization that identifies the object Γ with an exogenous set of priors. I finally connect Hannan regret to the literature by providing its first (to my knowledge) axiomatization. The connection is established by varying the “regret benchmark,” i.e. the utility frontier that generates regret, by varying the domain of an independence of irrelevant alternatives axiom. Section 3 concludes and offers comparisons to other notions of regret in the literature.

2 Axiomatic Analysis

2.1 Preliminaries and Axioms

The setup is inspired by Anscombe and Aumann (1963). There is a set \mathcal{S} of states of the world s , endowed with an algebra Σ of events E, F , etc.; a set \mathcal{X} of outcomes x ; and a set \mathcal{F} of possible acts f, g , etc. The only restrictions on \mathcal{S} , \mathcal{X} , and Σ are that \mathcal{X} must have at least two elements, that theorem 3 requires the existence of three distinct, nonempty events, and that axiom 7* (which is not used in the main results) requires \mathcal{X} to be compact metric. An act f is a Σ -measurable, finite step function $f : \mathcal{S} \rightarrow \Delta\mathcal{X}$ that maps states s onto finite outcome distributions $f(s)$. (In this paper, $\Delta(\cdot)$ generally denotes “finite mixtures over...”) An act is *constant* if f does not depend on s . I embed $\Delta\mathcal{X}$ in \mathcal{F} by writing p^* for the constant act associated with $p \in \Delta\mathcal{X}$. Mixtures of acts are identified with statewise mixtures, i.e. $h \equiv \lambda f + (1 - \lambda)g$ is generated by performing f with probability λ and g otherwise and is characterized by $h(s) = \lambda f(s) + (1 - \lambda)g(s)$. The word “convex” henceforth denotes closure under such mixture. The decision maker can choose from a finite, nonempty menu $M \subseteq \mathcal{F}$. Randomization is not allowed; this will be relaxed later. For any menu M , $\lambda M + (1 - \lambda)g$ denotes the menu generated by replacing every $f \in M$ with the analog mixture. I do not presume existence of a preference ordering

but of a choice correspondence C that maps every M onto some nonempty $C(M) \subseteq M$. (This is a good moment to emphasize that in this paper, \subseteq and \subset are distinct symbols.) I also define the problem of choosing an act after state s has been learned. The according choice correspondence will be labelled C_s . As is standard in the literature, I impose some notion of dynamic consistency by assuming that choice after revelation of s is equivalent to choice from constant acts; more formally, $f \in C_s(M)$ iff $(f(s))^* \in C(\{(g(s))^* : g \in M\})$. I call an act f *strictly potentially optimal* in M if there exists s s.t. $C_s(M) = \{f\}$. Finally, for future use, let $\Delta\mathcal{S}$ denote the set of Σ -measurable distributions on \mathcal{S} .

The following axioms on C are maintained throughout this paper.

Axiom 1 *Nontriviality*

$$\exists M : C(M) \subset M.$$

Axiom 2 *Monotonicity*

If $f \in M$, $g \in C(M)$, and $f \in C_s(\{f, g\})$ for all s , then $f \in C(M)$.

Axiom 3 *Independence*

$$C(\lambda M + (1 - \lambda)f) = \lambda C(M) + (1 - \lambda)f.$$

Axiom 4 *Ambiguity Aversion*

$C(M)$ is convex relative to M (the intersection of M with a convex set).

Axiom 5 *Independence of Irrelevant Alternatives (IIA) for Unambiguous Choices*

Let M and N consist of constant acts, then

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Axiom 6 *Independence of Never Strictly Optimal Alternatives (INA)*

Let M and N be such that $C_s(M \cup N) \cap M \neq \emptyset$ for all s . Then

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Axiom 7 *Mixture Continuity (Archimedean Property)*

$$h \in M \setminus C(M), g \in C(M), C(M \cup \{f\}) \cap \{f, g\} = \{f\}$$

$$\implies \exists \lambda, \gamma \in (0, 1) : C(M \cup \{\lambda f + (1 - \lambda)h\}) \cap \{\lambda f + (1 - \lambda)h, g\} = \{\lambda f + (1 - \lambda)h\},$$

$$C(M \cup \{\gamma f + (1 - \gamma)h\}) \cap \{\gamma f + (1 - \gamma)h, g\} = \{g\}.$$

Axiom 7* *Upper Hemicontinuity (Sequential Continuity)*

C is upper hemicontinuous, i.e. if $f_n \rightarrow f$, $f_n \in C(M_n)$ for all n , and $M_n \rightarrow M$, then $f \in C(M)$. Here, \rightarrow denotes uniform weak convergence.

Some remarks on the axioms are in order. Monotonicity states that if the agent would choose f from $\{f, g\}$ in every state of the world, then she cannot revealed prefer g over f in any menu. It is the revealed preference equivalent of the axiom of the same name in Gilboa and Schmeidler (1989) and other references. Independence requires that choice is invariant under mixing of *entire menus* with some act. One intuition for this comes from the following thought experiment. Suppose an agent chooses from a menu, but then learns that her choice will be actualized only conditional on heads in a previous coin toss; she has no control over what will happen conditional on tails. Then it can be argued that her choice behavior should not be affected.³ The same adaptation of independence was recently used by Eliaz and Ok (2006) and Ortoleva (2008).

Axioms 5 and 6 touch upon a crucial, and controversial, feature of minimax regret, namely that it violates independence of irrelevant alternatives (IIA). This axiom would translate into the present setting as

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}, \forall M, N,$$

meaning that preferences revealed by $C(M \cup N)$ cannot contradict those revealed by $C(M)$: The set chosen from $M \cup N$ is either strictly revealed preferred to $C(M)$, in which case it cannot contain elements of the latter, or it is revealed indifferent to $C(M)$, in which case it contains all of it.⁴

In this paper, IIA will only be imposed on restricted domains. Specifically, axioms 5 and 6, as well as several axioms to come, introduce a leitmotif of this paper: Comparison of $C(M)$ and $C(M \cup N)$ may reveal violations of IIA if the ambiguity perceived in a choice problem changes as one expands M to $M \cup N$. This cannot be the case if both M and N consist of constant acts, because then there is no ambiguity to begin with; hence axiom 5. Furthermore, axiom 6 (INA) specifies that it does not happen if acts added to menus are not strictly potentially optimal. To see this, and to understand the axiom's

³The leap from the thought experiment to the axiom relies on a hidden assumption of compound roulette lottery reduction. Here as in many other references, that axiom is implicit in the notation for mixture acts; see Seo (2008) for a treatment that makes it explicit.

⁴See Arrow (1959, definition C.4). To further underscore that this is a natural formulation of IIA, compare it to Sen's (1971) consistency requirements for choice correspondences. In the present paper's notation, these can be written as follows:

$$\begin{aligned} f \in C(M \cup N) \cap M &\implies f \in C(M), \\ \{f, g\} \in C(M) &\implies C(M \cup N) \cap \{f, g\} \in \{\{f, g\}, \emptyset\}. \end{aligned}$$

The former ("property α ", "contraction consistency") can be traced back at least to Chernoff (1954). It ensures that choice correspondences cannot shrink upon contraction of sets, but allows for them to expand. The second condition ("property β ", "expansion consistency") prevents this by imposing that revealed indifferences are not contradicted upon expansion of sets. In this paper's setting, the conjunction of both is equivalent to IIA as defined here.

name, note that menus M and N fulfil the axiom's hypothesis iff enlarging M to $M \cup N$ does not add any act a that would be strictly potentially optimal in $M \cup \{f\}$. An intuition for this restriction is that the agent's attitude to one and the same outcome in different states may be influenced by what could have been achieved in a state, hence the nature of an act's ambiguity may change with this information. This intuition is obviously related to the concept of regret, and INA accordingly plays a major role in enforcing regret-based choices.⁵

Mixture continuity usually presumes that $f \succ g \succ h$ within some menu and concludes that $\lambda f + (1 - \lambda)h \succ g \succ \gamma f + (1 - \gamma)h$ for some λ, γ . This cannot be done here for reasons explained in the introduction. To see that axiom 7 is a rather literal adaptation, note the hypothesis states that f is revealed preferred to g and g is revealed preferred to h , albeit in different menus. I also present the translation into choice correspondence language of sequential continuity. The result is the same continuity axiom that Hayashi (2008) uses. As with these axioms' preference counterparts, sequential continuity strengthens mixture continuity. Mixture continuity suffices for all results in this paper, but the effect of using sequential continuity will be reported. Finally, ambiguity aversion translates an axiom proposed by Schmeidler (1989); see Milnor (1954) for a precursor. It is equivalent to Hayashi's (2008) regret aversion. Note that unlike in most other contexts, the axiom is not a weakening of independence because it would follow from the latter only in conjunction with IIA.

Consider now the following axiom.

Axiom 8 *Symmetry*

For any menu M , let $E, F \in \Sigma \setminus \{\emptyset\}$ be any two disjoint events s.t. for any $a \in M$, $f(s)$ is constant on E as well as F . Define f' by

$$f'(s) = \begin{cases} f(s)|_{s \in E}, & s \in F \\ f(s)|_{s \in F}, & s \in E \\ f(s) & \text{otherwise} \end{cases} .$$

Let the function $(\cdot)': 2^{\mathcal{A}} \rightarrow 2^{\mathcal{A}}$ map every set of acts $N \subseteq \mathcal{A}$ onto $N' \equiv \{f' : f \in N\}$. Then

$$C(M') = (C(M))'.$$

In words, symmetry states the following. Take any two events such that all acts in a menu are constant on either event, then exchanging the consequences of the events for every act in the menu does not affect choices. This is certainly not an innocuous condition – the two events might be of very different size with respect to some measure on state space. Indeed, symmetry enforces a strong

⁵Krähmer and Stone (2006) give a similar, informational motivation for regret, although their technical notion of regret is outside this paper's scope. Essentially the same axiom as INA is used in Hayashi (2008), but the original source is Milnor (1954, "special row adjunction").

attitude of ignorance regarding elements of \mathcal{S} , specifically a refusal to weight them according to some importance criterion like subjective probability. The idea that prior ignorance about events should be modelled in this way is due to Arrow and Hurwicz (1972), who specify similar axioms for choice correspondences; see also Cohen and Jaffray (1980) for a preference-based formulation that is otherwise analog to the above.

Symmetry is implausible if one has available, and wishes to consider, prior information about states, respectively if one wishes to model agents who have and use such information. On the other hand, if no prior information exists, the axiom is compelling because decisions would otherwise be sensitive to manipulations of the state space, e.g. the relabeling of states or their duplication via conditioning on trivial events (Arrow and Hurwicz 1972). These observations are as they should be, given that symmetry will turn out to characterize prior-less minimax regret. For the intermediate case of *vague* prior information – not enough to commit to a prior but sufficient to cast doubt on symmetry –, I would point to the characterization of exogenous prior minimax regret below.

I finally consider the following axiom. Say that a menu has a *state independent outcome distributions* if the set $\{f(s) : f \in M\}$ does not vary with s . In words, the set of feasible outcome lotteries is constant across states. Notice that any menu consisting of constant acts induces state independent outcome distributions, but a menu can have this property without containing any constant act. Then one might consider the following.

Axiom 9 *C-Betweenness When Outcome Distributions are State Independent*

For any act f , constant act p^ , scalar $\lambda \in (0, 1)$, and menu $M \supseteq \{p^*, f, \lambda f + (1 - \lambda)p^*\}$ with state independent outcome distributions, if $p^* \notin C(M)$ and $f \notin C(M)$, then $\lambda f + (1 - \lambda)p^* \notin C(M)$.*

C-betweenness for state independent outcome distributions is related to betweenness (Chew 1983, Dekel 1986), which states that if two acts are ranked indifferent, then they are also ranked indifferent to any mixture between them. Rewriting the axiom in terms of choice correspondences requires some tweaking. One might want to think of revealed indifference as the absence of revealed strict preference, thus acts in a menu are revealed indifferent if either both or none are chosen. Betweenness would then require that any mixture of the two acts is revealed indifferent as well. If both acts are chosen, this is already implied by ambiguity aversion, so I merely need to impose it for the case that neither option is chosen.⁶

The motivation of c-betweenness is related to the usual motivation of c-independence: It limits the scope of ambiguity aversion, that is, of preferences for mixtures. Intuitively, a decision maker

⁶Intuitively, it may seem a stretch to think of this case as revealed indifference. While preferences over non-chosen acts are not axiomatized, one of those acts might be commonsensically very unattractive. However, when that happens, one would not worry about imposing the axiom's conclusion anyway.

might strictly prefer mixtures because they constitute a hedging of bets across ambiguous states. Both independence and betweenness stipulate that this will not happen in certain cases. C-independence states that mixture with a constant act cannot constitute a hedging of bets, just like purchase of a safe asset does not hedge any risks in a conventional portfolio problem. Axiom 9 further tightens the conditions under which a hedge is denied, thus weakening the axiom. The first tightening is that the menu must have a state independent outcome distributions. The idea here is to once again acknowledge that ambiguity can arise not just because outcomes differ across states, but also because the evaluation of outcomes might depend on what could have been achieved. The restriction shuts off the latter channel: Observed outcomes are still informative about which state occurred, and hence about which act should have been chosen, but if the set of ex post feasible outcome distributions is state independent, then none of this information matters for what could have been achieved. The second tightening is that the axiom does not apply to mixture of f and some other act g with a third, constant act, but only to mixture of f with a constant act that is already ranked indifferent to it. This is why the axiom restricts betweenness and not independence. Substantively, it sharpens the focus on preferences for or against randomization. By applying for all constant acts from very bad to very good ones, c-independence additionally limits the dependence of ambiguity attitudes upon stakes.

I conclude this section by stating a straightforward result. Specifically, a subset of axioms that will be maintained throughout implies that C extends some expected utility choice correspondence \tilde{C} in the sense of agreeing with it on unambiguous choice problems.

Lemma 1 *A choice correspondence fulfils axioms 1, 3, 5, and 7 iff choices from sets of constant acts are consistent with maximization of von Neumann-Morgenstern expected utility. Thus, there exists a unique (up to affine transformation), nonconstant function $U : \mathcal{X} \mapsto \mathbb{R}$ s.t. the restriction \tilde{C} of C to menus \tilde{M} consisting of constant acts p^* is*

$$\tilde{C}(\tilde{M}) = \arg \max_{p^* \in \tilde{M}} \int U(x) dp^*.$$

If mixture continuity is strengthened to sequential continuity, then U is furthermore continuous.

2.2 Characterizations of Minimax Regret

This section is devoted to characterizing different versions of minimax regret. The key to doing this is contained in the following lemma.

Lemma 2 *A choice function fulfils axioms 1 through 7 iff it can be represented as*

$$C(M) = \arg \min_{f \in \mathcal{M}} I(r \circ (f, M)),$$

where the finite step function $r \circ (f, M) : \mathcal{S} \mapsto \mathbb{R}^+$ is defined by

$$\begin{aligned} (r \circ (f, M))(s) &\equiv \max_{g \in M} u \circ g(s) - u \circ f(s) \\ u \circ f(s) &\equiv \int U(x) df(s) \end{aligned}$$

with U as in lemma 1, where the functional I , which maps functions of type $r \circ (f, M)$ into \mathbb{R}^+ , is quasiconvex, mixture continuous, weakly monotonic ($r \geq r'$ for all s implies $I(r) \geq I(r')$), and homothetic.

The formulation of lemma 2 is modelled on lemma 3.3 in Gilboa and Schmeidler (1989), with notational differences indicating substantive ones. The lemma tightly limits the ways in which ambiguity can affect choices. Specifically, choices from ambiguous menus must reveal a preference ordering \succsim , here represented by value functional I , over objects $r \circ (f, M)$ that one might call *regret acts*.

The proof of lemma 2 contains three crucial steps, the third of which is one of this paper's main insights.

- First, lemma 1, monotonicity, and INA jointly imply that acts can be identified with *utility acts* $u \circ f : \mathcal{S} \mapsto \mathbb{R}$. This idea is standard except for the observation that it requires merely INA instead of IIA.
- Second, independence can be used to restrict attention to the set \mathcal{M}_0 of join whose meet or ex-post utility frontier, $\{\max_{g \in M} u \circ g(s)\}_{s \in \mathcal{S}}$, is everywhere zero. This insight was anticipated more than half a century ago (Chernoff 1954, theorem 2) but seems to have gone unused since; e.g., it is missing in Milnor (1954) and Borodin and El-Yaniv (1998). An intuitive justification for it is that if there exists an act h with $u \circ h(s) = -\max_{g \in M} u \circ g(s)$, then independence implies that

$$f \in C(M) \iff \frac{1}{2}f + \frac{1}{2}h \in C\left(\frac{1}{2}M + \frac{1}{2}h\right),$$

but by construction, $\max_{g \in \frac{1}{2}M + \frac{1}{2}h} u \circ g(s) = 0$ for every s . Thus, C is determined by its restriction to \mathcal{M}_0 .

- Third, and most importantly, INA implies that one can construct a menu-independent preference ordering \succsim which rationalizes the restriction of C to \mathcal{M}_0 . Specifically, call an act choosable if it is chosen from some $M \in \mathcal{M}_0$ and define

$$\begin{aligned} f \succ_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in M \setminus C(M) \\ f \sim_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in C(M). \end{aligned}$$

that is, $f \succ_C g$ if f is strictly revealed preferred to g in some $M \in \mathcal{M}_0$ and $f \sim_C g$ if the two are revealed indifferent. The proof of lemma 2 establishes that \succsim_C is a preference ordering over

choosable acts. Furthermore, it generates the restriction of C to \mathcal{M}_0 as choice correspondence: For all $M \in \mathcal{M}_0$, it is true that $C(M) = \{a \in M : a \succsim_C b, \forall b \in M\}$. Finally, \succsim_C can be extended to an ordering \succsim over all nonpositive utility acts without affecting C .

- For convenience, the lemma finally identifies \succsim with a value functional I and collects some properties of it that are not central to the idea but will be used later.

Returning to the substantive importance of lemma 2, its upshot is as follows. The proofs in Gilboa and Schmeidler (1989), as well as many related papers, begin by stating that preferences can be represented by a value functional I operated on utility acts. Lemma 2 is analogous to this, except that I is operated on regret acts which absorb any menu dependence of the agent's evaluation of acts. Substantively, this leads to a separation of risk aversion as well as menu dependence of preferences, both of which are absorbed by $r \circ (f, M)$, and attitude to uncertainty about s , which is reflected in I .⁷ Formally, it re-instates a dualism between choice correspondences and preferences, albeit a more intricate one than is generated by IIA. The practical benefit of this dualism is that existing axiomatic results for preferences can be imported if \succsim can be shown to fulfil their if-sides.

This idea will now be exploited to generate different axiomatizations of choice correspondences. The first results are theorems 3 and 4, which characterize minimax regret choice correspondences with no respectively endogenous priors.

Theorem 3 *Prior-less Minimax Regret*

A choice correspondence fulfils axioms 1 through 8 iff it can be represented as

$$C(M) = \arg \min_{f \in M} \max_{s \in S} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2.

Given lemma 2, theorem 3 is established by applying theorem 1(iii) of Stoye (2006) to \succsim . Substantively, its message is as follows. The axioms except symmetry are the standard axioms of Bayesianism, with the caveat that independence of irrelevant alternatives has been weakened. If symmetry, i.e. a refusal to make likelihood judgments, were added to the full Bayesian axioms, a contradiction would be encountered.⁸ A known way to avoid this contradiction is to relax independence to c-independence,

⁷This separation is less clear in Gilboa and Schmeidler's (1989) lemma 3.3 because there, the preferences encoded by I may change if u is replaced with a positive affine transformation of itself; since this is considered an equivalence transformation, I is not really identified separately from u . To achieve this separation, one also has to impose c-independence of I (Ghirardato et al. 2005). The problem does not arise here: A positive affine transformation of u induces a positive *linear* transformation of $r \circ (f, M)$; since \succsim is homothetic, the choice correspondence cannot be affected.

⁸This follows immediately from theorem 3 since the minimax regret choice correspondence violates IIA.

leading to prior-less α -maximin utility, i.e. the “Hurwicz criterion,” and to prior-less maximin utility if ambiguity aversion is added (Milnor 1954, Stoye 2006). But we now see that one can avoid the contradiction while insisting on independence, namely by weakening independence of irrelevant alternatives. Maintaining independence yet avoiding priors is not a contradiction; it leads to a well known decision criterion, albeit at the price of menu dependence.

Endogenous prior minimax regret can be characterized by replacing symmetry with c-betweenness for menus with state independent outcome distributions.

Theorem 4 *Endogenous Prior Minimax Regret*

A choice correspondence fulfils axioms 1 through 7 and 9 iff it can be represented as

$$C(M) = \arg \min_{f \in M} \max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi$$

for some compact, convex $\Gamma \subseteq \Delta\mathcal{S}$. Here, Γ is unique and u is as in lemma 2.

To establish this result, one needs to show that \succsim is c-independent, after which Gilboa and Schmeidler (1989) can be invoked. Proof of c-independence has two main ingredients: First, axiom 9 together with ambiguity aversion ensures that \succsim fulfils a preference version of c-betweenness, i.e. if f is revealed indifferent to a constant act p^* , then it is also revealed indifferent to any mixture of the two. Intuitively, the upshot is that indifference sets of \succsim (in the space of regret acts) are collections of rays emanating from constant acts. Homogeneity of degree zero of \succsim insures that these rays cannot fan out or in, leading to c-independence. Thus, the “stake independence” aspect of c-independence is delivered by independence of C , which drives homogeneity of degree zero of \succsim .

Substantively, theorem 4 resembles a previous finding by Hayashi (2008). The least obviously similar axioms are c-betweenness here and “constant-regret independence of regret premium” there. Also, monotonicity displaces Hayashi’s (2008) admissibility axiom; as a result of this weakening, Γ need not intersect the interior of $\Delta\mathcal{S}$. Again, continuity can be weakened as well, at the cost of losing continuity of U . This does not strengthen Hayashi’s (2008) result since I use simple acts; it shows, however, that a weaker continuity axiom delivers the gist of the result in the present framework. Despite these differences, the contribution of theorem 4 lies less in the substantive result than in its derivation, specifically in exhibiting a tight link to Gilboa and Schmeidler (1989).

Theorems 3 and 4 do not exhaust the potential applications of lemma 2. For example, one can produce variations on theorem 4 by varying the domain of axiom 9. At an extreme, one could drop the requirement that p^* is constant and thereby assume a revealed preference equivalent of betweenness (whenever feasible outcome distributions are state independent). For an intermediate approach, betweenness could be imposed (with the same caveat) for any two acts f and g that have comonotonic regret profiles $r \circ (f, M)$, where comonotonicity is defined as in Schmeidler (1989); specifically, two

profiles are comonotonic if there exists an ordering of \mathcal{S} that renders both of them nondecreasing. The results of these variations are as follows.

Corollary 5 *If axiom 9 is imposed with p^* replaced by an arbitrary act g , then Γ is reduced to a singleton and the behavioral implications of expected utility theory are recovered.*

Corollary 6 *If axiom 9 is imposed with p^* replaced by an act g s.t. f and g have comonotonic regret profiles, then the choice correspondence can be characterized by minimization of Choquet expected regret, where the Choquet expectation is as in Schmeidler (1989).*

2.3 Minimax Regret when Agents can Randomize

Here as well as in Hayashi (2008), randomization was excluded so far, allowing for axioms to be asserted over arbitrary, and in particular nonconvex, sets.⁹ In many intended applications, agents can randomize however. This consideration is particularly pressing with respect to statistical decision theory. Optimal statistical decision rules are frequently randomized, so that by not permitting randomization, one excludes from characterization the very acts that will ultimately be chosen.

Unfortunately, allowing for randomization complicates matters because it convexifies all choice sets. Even an unrestricted independence of irrelevant alternatives assumption would then amount to imposing the weak but not the strong axiom of revealed preference; that is, it would fail to imply transitivity of revealed preference. As a result, lemma 2 appears to fail. However, it is possible to recover theorems 3 and 4.

Theorem 7 *Minimax Regret When Agents can Randomize*

Consider a setting exactly as in the preceding theorems, but where agents can randomize, thus choices are from convex hulls ΔM of finite menus M . Then theorems 3 and 4 continue to hold as stated.

While restricting attention to convex sets does not affect the two main theorems, much has changed below the surface. The revealed preference ordering \succsim_C is now highly incomplete even over choosable acts, and existence of a unique transitive completion \succsim of \succsim_C cannot, as far as this author can tell, be established at the level of generality of lemma 2. Rather, the adapted proofs of theorems 3 and 4 make heavy use of the additional axioms – c-betweenness respectively symmetry – to establish this extension, and then again to identify the extension with minimax regret. Theorem 7 therefore stands as the only main result in this paper that is not usefully thought of as a corollary of lemma 2.

⁹I thank a referee for raising the question answered in this section.

2.4 A Characterization of Perceived Ambiguity

Lemma 2 can also be used to develop a behavioral notion of perceived ambiguity in the framework of theorem 4. As it presupposes theorem 4, this notion will generalize to choice with randomization; a caveat is that it requires U to be unbounded, which was not otherwise necessary. The construction requires one preliminary step:

Definition 1 For any choice correspondence C , define its revealed unambiguous preference \succeq_C by

$$f \succeq_C g \text{ iff } [f \in C(M) \Rightarrow g \in C(M)], \forall M \supseteq \{f, g\}.$$

In words, $f \succeq_C g$ if g is revealed preferred to f in no menu. Of course, although \succeq_C can be shown to be transitive, it is not in general complete. I relate it to ambiguity to once again emphasize the conceptual link between menu dependence and ambiguity. The idea is that if $f \succeq_C g$, then the comparison between f and g is context independent, hence revealed to be unambiguous. It is, then, also intuitive to define comparative perceived ambiguity as follows.

Definition 2 The choice correspondence C reveals (weakly) more perceived ambiguity than C' if

$$f \succeq_C g \implies f \succeq_{C'} g.$$

Comparative ambiguity perception can be tightly characterized in terms of utility functions and sets of priors.

Theorem 8 Characterization of Comparative Ambiguity Aversion

Assume that theorem 4 applies and that U is unbounded. The choice correspondence C reveals (weakly) more perceived ambiguity than C' iff both can be represented by the same utility function U in conjunction with sets of priors Γ (to represent C) and Γ' (to represent C') s.t. $\Gamma \supseteq \Gamma'$.

The theorem lends further support to the identification of revealed unambiguous preference with ambiguity perception, because it tightly links perceived ambiguity to sets of priors, which prima facie represent the decision maker's perception of ambiguity in her environment. It should be kept in mind, however, that this paper is behavioral and hence, Γ is not claimed to map onto any real objects, including true sets of beliefs. One might therefore want to more cautiously interpret theorem 8 as a characterization of \succeq_C that illustrates conceptual consistency of this paper's terminology. (See also the similar discussion in Ghirardato et al. 2004.)

To understand why the theorem is true, recall that the link between C and \succsim from lemma 2 is partly established by mapping arbitrary menus into \mathcal{M}_0 via mixing with a "normalizing" act. This link can be used to show that $f \succeq_C g$ iff $\lambda f + (1 - \lambda)h \succsim \lambda g + (1 - \lambda)h$ for any probability $\lambda \in [0, 1)$ and

any act h (presuming that the resulting mixture acts have nonpositive utility range). But this means that $f \succeq_C g$ is the independence-abiding, incomplete preference ordering defined by Ghirardato et al. (2004, see also Bewley 2002) to capture perceived ambiguity. Theorem 8 then follows by importing their results. It implies that comparative ambiguity perception coincides with comparative regret aversion in Hayashi (2008). Once the link between \succsim and \succeq_C has been established, this result is expected because Hayashi's notion of comparative regret aversion adapts Ghirardato and Marinacchi's (2002) notion of comparative ambiguity aversion, which coincides with the one of Ghirardato et al. (2004) for maxmin utility preferences – but \succsim is just such a preference.

An interesting application of theorem 8 is axiomatization of minimax regret where the set of priors is informative but exogenous. Let there exist a compact, convex object $\Gamma^* \subseteq \Delta\mathcal{S}$ and consider linking it axiomatically to the object Γ in the preceding representation. As explained in the introduction, this approach is mainly motivated by statistical decision theory, where exogenous sets of priors are ubiquitous; specifically, corollary 9 below characterizes Γ -minimax regret. Under a revealed preference interpretation, the corollary specifies how Γ would be revealed to equal Γ^* . In either case, the necessary axioms are as follows.¹⁰

Axiom 10 Γ^* -Monotonicity

$$\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^* \implies f \succeq_C g.$$

Axiom 11 Consistency with Γ^* -Ambiguity

$$f \succeq_C g \implies \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

To understand Γ^* -monotonicity, recall that in view of lemma 1, the standard monotonicity axiom can be slightly rewritten: $f \in C_s(\{f, g\}), \forall s$ is equivalent to $u \circ f(s) \geq u \circ g(s), \forall s$. This reveals that Γ^* -monotonicity is intuitively similar to monotonicity, but strengthens it. It imposes that if the comparison of f and g is commonsensically unambiguous to anybody who accepts priors Γ^* , then revealed preferences should indeed not reveal any ambiguity in the sense of menu dependence, and should furthermore have the obvious direction. The same axiom is used, to a similar effect as here, by Gajdos et al. (2004).

Conversely, consistency with Γ^* -ambiguity stipulates that a revealed preference is unambiguous in the sense of menu-independent only if the according comparison of acts is commonsensically unambiguous given Γ^* . Equivalently, if the comparison of two acts under Γ^* is ambiguous in the sense that

¹⁰I find it most intuitive to state the axioms in terms of u , but of course, existence of this object depends on previous axioms. To avoid this, one can rephrase $\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*$ as $(\int f(s) d\pi)^* \in C\{(\int f(s) d\pi)^*, (\int g(s) d\pi)^*\}, \forall \pi \in \Gamma^*$. The statements are equivalent by lemma 1.

either act is favored by some elements of Γ^* , then the choice correspondence reflects this ambiguity in the sense of violating IIA. This relates consistency with Γ^* -ambiguity to the aforementioned leitmotif, namely that violations of IIA should be driven by ambiguity.

In short, Γ^* -monotonicity can be thought of as ensuring that the decision maker does not see more ambiguity than is encoded in Γ^* ; consistency with Γ^* -ambiguity can be thought of as ensuring that she does not see less. The axioms' effects accord with these intuitions.

Corollary 9 *Exogenous Priors Minimax Regret*

Assume that theorem 4 applies. Then:

- (i) A choice correspondence is consistent with Γ^* -monotonicity iff $\Gamma \subseteq \Gamma^*$.*
- (ii) A choice correspondence is consistent with Γ^* -ambiguity iff $\Gamma \supseteq \Gamma^*$.*

Corollary 8 is true because axioms 10 and 11 effectively restrict C to reveal less (respectively more) ambiguity than the minimax regret ordering with the same utility function and set of priors Γ^* . It builds on theorem 4, but also connects theorem 3 to theorem 4 because it identifies prior-less minimax regret as minimax regret with maximal perceived ambiguity. This yields an alternative characterization of prior-less minimax regret and again illustrates conceptual consistency of terminology. After all, we would surely think of the absence of any prior information as maximizing ambiguity. I finally note that corollary 8 is easily re-imported into the setting of Ghirardato et al. (2004) to yield a characterization of exogenous prior (or Γ -) maximin utility.

2.5 Minimax Regret with Different Benchmarks:

Axiomatizing Hannan Regret

This section considers variations of prior-less minimax regret where the “regret benchmark” may be generated not by the ex post best acts in M , but by the ex post best acts within a subset of M . The approach may seem unfamiliar to economic theorists, but can be specialized to yield a characterization of Hannan regret, which is frequently used in the statistics as well as computer science literatures. An informal motivation for it is that comparisons with all acts may generate benchmarks that are unreasonably high in some states. It is hard to imagine an axiomatization that would directly capture this concern without presupposing benchmarks, and therefore regret. However, nonstandard benchmarks can be characterized by varying the potential extent of menu dependence. I will first do this at some level of generality and then specialize the discussion to Hannan regret.

Assume there exists a subset $\mathcal{F}^* \subseteq \mathcal{F}$ of *special acts* with the following structure: (i) \mathcal{F}^* is closed under probabilistic mixture; (ii) it is closed under statewise recombination, i.e. $f, g \in \mathcal{F}^*$ and $E \in \Sigma$ imply that $f_{EG} \in \mathcal{F}^*$, where f_{EG} is the act that coincides with f on E and with g otherwise; (iii) there

exist constant acts $p^*, q^* \in \mathcal{F}^*$ s.t. $C(\{p^*, q^*\}) = \{p^*\}$.¹¹ Restrict attention to menus that contain some such act, i.e. menus M with $M \cap \mathcal{F}^* \neq \emptyset$. Suppose that the ambiguity inherent in a decision problem is not driven by all acts at the decision maker’s disposal, but only by the special ones. In a normative context, one would have to argue why this restriction is compelling; this argument would depend on the application at hand, and along with specification of \mathcal{F}^* , is left to users who wish to employ the model. In a revealed preference interpretation, this section shows how different degrees of menu dependence of choice behavior can be interpreted as revealing different ways of forming benchmarks; \mathcal{F}^* is then a set to be revealed behaviorally. In any case, the idea of equating ambiguity with menu independence leads to imposing axiom INA with respect to elements of \mathcal{F}^* and full independence of irrelevant alternatives otherwise. Technically, the new axiom would be:

Axiom 12 *INA for Special Acts*

Let M and N be such that $C_s([M \cup N] \cap \mathcal{F}^) \cap M \neq \emptyset$ for all s . Then*

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Replacing axiom 6 with axiom 12 affects lemma 2 and theorem 3 as follows.

Theorem 10 *Restrict attention to menus M s.t. $M \cap \mathcal{F}^* \neq \emptyset$. Then a choice correspondence fulfils axioms 1 through 5, 7, 8, and 12 iff it can be represented as*

$$C(M) = \arg \min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M \cap \mathcal{F}^*} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2.

In words, replacing INA with INA for special acts has the effect that the prior-less regret choice correspondence benchmarks not against $\max_{g \in M} u \circ g(s)$ but against $\max_{g \in M \cap \mathcal{A}^*} u \circ g(s)$. The ex post utility frontier is evaluated only over special acts.¹²

To elaborate the example that motivates this result, fix a finite-state, finite-action decision problem where action $\tilde{f} \in \tilde{\mathcal{F}}$ generates utility $u \circ \tilde{f}(\tilde{s})$ in state $\tilde{s} \in \tilde{\mathcal{S}}$ and consider the N -superproblem generated by replicating this problem N times. Let the state space for the superproblem be $\mathcal{S} \equiv \times_{n=1}^N \tilde{\mathcal{S}}$ with typical element $s \equiv (\tilde{s}^1, \dots, \tilde{s}^N)$; in particular, the true state of the world in the one-shot problem can vary arbitrarily across time periods. Let \tilde{M} denote the decision maker’s menu in the one-shot problem. A decision rule then specifies a decision maker’s act \tilde{f}^n in each of the N periods. This act

¹¹These conditions are simple and sufficient, but not necessary for results to go through. What is actually needed is the “outcome range overlap” assumption spelled out in Puppe and Schlag (2008), plus convexity and closure under statewise recombination of the image of \mathcal{F}^* under u .

¹²The result does not quite apply to endogenous prior regret. This extension would require an additional axiom that precludes revealed indifference over safe payments, e.g. Hayashi’s weak admissibility axiom. See the appendix for details.

may always be a function of the decision maker’s history of play, $(\tilde{f}^1, \dots, \tilde{f}^{n-1})$ and usually also of the history of states of the world $(\tilde{s}^1, \dots, \tilde{s}^{n-1})$; indeed, statistical learning is a core applications of the model. Assume that if she knew s , the decision maker would maximize $u \circ f(s) \equiv \sum_n u \circ \tilde{f}^n(\tilde{s}^n)$. Then a popular criterion by which to judge feasible decision rules is (the negative of)

$$\max_{s \in \mathcal{S}} \left\{ \max_{\tilde{g} \in \tilde{\mathcal{M}}} \sum_n u \circ \tilde{g}(\tilde{s}^n) - u \circ f(s) \right\},$$

the maximal regret incurred relative to constant decision rules. This criterion function was introduced by Hannan (1957), is accordingly known as *Hannan regret*, and is widely used in the machine learning literature as well as occasionally in economics.¹³ Hannan regret has not, to my knowledge, been axiomatized before. Assuming nontriviality of u , its characterization is a special case of theorem 9. To see this, let \mathcal{F} be the set of decision rules as explained, let \mathcal{F}^* be the closure under statewise recombination of the constant decision rules (noting that $\max_{g \in \mathcal{M} \cap \mathcal{F}^*} u \circ g(s)$ will always be achieved by a constant rule), and use u and s as defined in this paragraph. A caveat to this observation is that INA with respect to this very special class of acts has a certain degree of artifice. In this author’s personal judgment, the contribution is, therefore, to clarify how Hannan regret “fits in” with the previous developments and how one could characterize it behaviorally; with respect to normative foundations for the criterion, the finding may be more thought-provoking than positive.

3 Conclusion

This paper unified some of the recent, axiomatic literature on minimax regret. It adopted choice correspondences as general framework, but demonstrated that even without independence of irrelevant alternatives, there exists a tight link between axiomatizations of preference orderings respectively choice correspondences. Specifically, I showed that under restrictions shared by many regret-based approaches, results of the latter kind can be generated from existing results of the former kind. I used this insight to provide a number of minimax regret characterization, including a choice correspondence analog of a result in Stoye (2006), a reconstruction of a result in Hayashi (2008) as implication of Gilboa and Schmeidler (1989), the adaptation of results in Ghirardato et al. (2004) to characterize perceived ambiguity, and the extension of all of these results to choice from convex (through randomization) sets. Finally, I characterized minimax regret with exogenous priors as well as Hannan regret.

The framework is intended to be rather universal and covers all of the applications cited in the introduction as well as applications of Hannan regret in other literatures. Nonetheless, it is impossible

¹³See the textbook by Cesa-Bianchi and Lugosi (2006) and references therein. The authors define minimax regret with nonstandard benchmarks in section 2.10. The further specialization to Hannan regret underlies the notion of Hannan consistency (as in the last expression on p. 72), which is also used as optimality criterion by Hart and Mas-Colell (2001).

to unify in one paper every concept that has been labelled “regret.” I therefore conclude by clarifying the relation between minimax regret as formalized here and some other notions in economics and related fields.

Statistical decision theory is one motivation of this paper, and many but not all uses of regret there coincide with formalisms here. To comply with this paper’s notion of minimax regret, statistical decision rules or estimators must be compared to an “oracle estimator” which is best among the feasible ones, given hypothetical knowledge of the true state of the world. Examples include the treatment choice problems in Manski (2004, 2006, 2007), Schlag (2007), and Stoye (2007a, 2007b, 2008), where the oracle treatment rules are no-data rules that respond to true expectations, but also the estimation problems in Droge (1998, 2006), Eldar et al. (2004), and Hansen (2005), where they are the ex post best from certain classes of estimators. An incompatible example, however, is the “predictive entropy regret” approach of Sweeting et al. (2006), who propose to find a noninformative prior by minimizing maximal regret against a particular, pre-assigned prior.¹⁴ As a result, one of the distinguishing features of minimax regret, namely its menu dependence, is lost. Indeed, while I find the approach fascinating, I suspect that from a decision theoretic point of view, it should be conceived not as minimax regret but as maximin utility with a specific, state dependent (but not menu dependent) utility function.

The word “regret” resounds in everyday language, and some readers may accordingly be interested in it from the vantage point of psychology or behavioral economics. From that perspective, one may critically remark that regret here benchmarks against an “omniscient” ex post stage in which the true state of the world has been revealed. This does not correspond to a situation that the decision maker anticipates to actually experience, so one may hesitate to identify this paper’s notion of regret with anticipated feelings. If the latter are a core motivation, one may want to explore regret preferences that benchmark against what the decision maker will, in fact, learn from outcome realizations. This is the motivation of Krähmer and Stone (2006). Interestingly, it can lead to preferences against information: Of two otherwise identical acts, the one whose outcomes are less correlated with, and hence less informative about, other acts’ potential outcomes may be strictly preferred. The approach has not, to my knowledge, been axiomatized.

A well known invocation of regret in economic theory is due to Loomes and Sugden (1982; see also Fishburn 1989, Sugden 1993). This approach has in common with the current one that regret is evaluated from an omniscient view; some papers partially justify this by imposing independence of outcome lotteries across acts, thus removing one wedge between the “realistic ex post” and the omniscient information stage. Major differences to the present perspective are the imposition of a concave transformation of regret and the use of subjective priors rather than maxmin-operators.

¹⁴This is not a special case of theorem 10 with \mathcal{A}^* being a singleton. Singletons fail the conditions mentioned in section 2.5.

Indeed, the dichotomy of uncertainty/risk versus ambiguity/Knightian uncertainty is not emphasized in this approach, and the relevant papers switch between imposing objective probabilities (Loomes and Sugden 1982) and a Savage environment (Sugden 1993).

Finally, Sarver (2008) proposed a model of regret that embeds it in the recent literature on menu dependent preferences. One difference to the present approach is that Sarver’s utility function combines conventional utility with an additive regret penalty. More importantly, he follows Kreps (1979) and Gul and Pesendorfer (2001) in axiomatizing preferences over menus; in the language of utility maximization, the axiomatization is of the value functional. For Gul and Pesendorfer’s (2001) as well as Sarver’s (2008) motivation, this is an ingenious device. For the present paper’s notion of regret, its use would be less obvious because the minimax regret value functional does not have a clean interpretation. If choice problem M causes more minimax regret than problem N , this may mean that learning is more valuable in M , and this intuition may well inform a future axiomatization – but it does not imply that N is more desirable by any commonsensical standard. Indeed, it could easily be the case that any option in M dominates any element of N in utility terms. An instructive axiomatization of this value functional would, accordingly, have to be quite different from the present contribution.

A Proofs

Lemma 1 Follows from standard results after defining $p^* \succsim q^* \Leftrightarrow p^* \in C(\{p^*, q^*\})$ for constant acts. It is well known that mixture continuity suffices for a von Neumann-Morgenstern representation, while sequential continuity enforces continuity of U .

Lemma 2 In this and the next two proofs, I only show “only if.” Recall that lemma 1 applies. For any act f , define the mapping (“utility act”) $u \circ f : \mathcal{S} \mapsto \mathbb{R}$ by $u \circ f(s) \equiv \int U(x)df(s)$ and use $\geq [\gg]$ as shortcut for weak [strict] dominance with respect to this mapping, i.e. $f \geq [\gg]g \Leftrightarrow u \circ f(s) \geq [\gg]u \circ g(s), \forall s$. Observe that in the statement of monotonicity, $[f \in C_s(\{f, g\}), \forall s]$ can now be written as $f \geq g$.

Step 1: No information is lost by identifying every act f with $u \circ f$. To see this, fix any menus M and M' such that $u \circ M = u \circ M'$. The claim is that $u \circ C(M) = u \circ C(M')$. To see this, consider $C(M \cup M')$. By INA, $C(M \cup M') \cap M \in \{C(M), \emptyset\}$ and $C(M \cup M') \cap M' \in \{C(M'), \emptyset\}$. These statements are consistent with monotonicity and nonemptiness of C only if $C(M \cup M') = C(M) \cup C(M')$, but then either of $C(M) \subset C(M')$ or $C(M) \supset C(M')$ would violate monotonicity. With abuse of notation, I henceforth identify acts with utility acts whenever convenient.

Nonconstancy of U is necessary for monotonicity and nontriviality to be mutually consistent. It

implies that after normalization, $U^{-1}(-1)$ and $U^{-1}(1)$ can be assumed to exist. Hence, any finite, Σ -measurable step function $u : \mathcal{S} \rightarrow [-1, 1]$ can be identified with a feasible act f . This specifically includes p_0^* , the constant act with utility value 0. Independence implies that $C(\lambda M + (1 - \lambda)p_0^*) = \lambda C(M) + (1 - \lambda)p_0^* = \lambda C(M)$. Hence, C is homogeneous of degree zero: For any menu M and scalar $\lambda \in (0, 1)$, the menu λM exists and $C(\lambda M) = \lambda C(M)$.

Step 2: For any menu M , let \bar{f}_M denote the act with $u \circ \bar{f}_M(s) = \max_{f \in M} u \circ f(s)$. This “oracle act” or join always exists, although it need not be an element of M . Let \mathcal{M}_0 denote the set of menus M s.t. $\bar{f}_M = p_0^*$, i.e. the ex post best possible utility is zero in every state. In this and the next step, restrict attention to menus $M \in \mathcal{M}_0$. Call an act f choosable if there exists $M \in \mathcal{M}_0$ s.t. $f \in C(M)$, and let \mathcal{C} denote the set of choosable acts. (Of course, every act in \mathcal{C} has nonpositive utility range.) Define the relation \succ_C on $\mathcal{C} \times \mathcal{C}$ as follows:

$$\begin{aligned} f \succ_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in M \setminus C(M) \\ f \sim_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in C(M). \end{aligned}$$

I now collect some properties of \succ_C . Weak monotonicity and nontriviality follow immediately from the according axioms. Axiom INA straightforwardly implies that \succ_C is antisymmetric, and also that \succ_C and \sim_C are disjoint. To see completeness, fix any $f \in \mathcal{C}$ and consider any $M \in \mathcal{M}_0$ from which f is chosen, then by axiom INA, choice from $M \cup \{g\}$ must define \succ_C . To see transitivity ($f \succ_C g, g \succ_C h \Rightarrow f \succ_C h$), consider its contrapositive: $h \succ_C f \Rightarrow (h \succ_C g \vee g \succ_C f), \forall g$. Assume $h \succ_C f$, hence there exists M with $h \in C(M)$ and $f \in M \setminus C(M)$. Then either $g \in C(M \cup \{g\})$, in which case $g \succ_C f$, or $g \notin C(M \cup \{g\})$, in which case $h \succ_C g$.

Mixture continuity implies the analog property of \succ_C . Let $f \succ_C g \succ_C h$, thus there exists $M \supseteq \{g, h\}$ with $C(M) \cap \{g, h\} = \{g\}$ and $N \supseteq \{f, g\}$ with $C(N) \cap \{f, g\} = \{f\}$. To derive the then-side of mixture continuity from axiom 7, it suffices to show $C(M \cup \{f\}) = \{f\}$. To see “ \supseteq ,” suppose that $f \notin C(M \cup \{f\})$, thus $C(M \cup \{f\}) \cap M = C(M)$, thus $g \in C(M \cup \{f\})$, thus $g \succ_C f$, a contradiction. To see “ \subseteq ,” consider $C(M \cup N) \cap N = C((M \cup \{f\}) \cup N) \cap N$. Applying INA to the expanded expression, one finds that $\{f\} \in C(M \cup N)$, hence $C(M \cup N) \cap C(N) = C(N)$, hence $g \notin C(M \cup N)$, hence $g \notin C(M \cup \{f\})$, hence $C(M \cup \{f\}) = \{f\}$.

Ambiguity aversion implies that $f \sim_C g \Rightarrow \lambda f + (1 - \lambda)g \succ_C f$ for all $\lambda \in (0, 1)$. Fix any $M \supseteq \{f, g\}$, then $f \sim_C g$ implies that either $\{f, g\} \in C(M)$ or $C(M) \cap \{f, g\} = \emptyset$. In the latter case, M vacuously fulfils the condition defining $\lambda f + (1 - \lambda)g \succ_C f$. In the former case, ambiguity aversion implies that $\lambda f + (1 - \lambda)g \in C(M \cup \{\lambda f + (1 - \lambda)g\})$, thus the condition is fulfilled as well.

Step 3: Let \mathcal{F}_- denote the set of finite, Σ -measurable step functions from \mathcal{S} into \mathbb{R}_- , irrespective of whether their range is contained in the range of U . Then $\mathcal{C} \subseteq \mathcal{F}_-$, hence $\succsim_{\mathcal{C}}$ can be considered a (potentially incomplete) relation on $\mathcal{F}_- \times \mathcal{F}_-$. Any completion \succsim of $\succsim_{\mathcal{C}}$ on $\mathcal{F}_- \times \mathcal{F}_-$ induces C as choice correspondence: $C = C_{\succsim} \equiv \{f \in M : g \in M \Rightarrow f \succsim g\}$ for all $M \in \mathcal{M}_0$. To see this, fix M . Let $f \in C(M)$, then $f \succsim_{\mathcal{C}} g$ for all $g \in M$, hence $f \succsim g$ for all $f \in M$. Let $h \in M \setminus C(M)$, then by nonemptiness of C , there exists $f \in M$ with $f \succ_C h$, hence $f \succ h$, hence $h \notin \{f \in M : g \in M \Rightarrow f \succsim g\}$.

Step 4: I now specify a particular completion of $\succsim_{\mathcal{C}}$. Assume that \mathcal{C} contains a constant act $p^* \ll p_0^*$; the case where this fails will be handled separately in step 6. Then monotonicity implies that $f \in \mathcal{C}$ for any $f \geq p^*$. Furthermore, homogeneity of degree zero of C yields $f \succsim_{\mathcal{C}} g \Rightarrow \lambda f \succsim_{\mathcal{C}} \lambda g$ for all $\lambda \in (0, 1)$. Now define \succsim by

$$f \succsim g \Leftrightarrow \lambda f \succsim_{\mathcal{C}} \lambda g \text{ for some } \lambda \in (0, 1].$$

\succsim is complete because $\lambda f, \lambda g \geq p^*$ for λ small enough. It is unique, and its strict part is antisymmetric, because by this step's previous conclusion, different choices of λ cannot lead to contradictory assignments of \succsim . For the same reason, it is homogeneous of degree zero: $f \succsim g$ iff $\lambda f \succsim \lambda g$ for all $\lambda > 0$. Finally, \succsim completes $\succsim_{\mathcal{C}}$ and inherits all properties of $\succsim_{\mathcal{C}}$ established in step 2. They imply that \succsim can be represented by a homothetic value functional J that maps utility acts into \mathbb{R}^- . To make J unique, normalize it such that any constant act is mapped onto its utility value.

Step 5: Independence can be used to generate all choice correspondences from choice correspondences over menus in \mathcal{M}_0 . To see this, fix any menu M , let the scalar $\lambda > 0$ be small enough s.t. $\lambda M + (-\lambda \bar{f}_M) \subseteq \mathcal{C}$, and use independence to conclude that

$$\lambda (C(M) + (-\bar{f}_M)) = C(\lambda M + (-\lambda \bar{f}_M)) = \lambda C_{\succsim}(M - \bar{f}_M),$$

where the last step uses that \succsim is defined on \mathcal{F}_- and that it is homothetic, and where C_{\succsim} was defined in step 3. It follows that

$$\begin{aligned} C(M) &= C_{\succsim}(M - \bar{f}_M) + \bar{f}_M \\ &= \arg \max_{f \in M} J(u \circ f - u \circ \bar{f}_M) + u \circ \bar{f}_M \\ &= \arg \min_{f \in M} I(r \circ (f, M)), \end{aligned}$$

where $I \equiv -J$ has the properties asserted in the lemma.

Step 6: Assume that \mathcal{C} contains no act $p^* \ll p_0^*$, then by monotonicity, \mathcal{C} contains no act $f \ll p_0^*$. But ambiguity aversion implies that \mathcal{C} is convex, hence there exists a state s^* s.t. $\mathcal{C} \subseteq \{f : u \circ f(s^*) = 0\}$.

Fix any act f with $u \circ f(s^*) = 0$. Define the act g by $u \circ g(s^*) = -1$ and $u \circ g(s) = 0$ for any state s that is distinct from s^* (i.e., there exists an event Σ separating s and s^*). Then $g \notin \mathcal{C}$ and $\{f, g\} \in \mathcal{M}_0$, hence $C(\{f, g\}) = \{f\}$, hence $f \in \mathcal{C}$. It follows that $\mathcal{C} = \{f : u \circ f(s^*) = 0\}$.

\mathcal{C} is an indifference set of $\succsim_{\mathcal{C}}$. To see this, fix any act $f \in \mathcal{C}$. Suppose by contradiction that $f \prec_{\mathcal{C}} p_0^*$, thus there exists $M \in \mathcal{M}_0$, $M \supseteq \{f, p_0^*\}$, s.t. $C(M) \cap \{f, p_0^*\} = \{p_0^*\}$. Consider $M' = [M \setminus \mathcal{C}] \cup \{f, p_0^*, g, p^*\}$, where g is as defined in the preceding paragraph and where $p^* \ll p_0^*$, then axiom INA and $g, p^* \notin \mathcal{C}$ imply that $C(M') = \{p_0^*\}$ and that $C(M' \setminus \{p_0^*\}) = \{f\}$. Now continuity implies existence of $\lambda > 0$ s.t. $C(M' \setminus \{p_0^*\} \cup \{\lambda p^*\}) = \{\lambda p^*\}$. Hence $\lambda p^* \in \mathcal{C}$, a contradiction.

It follows that $C(M) = \{f : u \circ f(s^*) = 0\}$ for any $M \in \mathcal{M}_0$. This choice correspondence can be represented as maximizing $u \circ f(s^*)$, thus it fulfils all conclusions from step 5.

Theorem 3 Apply lemma 2. The choice correspondence discovered in step 6 of lemma 2 violates symmetry; hence \mathcal{C} contains some constant act $p^* \ll p_0^*$. The previous proof established that $\succsim_{\mathcal{C}}$ fulfils all axioms used in theorem 1(iii) of Stoye (2006) except for symmetry, which follows from the symmetry axiom imposed here as follows. Fix any acts $f, g \in \mathcal{C}$ and events $E_1, E_2 \in \Sigma$ s.t. f and g are constant on E and F . Define $E = E_1 \cup E_2$ and $F = S \setminus E$. Let the constant act q^* have utility value $q = 2 \min_{s \in S} \min\{u \circ f(s), u \circ g(s)\}$; if $q^* \ll p^*$, use homogeneity of degree zero of \mathcal{C} to complete the argument. Consider $M \equiv \{p_{0E}^* q^*, q_E^* p_0^*, q^*/2\}$, then symmetry and ambiguity aversion jointly imply that $q^*/2 \in C(M)$. Consider now $N \equiv M \cup \{f, g\}$. Noting that $q^*/2$ is dominated by both f and g , axiom INA and monotonicity jointly imply that $C(N) \cap \{f, g\} \neq \emptyset$, hence $f \succsim_{\mathcal{C}} g \Leftrightarrow f \in C(N)$. But now Stoye's (2006) symmetry axiom is implied upon comparing N and N' , the menu generated from N by interchanging the consequences of E_1 and E_2 .

Thus $\succsim_{\mathcal{C}}$ (and by extension \succsim) is priorless maximin utility: $f \succsim_{\mathcal{C}} g$ iff $\min_{s \in S} u \circ f(s) \geq \min_{s \in S} u \circ g(s)$. Substituting into lemma 2 yields

$$C(M) = \arg \min_{f \in M} \max_{s \in S} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

as required. For necessity of three events as well as individual necessity of axioms, see Stoye (2006).

Theorem 4 The choice correspondence identified in step 6 of the proof of lemma 2 fulfils this theorem's conclusion. Hence, restrict attention to the case where \mathcal{C} contains a constant act $p^* \ll p_0^*$. I show that $\succsim_{\mathcal{C}}$ is c-independent, after which the proof of Gilboa and Schmeidler can be mimicked (beginning at their lemma 3.3). Define the relation $\succsim_{\mathcal{C}}^*$ on $\mathcal{C} \times \mathcal{C}$ by

$$f \succ_{\mathcal{C}}^* g \iff \exists M \in \mathcal{M}_0^* : f \in C(M), g \in M \setminus C(M)$$

$$f \sim_{\mathcal{C}}^* g \iff \exists M \in \mathcal{M}_0^* : f \in C(M), g \in C(M).$$

as before, but where \mathcal{M}_0^* is the set of menus with join p_0^* that also have state independent outcome distributions. I will first argue that $\succsim_C = \succsim_C^*$ and then show that \succsim_C^* is c-independent.

Fix any menu $M \in \mathcal{M}_0$. Let $\Sigma_0 \subset \Sigma$ be some partition of \mathcal{S} s.t. $\bigcap_{s \in E} \arg \max_{a \in M} u \circ f(s) \neq \emptyset$ for any $E \in \Sigma_0$; in words, there exists a weakly dominating act conditional on any element of Σ_0 . (Finiteness of acts and menus implies that Σ_0 can be chosen to be finite.) Define the act \underline{f}_M by $u \circ \underline{f}_M(s) = \min_{f \in M} u \circ f(s)$, define $\mathcal{U}_M \equiv \{v \in [-1, 0] : u \circ f(s) = v \text{ for some } f \in M, s \in \mathcal{S}\}$, and let $M^* = M \cup \left\{ v_E \underline{f}_M : E \in \Sigma_0 \cup \{\emptyset\}, v \in \mathcal{U}_M \right\}$. Every element of M^* is weakly dominated by some element of M , hence INA and monotonicity jointly imply that $C(M \cup M^*) \cap M = C(M)$. But M^* has state independent outcome distributions. Inspection of the definitions of \succsim_C^* respectively \succsim_C now reveals that $\succsim_C = \succsim_C^*$.

For use in the corollaries, I isolate the following lemma.

Lemma 11 *Under assumptions maintained in this proof, $f \succsim_C^* g \Leftrightarrow \lambda f + (1 - \lambda)p^* \succsim_C^* \lambda g + (1 - \lambda)p^*$ for all acts $f, g, p^* \in \mathcal{C}$ and scalars $\lambda \in (0, 1)$. In words, \succsim_C^* is c-independent.*

Proof. To begin, \succsim_C^* has the property that $f \gg g$ implies $f \succ_C^* g$, henceforth called increasingness. Suppose it fails, then there exist $f, g \in \mathcal{C}$ with $f \gg g$ but $g \succsim_C^* f$. Then there exists $\gamma < 1$ with $f \gg \gamma g$, thus $f \succsim_C^* \gamma g \succsim_C^* g$ by monotonicity, thus $g \sim_C^* \gamma g$ by transitivity. Homogeneity of degree 0 and transitivity of \succsim_C^* now jointly imply that $g \sim_C^* \rho g$ for any $\rho \in (0, 1)$. Now consider any acts $h, j \in \mathcal{C}$ s.t. $p_0^* \gg h, j$. One then has $h \succsim_C^* j$ iff $\rho h \succsim_C^* \rho j$ for any $\rho \in (0, 1)$. Appropriate choices of γ and ρ yield $\gamma g \geq \rho g, \rho j \geq g$, hence monotonicity and transitivity imply $\rho h \sim_C^* \rho j$, hence $h \sim_C^* j$, hence $\{f \in \mathcal{C} : f \ll p_0^*\}$ constitutes an indifference set. I now extend this indifference set to $\mathcal{C} \setminus \{p_0^*\}$. To do so, fix an arbitrary act g s.t. $p^* \leq g \neq p_0^*$ but not $g \ll p_0^*$ as well as an arbitrary act f with $p^* \leq f \ll p_0^*$. I will show that $f \sim_C^* g$. First, $g \geq p^*$, hence monotonicity, transitivity, and indifference of all acts $f \ll p_0^*$ jointly imply $g \succsim_C^* f$. Suppose that $p_0^* \succ_C^* g \succ_C^* f$, then continuity implies that $\lambda f \succ_C^* g$ for some $\lambda > 0$, but $\lambda f \notin \mathcal{C}$. Suppose that $p_0^* \sim_C^* g \succ_C^* f$, then homogeneity of C implies that $p_0^* \sim_C^* \gamma g$ for any $\gamma \geq 0$, and the choice correspondence would be the one from lemma 2, step 6. Thus, \succsim_C^* can be represented by two indifference sets, $\{p_0^*\}$ and $\mathcal{C} \setminus \{p_0^*\}$ (with the obvious ordering). This violates continuity however: Let there be two states and identify utility acts with vectors $(u, v) \in \mathbb{R}^2$ with obvious interpretation, then $C(\{(0, 0), (1, 1), (1, 0)\}) = \{(1, 1)\}$, $C(\{(0, 0), (1, 0)\}) = \{(1, 0)\}$, yet $C(\{(0, 0), (\lambda, \lambda), (1, 0)\}) = \{(0, 0), (\lambda, \lambda), (1, 0)\}$ for any $\lambda \in (0, 1)$.

Next, \succsim_C^* fulfils what might be called c-betweenness for preferences: For any act f and constant act p^* , $f \sim_C^* p^*$ iff $f \sim_C^* \lambda f + (1 - \lambda)p^*$ for all $\lambda \in (0, 1)$. To see “only if,” assume that $f \sim_C^* p^*$, then $f \succsim_C^* \lambda f + (1 - \lambda)p^*$ follows from ambiguity aversion. Assume by contradiction that $f \prec_C^* \lambda f + (1 - \lambda)p^*$, then there exists $M \in \mathcal{M}_0^*$ s.t. $\lambda f + (1 - \lambda)p^* \in C(M)$ but $f \in M \setminus C(M)$. Consider now $M \cup \{p^*\}$, noting that $M \cup \{p^*\} \in \mathcal{M}_0^*$. Two consecutive uses of INA yield $f \notin C(M \cup \{p^*\})$ and then $p^* \notin C(M \cup \{p^*\})$.

Nonemptiness of C and INA now jointly imply that $\lambda f + (1 - \lambda)p^* \in C(M \cup \{p^*\})$, contradicting axiom 9. Now suppose that $f \sim_C^* \lambda f + (1 - \lambda)p^*$ for *some* $\lambda \in (0, 1)$. This implies $f \sim_C^* p^*$, establishing “if.” To see this, note that monotonicity and continuity jointly imply existence of a constant act q^* s.t. $f \sim_C^* q^*$, hence $f \sim_C^* \lambda f + (1 - \lambda)q^*$ by “only if,” hence $\lambda f + (1 - \lambda)p^* \sim_C^* \lambda f + (1 - \lambda)q^*$ by transitivity – but this is consistent with increasingness only if $p^* = q^*$.

I will now derive c-independence when $f \sim_C^* g$, i.e. $f \sim_C^* g \Leftrightarrow \lambda f + (1 - \lambda)p^* \sim_C^* \lambda g + (1 - \lambda)p^*$. For this and the following step, assume that $f, g \ll p_0^*$; the preceding paragraph’s result can be used to extend indifference sets to boundary acts. If $p^* = p_0^*$, then the claim is immediate from homogeneity of degree 0 of C . Else, suppose $f \sim_C^* g$, then by monotonicity and continuity, there exists $\gamma > 0$ s.t. $f \sim_C^* g \sim_C^* \gamma p^*$. By the “only if”-direction of c-betweenness and transitivity, $\rho f + (1 - \rho)\gamma p^* \sim_C^* \rho g + (1 - \rho)\gamma p^*$ for any $\rho \in (0, 1)$. Letting $\delta \equiv (1 - \lambda + \gamma\lambda) / \gamma$ and $\rho \equiv \gamma\lambda / (1 - \lambda + \gamma\lambda)$, then $\lambda f + (1 - \lambda)p^* = \delta[\rho f + (1 - \rho)\gamma p^*]$ and $\lambda g + (1 - \lambda)p^* = \delta[\rho g + (1 - \rho)\gamma p^*]$. Homogeneity of degree 0 of C therefore implies that $\lambda f + (1 - \lambda)p^* \sim_C^* \lambda g + (1 - \lambda)p^*$. The converse follows from the reverse argument, using the “if”-direction of c-betweenness.

Assume now that $f \prec_C^* g$. Then by increasingness and continuity, there exists $\gamma \in (0, 1)$ s.t. $\gamma f \sim_C^* g$. The preceding paragraph’s conclusion implies that $\lambda \gamma f + (1 - \lambda)p^* \sim_C^* \lambda g + (1 - \lambda)p^*$, which in turn implies $\lambda f + (1 - \lambda)p^* \prec_C^* \lambda g + (1 - \lambda)p^*$ by increasingness and transitivity. If $f \succ_C^* g$, use the same argument with the roles of f and g reversed. Finally, assume $\lambda f + (1 - \lambda)p^* \prec_C^* \lambda g + (1 - \lambda)p^*$, then $f \sim_C^* g$ would violate the preceding paragraph’s conclusion, and $f \succ_C^* g$ would violate this paragraph’s preceding conclusion, hence $f \prec_C^* g$. ■

Lemma 10 delivers c-independence of \succ_C^* , hence \succ_C^* (and by extension \succ^*) can be represented by $\min_{\pi \in \Gamma} \int v \circ f(s) d\pi$, where $\Gamma \subseteq \Delta\mathcal{S}$ is a unique, convex, compact set of priors, and v is an object entirely analogous to u . Indeed, v can be identified with u through lemma 1 and monotonicity.

Individual necessity of most axioms is established in the working paper version of Hayashi (2008). For necessity of c-betweenness, consider a choice correspondence as in lemma 2, but with $I(r \circ (f, M)) = [f(r \circ (f, M))^2(s) d\pi(s)]^{1/2}$, where $\pi \in \Delta\mathcal{S}$ is a prior.

Corollaries 5 and 6 Follow from the above by adapting lemma 11.

Theorem 7 Recall that for this theorem, all choice sets are convex hulls ΔM of menus M . Lemma 1 continues to hold. This is shown in Stoye (2008b), the argument is repeated here for completeness. Define the relation \succeq_C on constant acts by $p^* \succeq_C q^*$ iff $p^* \in C(\Delta\{p^*, q^*\})$. Then \succeq_C is complete: Assume by contradiction that $C(\Delta\{p^*, q^*\}) \cap \{p^*, q^*\} = \emptyset$, then there exists $r^* = \lambda p^* + (1 - \lambda)q^* \in C(\Delta\{p^*, q^*\})$, thus $r^* \in C(\Delta\{p^*, r^*\})$ by IIA, thus $q^* \in C(\Delta\{p^*, q^*\})$ by independence, used with p^* as mixing act. \succeq_C is also transitive: Suppose by contradiction that $p^* \succeq_C q^* \succeq_C r^* \succ_C p^*$. The latter

implies that $\lambda r^* + (1-\lambda)q^* \succ_C \lambda p^* + (1-\lambda)q^*$ and then $(1-\gamma+\lambda\gamma)r^* + \gamma(1-\lambda)q^* \succ_C (1-\gamma)r^* + \gamma\lambda p^* + \gamma(1-\lambda)q^*$ for all $\gamma, \lambda \in (0, 1]$, where I used independence twice. IIA now implies $C(\Delta\{p^*, q^*, r^*\}) \subseteq \Delta\{q^*, r^*\}$, hence $\lambda r^* + (1-\lambda)q^* \in C(\Delta\{p^*, q^*, r^*\})$ for some $\lambda \in (0, 1)$. Independence and IIA then imply that $q^* \in C(\Delta\{p^*, q^*, r^*\})$, after which IIA implies that $p^* \in C(\Delta\{p^*, q^*, r^*\})$, a contradiction. Since \succeq_C is complete and transitive, it induces choice correspondence C (Arrow 1959). Furthermore, \succeq_C is von Neumann-Morgenstern utility by Herstein and Milnor (1953).

Now, steps 1 and 6 of lemma 2 go through unchanged for convex sets. Hence, I can again assume that some $p^* \ll p_0^*$ is choosable; note that by monotonicity, any $q^* \geq p^*$ is then choosable as well. Let \mathcal{C} collect all finite, Σ -measurable step functions from \mathcal{S} to $[p, 0]$, i.e. all utility acts with range in $[p, 0]$. Define the relation \succsim_C on $\mathcal{C} \times \mathcal{C}$ by

$$\begin{aligned} f \succ_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(\Delta M), g \in M \setminus C(\Delta M) \\ f \sim_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(\Delta M), g \in C(\Delta M), \end{aligned}$$

i.e. the adaptation to convex sets of the preceding definition. There are some complications: \succsim_C is potentially incomplete even on \mathcal{C} and not all acts in \mathcal{C} need be choosable (which is why \mathcal{C} had to be defined in a slightly different manner), and I am unable to recover lemma 2. Step 4 of the lemma's proof still applies however, hence every completion of \succsim_C induces C as choice correspondence. The theorems can be recovered by appropriately completing \succsim_C on \mathcal{C} . To limit the number of subscripts, this extension of \succsim_C will be labeled \succsim in this proof. The additional extension to \mathcal{F}_- via homotheticity is as before and will be omitted. Finally, note that the proof of increasingness presented in the first paragraph of lemma 11 goes through essentially unchanged. To keep notation manageable, I introduce some shortcuts. If v^* is a constant act, then the scalar v is its utility value, and the act $u_E^* v^*$ is more simply denoted $u_E v$.

From here on, recovery of theorem 3 and 4 go different ways, but there is a shared intuition: \succsim_C is incomplete even on \mathcal{C} . It can be completed by associating with every act $f \in \mathcal{C}$ a suitably defined certainty equivalent $c(f) \in \Delta\mathcal{X}$. Specifically, $c(f)$ will be the constant act q^* with utility $q \equiv \inf\{p : p^* \succ_C f\}$, with the convention that $c(f) = p_0^*$ if $\{p : p^* \succ_C f\} = \emptyset$. The completion will be to define $f \succsim g$ iff $c(f) \geq c(g)$. Further details differ across theorems, however, because the axioms used for lemma 2 are not sufficient to establish that \succsim extends \succsim_C . Symmetry respectively c-betweenness must be invoked for this. In fact, recovering theorem 3 leads to a self-contained result, whereas recovery of theorem 4 continues to culminate in an invocation of Gilboa and Schmeidler (1989).

Recovering Theorem 3

Step 1: This step establishes some preliminary findings for acts of the form $u_E v$, where $u \geq v$ are scalars and $E \in \Sigma \setminus \{\emptyset, \mathcal{S}\}$. Specifically: (i) $u_E v$ and q^* are \succsim_C -comparable for any constant act $q^* \gg v^*$; (ii) $c(u_E v) = c(u_F v)$ for any $F \in \Sigma \setminus \{\emptyset, \mathcal{S}\}$.

Fix an act $u_E v \in \mathcal{C}$ and any constant act $q^* \gg v^*$. Let $w^* \equiv (u^* + v^*)/2$. Suppose that $u^* \gg q^* \gg w^*$. Consider choice from $\Delta\{0_E z, q^*, z_E 0\}$, where $z = (v - u)/(1 - u/q)$, noting that $u_E v$ is a linear combination of $0_E z$ and q^* . (If z is below the range of U , use homogeneity of degree zero of C to complete the argument.) Suppose by contradiction that $C(\Delta\{0_E z, q^*, z_E 0\})$ contains some act $x_{EY} \neq q^*$. Then $y_{EX} \in C(\Delta\{0_E z, q^*, z_E 0\})$ by symmetry, thus $C(\Delta\{0_E z, q^*, z_E 0\})$ contains a constant act $r^* \ll q^*$ by convexity, contradicting increasingness. Hence, $C(\Delta\{0_E z, q^*, z_E 0\}) = \{q^*\} \Rightarrow q^* \succ_C u_E v$. (It now also follows easily that $q^* \succ_C u_E v$ whenever $q^* \geq u^*$.)

Now let $w^* \geq q^* \gg v^*$ and consider choice from $\Delta\{0_E z, u_E v, z_E 0\}$, where $z = q(u - v)/(q - v)$, noting that q^* is now a linear combination of $u_E v$ and $z_E 0$. Monotonicity implies that $C(\Delta\{0_E z, u_E v, z_E 0\})$ intersects $\Delta\{0_E z, u_E v\} \cup \Delta\{u_E v, z_E 0\}$. I will show that more specifically, $C(\Delta\{0_E z, u_E v, z_E 0\})$ intersects $\Delta\{u_E v, q^*\}$. Suppose by contradiction that $C(\Delta\{0_E z, u_E v, z_E 0\})$ contains an act $x_{EY} \in \Delta\{u_E v, z_E 0\} \setminus \Delta\{u_E v, q^*\}$. Then INA implies that $x_{EY} \in C(\Delta\{0_E z, q^*, z_E 0\})$, but then convexity and symmetry lead to a contradiction just as in the preceding paragraph. Suppose by contradiction that $C(\Delta\{0_E z, u_E v, z_E 0\})$ contains an act $x_{EY} \in \Delta\{0_E z, u_E v\} \setminus \{u_E v\}$ but does not intersect $\Delta\{u_E v, q^*\}$. Let $r^* = (x^* + y^*)/2$. If $r^* \leq q^*$, one again gets an immediate contradiction with symmetry and convexity, hence $r^* \gg q^*$. Now define the act g by $\{g\} = \Delta\{u_E v, q^*\} \cap \Delta\{x_{EY}, r^*\}$, then $x_{EY} \succ_C g$. At the same time, another simple argument from symmetry and convexity reveals that $r^* \succsim_C x_{EY}$. $r^* \sim_C x_{EY}$ would contradict $x_{EY} \succ_C g$ in light of convexity and INA, so $r^* \succ_C x_{EY}$. Define $\lambda^* \equiv \sup\{\lambda : x_{EY} \in C(\Delta\{0_E z, x_{EY}, \lambda x_{EY} + (1 - \lambda)r^*, z_E 0\})\}$, observing that by INA, $x_{EY} \in C(\Delta\{0_E z, x_{EY}, \lambda x_{EY} + (1 - \lambda)r^*, z_E 0\})$ for all $\lambda < \lambda^*$. Consider the act $h \equiv \lambda^* r^* + (1 - \lambda^*)x_{EY}$, noting that h is also a linear combination of g and r^* . Let $M_h \equiv \{0_E z, x_{EY}, \lambda x_{EY} + (1 - \lambda)r^*, z_E 0\}$. Suppose that $\{h, x_{EY}\} \subseteq C(\Delta M_h)$, then convexity would imply $g \in C(\Delta M_h)$, hence $x_{EY} \sim_C g$. Suppose that $C(\Delta M_h) \cap \{h, x_{EY}\} = \{h\}$, thus $h \succ_C x_{EY}$, then continuity (used with $x_{EY} \succ_C g$) implies that $\lambda x_{EY} + (1 - \lambda)r^* \succ_C x_{EY}$ for some $\lambda < \lambda^*$, a contradiction. It follows that $h \notin C(\Delta M_h)$. Suppose now first that $x_{EY} \notin C(\Delta M_h)$, then INA implies that $C(\Delta M_h) \cap \Delta\{0_E z, u_E v, z_E 0\} = \emptyset$. Monotonicity then implies that $C(\Delta\{0_E z, x_{EY}, h, z_E 0\})$ intersects $\Delta\{g, h\} \cup \Delta\{h, 0_E z\}$. If $C(\Delta M_h)$ intersected $\Delta\{g, h\}$, one would again have $\lambda x_{EY} + (1 - \lambda)r^* \succ_C x_{EY}$ for some $\lambda < \lambda^*$, a contradiction. It follows that $h_\lambda \equiv \lambda h + (1 - \lambda)z_E 0 \in C(\Delta M_h)$ for some $\lambda \in (0, 1)$. This implies $h_\lambda \succ_C x_{EY}$, hence continuity (used with $x_{EY} \succ_C g$) implies that $\gamma h_\lambda + (1 - \gamma)g \succ_C x_{EY}$ for some $\gamma \in (0, 1)$. But any linear combination of h_γ and g is contained in some menu of the form $\Delta\{0_E z, x_{EY}, \lambda' x_{EY} + (1 - \lambda')r^*, z_E 0\}$, where $\lambda' < \lambda^*$, thus x_{EY} cannot be chosen from that set or any similar set with $\lambda > \lambda'$, a contradiction. Suppose finally that $x_{EY} \in C(\Delta M_h)$, thus $x_{EY} \succ_C h$. Continuity, used with $r^* \succ_C x_{EY}$, then implies

that $x_E y \in C(\Delta\{0_E z, x_E y, \lambda x_E y + (1 - \lambda)r^*, z_E 0\})$ for some $\lambda > \lambda^*$, a contradiction.

We now have that $C(\Delta\{0_E z, u_E v, z_E 0\})$ intersects $\Delta\{u_E v, q^*\}$. Suppose it contains neither of $\{u_E v, q^*\}$, then it contains some act $g \equiv \lambda u_E v + (1 - \lambda)q^*$. Increasingness implies that $u^* \succ_C g$, thus continuity implies that $g \succ_C \gamma u^* + (1 - \gamma)q^*$ for some $\gamma \in (0, 1)$. However, $\gamma u^* + (1 - \gamma)q^* \succ_C u_E v$ for any such act by definition of v^* , a contradiction because any menu in which this preference could be revealed also contains acts that dominate g . Thus, $C(\Delta\{0_E z, u_E v, z_E 0\})$ intersects $\{u_E v, q^*\}$. It follows that one of $u_E v \succ_C [\sim_C, \prec_C]q^*$ is true, i.e. $u_E v$ and q^* are comparable.

To see claim (ii), note that symmetry, applied to $C(\Delta\{0_E z, u_E v, z_E 0\})$ and $C(\Delta\{0_F z, u_F v, z_F 0\})$, implies $u_E v \succsim_C q^* \Leftrightarrow u_F v \succsim_C q^*$ for any $q^* \gg v^*$. Suppose that $c(u_E v) \gg v^*$, then this directly implies $c(u_E v) = c(u_F v)$. Suppose that $c(u_E v) \leq v^*$, then it still implies $c(u_F v) \leq v^*$, and monotonicity then yields $c(u_E v) = c(u_F v) = v^*$.¹⁵

Step 2: Fix any Σ -measurable partition of \mathcal{S} into three events (E_1, E_2, E_3) and identify utility acts with vectors $(u, v, w) \in [p, 0]^3$ with obvious interpretation. I will show that for any $u > v$, $c((u + v)/2, (u + v)/2, v) \geq c(u, v, v)$. To see this, consider choice from $M \equiv \{(u, v, v), (v, u, v), c(u, v, v), ((u + v)/2, (u + v)/2, v), (0, 2w, 2w), (2w, 0, 2w), (2w, 2w, 0)\}$ with w as before; as before, use homogeneity of degree zero if $2w$ lies outside the range of U . Noting that M is invariant under interchanging the consequences of E_1 and E_2 , symmetry, convexity, and monotonicity jointly imply that $C(\Delta M) \cap N \neq \emptyset$, where

$$N = [\Delta\{(w, w, 2w), (c((u + v)/2, (u + v)/2, v))\} \\ \cup \Delta\{c((u + v)/2, (u + v)/2, v), c(u, v, v)\} \cup \Delta\{c(u, v, v), (w, w, 0)\}].$$

Note that N entirely resembles the set constructed in step 1. Arguments from that step now imply that $C(\Delta M)$ intersects $\{c((u + v)/2, (u + v)/2, v), c(u, v, v)\}$. Suppose $c(u, v, v) \in C(\Delta M)$, then $\{(u, v, v), (v, u, v)\} \in C(\Delta M)$ by INA, hence $((u + v)/2, (u + v)/2, v) \succsim_C c(u, v, v)$, hence the claim.

Step 3: The preceding steps imply that $c(u, v, v) = c(u, u, v)$, that $c((u + v)/2, (u + v)/2, v) = c((u + v)/2, v, v)$, and that $c((u + v)/2, (u + v)/2, v) \geq c(u, v, v)$. Monotonicity immediately implies that $c(u, u, v) \geq c((u + v)/2, (u + v)/2, v)$. Taking these findings together, one finds that $c(u, v, v) = c((u + v)/2, v, v)$. Iterating this argument and using monotonicity, one finds that $c(u, v, v) = c(w, v, v)$ for any $w \in (v, 0)$. Hence, $c(u, v, v) \sim_C (w, v, v)$ for any $w > v$, but this is consistent with monotonicity only if $c(u, v, v) \leq v^*$. At the same time, $c(u, v, v) \geq v^*$ is immediate from monotonicity, hence $c(u, v, v) = v^*$.

¹⁵The somewhat circuitous argument is needed because $u_E v \sim_C v^*$ can fail, specifically (under the preference ordering characterized here) if $\mathcal{S} \setminus E$ is an atom of Σ , in which case $u_E v$ is not choosable.

This finding can be extended to general acts: Fix any act $f \in \mathcal{C}$, define $\underline{f}[\bar{f}] = \min_{s \in \mathcal{S}}[\max_{s \in \mathcal{S}}]u \circ f(s)$, and let E be the event on which $u \circ f(s) = \underline{f}$. Then $\underline{f}_E \bar{f} \geq f$, hence $c(f) \leq (\underline{f}_E \bar{f}) = \underline{f}^*$, but also $f \geq \underline{f}^* \Rightarrow c(f) \geq \underline{f}^*$. It follows that $c(f) = \underline{f}^*$.

Step 4: Define $f \succsim g$ iff $c(f) \geq c(g)$. In view of the last step's conclusion, \succsim is priorless maximin utility. The proof is concluded by showing that \succsim extends \succsim_C . Suppose $f \succ_C g$ and let M be the menu in which this preference is observed. Consider choice from $N \equiv M \cup \{c(f), c(g)\}$. Step 5 implies that $f \geq c(f)$ and $g \geq c(g)$, hence repeated uses of INA yield $f \in C(\Delta N) \Rightarrow c(f) \in C(\Delta N)$ and $g \notin C(\Delta N) \Rightarrow c(g) \notin C(\Delta N)$, hence $c(f) \gg c(g)$. The argument for $f \sim_C g \Rightarrow f \sim g$ is similar.

Recovering Theorem 4

Step 1: I first establish that if f is choosable, then $f \sim_C c(f)$. To see this, fix f and let M be s.t. $f \in C(\Delta M)$. For any constant act $q^* \gg c(f)$, one has $f \notin C(\Delta(M \cup \{q^*\}))$ and $q^* \in C(\Delta(M \cup \{q^*\}))$. The first of these follows from $q^* \succ_C f$ and INA. To see the second one, note that $f \notin C(\Delta(M \cup \{q^*\}))$ implies $C(\Delta(M \cup \{q^*\})) \cap \Delta M = \emptyset$. If also $q^* \notin C(\Delta(M \cup \{q^*\}))$, c-betweenness would imply $C(\Delta(M \cup \{q^*\})) = \emptyset$, a contradiction.

Suppose now first that $f \prec_C c(f)$. Let M be as in the preceding paragraph, then $C(\Delta(M \cup \{p^*\})) = C(\Delta M)$, thus $f \succ_C p^*$. Continuity now implies that $f \prec_C \lambda c(f) + (1 - \lambda)p^*$ for some $\lambda \in (0, 1)$, contradicting the definition of $c(f)$. Suppose now that $f \succ_C c(f)$, then continuity would imply that $f \succ_C \lambda c(f)$ for some $\lambda < 1$, contradicting the previous paragraph's finding. Suppose that f and $c(f)$ are noncomparable, then $f \notin C(\Delta(M \cup \{c(f)\}))$ but also $c(f) \notin C(\Delta(M \cup \{c(f)\}))$. The former implies that $C(\Delta(M \cup \{c(f)\})) \cap \Delta M = \emptyset$, but c-betweenness then implies that $C(\Delta(M \cup \{c(f)\})) = \emptyset$, a contradiction.

Step 2: To see that \succsim completes \succsim_C , assume first that $f \succ_C g$. Fix M s.t. $C(\Delta M) \cap \{f, g\} = \{f\}$. Consider choice from ΔN , where $N \equiv M \cup \{c(g)\}$. If $c(g)$ were chosen, g would have to be chosen by INA, a contradiction. Thus $c(g)$ is not chosen. Now if $C(\Delta N) \cap \Delta M = \emptyset$, c-betweenness would imply $C(\Delta N) = \emptyset$; thus $C(\Delta N) \cap \Delta M = C(\Delta M)$, in particular $C(\Delta N) \cap \{f, g\} = \{f\}$. Now suppose by contradiction that $c(f) \leq c(g)$, then choice from $\Delta(N \cup \{c(f)\})$ would have to contradict either INA or monotonicity (recalling that $f \succ_C g$ implies f choosable, hence $f \sim_C c(f)$). Thus, $c(f) \gg c(g)$ and hence $f \succ g$. A similar argument establishes that $f \sim_C g \Rightarrow f \sim g$ (again, recall that $f \sim_C g$ implies f, g choosable, hence $f \sim_C c(f), g \sim_C c(g)$).

Step 3: \succsim fulfils Gilboa and Schmeidler's (1989) axioms. It is complete and transitive by construction. C-independence follows by mimicking the proof of lemma 10, all steps of which are easily

adapted to convex menus. Nontriviality follows immediately from nontriviality of C . To see continuity, suppose by contradiction that $f \succ g \succ h$ yet $\lambda f + (1-\lambda)h \precsim g$ for all $\lambda \in (0, 1)$. Let p^* be a constant act s.t. $c(f) \gg p^* \gg c(g)$. Fix a menu $M \supseteq \{f, h\}$ s.t. $C(\Delta M) \cap \{f, h\} = \{f\}$ and consider $N = M \cup \{p^*\}$. $p^* \notin C(\Delta N)$ because \succsim induces choice function C . For the same reason, $h \notin C(\Delta(N \setminus \{f\}))$. Also using c-betweenness and INA, one finds that $p^* \in C(\Delta(N \setminus \{f\} \cup \{\lambda f + (1-\lambda)h\}))$ for every $\lambda \in (0, 1)$, contradicting continuity of C .

Theorem 9 Recall that theorem 4 applies. First restrict attention to acts with nonpositive utility range. For these, define the incomplete ordering \succeq by

$$f \succeq g \iff \lambda f + (1-\lambda)h \succsim \lambda g + (1-\lambda)h, \forall \lambda \in (0, 1], h \in \mathcal{F}_-$$

Then by proposition 5 in Ghirardato et al. (2004; see also Bewley 2002), there exists a unique, compact, convex, set of probabilities $\tilde{\Gamma}$ s.t.

$$f \succeq g \iff \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \tilde{\Gamma}.$$

Now, proposition 19 of the same paper (used with $\alpha = 1$) implies that $\tilde{\Gamma} = \Gamma$. It immediately follows that $f \succeq g \Rightarrow f \succeq_C g$. To show the converse, fix f and g and choose any h and λ . For any $M \in \mathcal{M}_0$, let $M' = (M - (1-\lambda)h)/\lambda$. (Unbounded utility is required for existence of M' .) Then independence implies that $f \in C(M) \Leftrightarrow \lambda f + (1-\lambda)h \in C(M')$ and $g \in C(M) \Leftrightarrow \lambda g + (1-\lambda)h \in C(M')$, thus the claim.

To extend the result to acts with positive utility range, fix any acts f and g and define the act $f \vee g$ by $u \circ (f \vee g)(s) = \max\{u \circ f(s), u \circ g(s)\}$. Then for any menu M , independence implies that $f \in C(M) \Leftrightarrow (f - f \vee g)/2 \in C((M - f \vee g)/2)$ and similarly for g , but $(f - f \vee g)/2, (g - f \vee g)/2$ have nonpositive utility range.

Corollary 9 Define \succeq^* by

$$f \succeq^* g \iff \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

Then axiom 10 states that $f \succeq^* g \Rightarrow f \succeq g$, whereas axiom 11 states that $f \succeq g \Rightarrow f \succeq^* g$. The claim then follows from theorem 7.

Theorem 10 To begin, the proof of lemma 1 goes through unchanged. For any menu M , redefine the ‘‘oracle act’’ \bar{f}_M by $u \circ \bar{f}_M(s) \equiv \max_{f \in M \cap \mathcal{A}^*} u \circ f(s)$. In the proof of lemma 2, the second paragraph of step 1 must be modified as follows: ‘‘Nonconstancy of U and existence of $p^*, q^* \in \mathcal{F}^*$ with $C(\{p^*, q^*\}) = p^*$ implies that (i) after normalization, $U^{-1}(-1)$ and $U^{-1}(1)$ can be assumed to exist,

(ii) one can assume $u \circ p^*(s) = 1$ and $u \circ q^*(s) = -1$ for all s . Closure under statewise recombination and mixture of \mathcal{A}^* now implies that any finite, Σ -measurable step function $u \circ f : \mathcal{S} \rightarrow [-1, 1]$ can be identified with an act $f \in \mathcal{F}^*$. This specifically includes p_0^* , the constant act with utility value 0. Independence implies that $C(\lambda M + (1 - \lambda)p_0^*) = \lambda C(M) + (1 - \lambda)p_0^*$. However, for any act f , the act $\lambda f + (1 - \lambda)p_0^*$ is characterized by $u \circ (\lambda f + (1 - \lambda)p_0^*)(s) = \lambda u \circ f(s)$ for all s , hence C is homogeneous of degree zero. For future use, write λf for the mixture act just defined, then for any menu M and scalar $\lambda \in (0, 1)$, the menu λM exists and $C(\lambda M) = \lambda C(M)$." The remainder of the paragraph goes through unchanged. Note finally that step 4 now also uses the closure properties of \mathcal{F}^* . The proof of theorem 3 is unchanged.

Axiom 9 does not yield the analog result because c-independence can only be derived separately for the upper and lower \succsim -contour sets of p_0^* (corresponding to the restrictions of C to menus M s.t. $\bar{f}_M \in C(M \cup \{\bar{f}_M\})$ respectively to menus M with $\bar{f}_M \notin C(M \cup \{\bar{f}_M\})$). As a result, preferences within either contour set can be identified with maxmin expected utility, where furthermore u is constant across contour sets, but the set of priors Γ can change. While disjointness of contour sets enforces that Γ increases as one moves from lower to upper contour, constancy is not implied. Note that strict expansion of the set of priors implies that the indifference set containing p_0^* is thick, hence the case can be excluded by any axiom that precludes thick indifference sets.

References

- [1] Anscombe, F.J. and R.J. Aumann (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 34: 199-205.
- [2] Arias, J.P., J. Hernández, J. Martín, and A. Suárez (2003): "Bayesian Robustness with Quantile Loss Functions," in J.M. Bernard, T. Seidenfeld, and M. Zaffalon (Eds.), *Proceedings of the Third International Symposium on Imprecise Probabilities and their Applications*.
- [3] Arrow, K.J. and L. Hurwicz (1972): "An Optimality Criterion for Decision-Making under Ignorance," in C.F. Carter and J.L. Ford (Eds.), *Uncertainty and Expectations in Economics: Essays in Honour of G.L.S. Shackle*. Oxford: Basil Blackwell.
- [4] Arrow, K.J. (1959): "Rational Choice Functions and Orderings," *Economica* 26: 121-127.
- [5] Bergemann, D. and K.H. Schlag (2007): "Robust Monopoly Pricing," Cowles Foundation Discussion Paper.
- [6] — (2008): "Pricing without Priors," *Journal of the European Economic Association (Papers and Proceedings)* 6: 560-569.

- [7] Bewley, T. (2002): “Knightian Decision Theory: Part I,” *Decisions in Economics and Finance* 25: 79–110.
- [8] Borodin, A. and R. El-Yaniv (1998): *Online Computation and Competitive Analysis*. Cambridge, New York: Cambridge University Press.
- [9] Brock, W.A. (2006): “Profiling Problems with Partially Identified Structure,” *Economic Journal* 92: F427-F440.
- [10] Cesa-Bianchi, N. and G. Lugosi (2006): *Prediction, Learning, and Games*. Cambridge University Press.
- [11] Chamberlain, G. (2000): “Econometrics and Decision Theory,” *Journal of Econometrics* 95: 255-283.
- [12] Chernoff, H. (1954): “Rational Selection of Decision Functions,” *Econometrica* 22: 422-443.
- [13] Chew, S.H. (1983): “A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox,” *Econometrica* 51: 1065-1092.
- [14] Cohen, M. and J.-Y. Jaffray (1980): “Rational Behavior under Complete Ignorance,” *Econometrica* 48: 1281-1299.
- [15] DasGupta, A. and W. Studden (1991): “Robust Bayesian Experimental Designs in Normal Linear Models,” *Annals of Statistics* 19: 1244-1256.
- [16] Dekel, E. (1986): “An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom,” *Journal of Economic Theory* 40: 304-318.
- [17] Droge, B. (1998): “Minimax Regret Analysis of Orthogonal Series Regression Estimation: Selection Versus Shrinkage,” *Biometrika* 85: 631-643.
- [18] — (2006): “Minimax Regret Comparison of Hard and Soft Thresholding for Estimating a Bounded Normal Mean,” *Statistics and Probability Letters* 76: 83-92.
- [19] Eldar, Y.C., A. Ben-Tal, and A. Nemirovski (2004): “Linear Minimax Regret Estimation of Deterministic Parameters with Bounded Data Uncertainties,” *IEEE Transactions on Signal Processing* 52: 2177-2188.
- [20] Eliaz, K. and E.A. Ok (2006): “Indifference or Indecisiveness? Choice-theoretic Foundations of Incomplete Preferences,” *Games and Economic Behavior* 56: 61-86.
- [21] Eozenou, P., J. Rivas, and K.H. Schlag (2006): “Minimax Regret in Practice – Four Examples on Treatment Choice,” mimeo, European University Institute.

- [22] Fishburn, P.C. (1989): “Non-transitive Measurable Utility for Decisions under Uncertainty,” *Journal of Mathematical Economics* 18: 187-207.
- [23] Gajdos, T., J.-M. Tallon, and J.-C. Vergnaud (2004): “Decision Making with Imprecise Probabilistic Information,” *Journal of Mathematical Economics* 40: 647-681.
- [24] Gajdos, T., T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud (2008): “Attitude Toward Imprecise Information,” *Journal of Economic Theory* 140: 27-65.
- [25] Ghirardato, P., F. Maccheroni, and M. Marinacci (2004): “Differentiating Ambiguity and Ambiguity Attitude,” *Journal of Economic Theory* 118: 133-173.
- [26] — (2005): “Certainty Independence and the Separation of Utility and Beliefs,” *Journal of Economic Theory* 120: 129-136.
- [27] Ghirardato, P. and M. Marinacci (2002): “Ambiguity Made Precise: A Comparative Foundation,” *Journal of Economic Theory* 102: 251-289.
- [28] Gilboa, I. and D. Schmeidler (1989): “Maxmin Expected Utility with Non-unique Prior,” *Journal of Mathematical Economics* 18: 141-153.
- [29] Gul, F. and W. Pesendorfer (2001): “Temptation and Self-Control,” *Econometrica* 69: 1403-1435.
- [30] Hannan, J. (1957): “Approximation of Bayes Risk in Repeated Play,” in M. Dresher, A.W. Tucker, and P. Wolfe (Eds.), *Contributions to the Theory of Games (Vol. III)*. Princeton: Princeton University Press.
- [31] Hansen, B.E. (2005): “Exact Mean Integrated Squared Error of Higher-Order Kernels,” *Econometric Theory* 21: 1031-1057.
- [32] Hart, S. and A. Mas-Colell (2001): “A General Class of Adaptive Strategies,” *Journal of Economic Theory* 98: 26-54.
- [33] Hayashi, T. (2008): “Regret Aversion and Opportunity-dependence,” *Journal of Economic Theory* 139: 242-268.
- [34] Krämer, D. and R. Stone (2006): “Regret and Ambiguity Aversion,” mimeo, Freie Universität Berlin and University of Leicester.
- [35] Kreps, D. (1979): “A Preference for Flexibility,” *Econometrica* 47: 565-576.
- [36] Loomes, G. and R. Sugden (1982): “Regret Theory: An Alternative Theory of Rational Choice Under Uncertainty,” *Economic Journal* 92: 805-824.
- [37] Manski, C.F. (2000): “Identification Problems and Decisions Under Ambiguity: Empirical Analysis of Treatment Response and Normative Analysis of Treatment Choice,” *Journal of Econometrics* 95: 415-442.

- [38] — (2004): “Statistical Treatment Rules for Heterogeneous Populations,” *Econometrica* 72: 1221-1246.
- [39] — (2006): “Search Profiling with Partial Knowledge of Deterrence,” *Economic Journal* 116: F385-F401.
- [40] — (2007): “Minimax-Regret Treatment Choice with Missing Outcome Data,” *Journal of Econometrics* 139: 105-115.
- [41] Milnor, J. (1954): “Games Against Nature,” in R.M. Thrall, C.H. Coombs, and R.L. Davis (Eds.), *Decision Processes*. New York: Wiley.
- [42] Ortoleva, P. (2008): “Status Quo Bias, Ambiguity and Rational Choice,” mimeo, New York University.
- [43] Puppe, C. and K.H. Schlag (2008): “Choice under Complete Uncertainty when Outcome Spaces are State-Dependent,” forthcoming in *Theory and Decision*.
- [44] Renou, L. and K.H. Schlag (2008): “Minimax Regret and Strategic Uncertainty,” mimeo, University of Leicester and Universitat Pompeu Fabra.
- [45] Sarver, T. (2008): “Anticipating Regret: Why Fewer Options May be Better,” *Econometrica* 76: 263-305.
- [46] Savage, L.J. (1951): “The Theory of Statistical Decision,” *Journal of the American Statistical Association* 46: 55-67.
- [47] Schlag, K.H. (2003): “How to Minimize Maximum Regret under Repeated Decision-Making,” mimeo, European University Institute.
- [48] — (2007): “Eleven – Designing Randomized Experiments under Minimax Regret,” mimeo, European University Institute.
- [49] Schmeidler, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica* 57: 571-587.
- [50] Sen, A.K. (1971): “Choice Functions and Revealed Preference,” *Review of Economic Studies* 38: 307-317.
- [51] Seo, K. (2008): “Ambiguity and Second-Order Belief,” forthcoming in *Econometrica*.
- [52] Stoye, J. (2006): “Statistical Decisions under Ambiguity,” mimeo, New York University.
- [53] — (2007a): “Minimax Regret Treatment Choice with Finite Samples,” mimeo, New York University.
- [54] — (2007b): “Minimax Regret Treatment Choice with Incomplete Data and Many Treatments,” *Econometric Theory* 23: 190-199.

- [55] — (2008a): “Minimax Regret Treatment Choice with Missing Data: An Application to Young Offenders,” forthcoming in the *Journal of Statistical Theory and Practice*.
- [56] — (2008b): “Revealed Preference when Agents can Randomize,” mimeo, New York University.
- [57] Sugden, R. (1993): “An Axiomatic Foundation for Regret Theory,” *Journal of Economic Theory* 60: 159-180.
- [58] Sweeting, T.J., G.S. Datta, and M. Ghosh (2006): “Nonsubjective Priors via Predictive Relative Entropy Regret,” *Annals of Statistics* 34: 441-468.
- [59] Wald, A. (1950): *Statistical Decision Functions*. New York: Wiley.
- [60] Wasserman, L. and J.B. Kadane (1992): “Computing Bounds on Expectations,” *Journal of the American Statistical Association* 87: 516-522, 1992.
- [61] Zen, M.-M. and A. DasGupta (1993): “Estimating a Binomial Parameter: Is Robust Bayes Real Bayes?,” *Statistics and Decisions* 11: 37-60.