

Axioms for Minimax Regret Choice Correspondences

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Abstract

This paper unifies and extends the recent axiomatic literature on minimax regret. It compares several models of minimax regret, shows how to characterize the according choice correspondences in a unified setting, extends them to choice from convex sets, connects them by defining a behavioral notion of perceived ambiguity, and provides an axiomatization of Hannan regret. Substantively, a main idea is to behaviorally identify ambiguity with failures of independence of irrelevant alternatives. Regarding proof technique, the core contribution is to uncover a dualism between choice correspondences and preferences in an environment where this dualism is not obvious. This insight can be used to generate results by importing findings from the existing literature on preference orderings.

Keywords: Minimax regret, Hannan regret, ambiguity, multiple priors, choice correspondences.

JEL classification codes: C44, D81.

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1 Introduction

1.1 Motivation

The minimax regret decision criterion was first suggested in Savage’s (1951) reading of Wald (1950) and has since seen occasional use in statistics (DasGupta and Studden (1991), Droge (1998, 2006), Eldar et al. (2004)). Interest in minimax regret has recently increased among econometricians (see Stoye (2009c) for a compilation of references) as well as economic theorists (Bergemann and Schlag (2007, 2008), Eozenou et al. (2006), Schlag (2003, 2007); substantively, some of these papers overlap with econometrics). Its decision theoretic foundations, the classic reference for which is Milnor (1954), were revisited as well, specifically in recent work by Hayashi (2008) and Stoye (2006).¹

The present paper provides further insight into axiomatic characterizations of minimax regret. It is partly motivated by the observation that, although all of the aforementioned authors talk about something they call “minimax regret,” there are significant differences between their conception of this object. I hope to clarify some discussions of regret by elaborating on these differences. Building on this, I provide a unified characterization of different “minimax regret” objects in a common framework and axiomatic system. I also give a behavioral characterization of perceived ambiguity and show how to vary the benchmark according to which regret is computed. These are this paper’s main substantive contributions. One leitmotif that connects them is the idea to identify perceived ambiguity with violations of independence of irrelevant alternatives (IIA). Insofar as IIA is the standard axiom that will not be imposed in this paper, this identification resembles Ghirardato et al.’s (2004) identification of perceived ambiguity with violations of von Neumann-Morgenstern independence. This similarity will turn out to be much more than semantic. The paper’s main technical contribution is a proof technique which recovers a dualism between choice correspondences and preference orderings in an environment where this dualism is not obvious. This allows one to adapt existing results on preferences to statements about regret-based choice correspondences. A major exception to this is the extension of most results to convex sets, i.e. agents who can randomize. This is the second main contribution of this paper and requires new ideas beyond said dualism.

I unify previous axiomatizations of regret along two dimensions. First, one can think of a minimax regret preference ordering or of the according choice correspondence. All applications are phrased in terms of the preference ordering, and it is this ordering that Stoye (2006) analyzes. However, this ordering is menu dependent, i.e. the ranking of acts can depend on the feasible set within which they are compared. As a result, axiomatizations of minimax regret preferences are at tension with the revealed preference paradigm. To illustrate, consider the statement $f \succ g \succ h$, where the menu in question is

¹Both also do other things to which this paper is less related – Stoye (2006) by looking at more preference orderings, Hayashi (2008) by formalizing a notion of smooth (non-minimax) regret aversion.

$\{f, g, h\}$. Choice from the menu reveals that $f \succ g$ and $f \succ h$, but not that $g \succ h$. If preferences do not depend on menus, an obvious dualism between preferences and choice correspondences resolves the problem – $g \succ h$ can be inferred from choices over $\{g, h\}$. With menu-dependent preferences, this dualism breaks down, and choice from $\{g, h\}$ will not reveal the ranking of those same acts within $\{f, g, h\}$. Indeed, the second part of $f \succ g \succ h$ need not map onto revealed preference for g over h in *any* menu. One could accordingly prefer to not axiomatize it, that is, one could restrict attention to the minimax regret choice correspondence, as is done by Hayashi (2008). The present paper adopts the choice correspondence approach, but on a deeper level, shows that the two perspectives continue to be tightly related because a somewhat different dualism can be uncovered.

Second, minimax regret can be thought of as presuming no priors, endogenous priors, or exogenous priors.

(i) No priors: Milnor (1954) and Stoye (2006) axiomatize the preference ordering represented by (the negative of)

$$\max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\},$$

where f and g are acts in a menu M , \mathcal{S} is a state space, and u is an expected utility functional; I will explain notation in detail below. The idea here is that \mathcal{S} reflects the objective ambiguity inherent in a situation.

In applications, prior-less minimax regret was recently used by Bergemann and Schlag (2008), Manski (2004, 2005, 2009), Schlag (2003, 2007), and Stoye (2007a, 2009a) and also underlies Hannan regret (to be elaborated later). For example, Manski considers the problem of treatment choice – be it assignment to on-the-job training programs or to medical treatment – as a statistical decision problem and compares the risk functions generated by different statistical treatment rules. He advocates the use of minimax regret risk as decision criterion, but certainly not the use of priors.² Indeed, as this version of minimax regret is the only one that can be interpreted without any notion of priors, it is the one that frequentist statisticians must have in mind and that corresponds to Savage’s (1951) original suggestion.

(ii) Endogenous Priors: Hayashi (2008) axiomatizes minimization of

$$\max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi,$$

²Notation in statistics is typically somewhat different. In short, statisticians postulate a set of conceivable data generating processes and a loss function. A risk function maps any combination of true data generating process and statistical decision rule onto the implied expectation of loss. Noting that data generating processes correspond to states of the world, statistical decision rules to acts, and loss to (the negative of) utility, risk functions are seen to map onto the functional u (i.e., utility acts). The parallels are explained in detail in Stoye (2009b).

where the “set of priors” $\Gamma \subseteq \Delta\mathcal{S}$ is behavioral or “as if.” Mathematically, this generalizes the previous expression because Γ could equal $\Delta\mathcal{S}$. This approach is in line with Gilboa and Schmeidler (1989) and the large literature that builds on them. It may be the most interesting one for a theorist since we cannot typically observe people’s (sets of) beliefs; hence, these axiomatizations are most revealing regarding a theory’s observable implications. Endogenous prior minimax regret was recently applied to the modelling of games by Renou and Schlag (2008).

(iii) Exogenous Priors: An intermediate possibility is that the representation is as in (ii) but Γ is a feature of the environment. This might appear unusual to decision theorists, but is faithful to many applications. The quintessential example is the robust (multi-prior) Bayesian literature (Berger (1985)), which first specifies a set of priors and hence the extent of ambiguity in a decision situation and subsequently thinks about how to make decisions. Indeed, many contributions make the first and not the second step (e.g., Wasserman and Kadane (1992)), and the literature “does not as yet contain substantial work on how exactly a specific action should be chosen” (Zen and DasGupta (1993); see also Arias et al. (2003)). If specification of set valued beliefs precedes the contemplation of action, a faithful model of the latter requires that the beliefs are part of the decision theoretic environment, and that beliefs revealed by choices are axiomatically linked to them. Hence, a characterization of minimax regret with exogenous priors would provide axiomatic foundations for Γ -minimax regret (Berger (1985); see Bergemann and Schlag (2007) or Chamberlain (2000) for applications in economics). Indeed, I use the symbol Γ for sets of priors to emphasize this link. Exogenous sets of priors were recently considered by Gajdos et al. (2004) and Gajdos et al. (2008), but these papers are not about regret.

To repeat, which of these possibilities appears most interesting depends on the desired application. The typical, namely descriptive and behavioral, application in economic theory will rely on case (ii). On the other hand, although this paper is firmly rooted in the revealed preference paradigm, it is partly motivated by statistical and econometric applications. Any frequentist application avoids priors and consequently, is an example of case (i). When statistical applications do use (sets of) priors, these priors should typically be thought of as exogenous; these cases, most notably Γ -minimax regret, therefore fall under case (iii). I will characterize all three cases.

1.2 Overview and Brief Summary

This paper provides and compares characterizations of minimax regret choice correspondences for all of (i)-(iii) above. The representations can be connected via a behavioral characterization of perceived ambiguity. They rely on a proof technique which puts into sharp focus the role of certain axioms, recovers a duality between choice correspondences and preference orderings, and can be extended to

generate more results. The results are new with the exception of theorem 4, which resembles a core finding in Hayashi (2008). In that theorem’s case, the main contribution lies in the new proof and the embedding of results. The extension to convex sets is entirely new.

An overview of the paper’s structure and brief summary of results goes as follows. Section 2 begins by describing the decision theoretic environment and stating axioms. I then establish a lemma that generates the aforementioned connection between choice correspondences and preference orderings. Characterizations of prior-less minimax regret (by importing a result of Stoye (2006)) and endogenous prior minimax regret (by importing Gilboa and Schmeidler 1989) follow easily. The former result is a choice correspondence analog of Stoye’s (2006) minimax regret representation. The latter result is substantively similar to one in Hayashi (2008), so I essentially derive his representation from Gilboa and Schmeidler (1989).

Some extensions follow. First, these results can be recovered under the assumption that agents can randomize, which convexifies all choice sets and substantially reduces the domain on which revealed preference arguments can be applied. One can also use a link to previous work on multi-prior Pareto criteria by Bewley (2002) and Ghirardato et al. (2004) to characterize a notion of ambiguity perception. This technique also leads to an axiomatization that identifies the object Γ with an exogenous set of priors. I finally connect Hanman regret to the literature by providing its first (to my knowledge) axiomatization. The connection is established by varying the “regret benchmark,” i.e. the utility frontier that generates regret, by varying the domain of an independence of irrelevant alternatives axiom. Section 4 concludes and offers comparisons to other notions of regret in the literature.

2 Axiomatic Analysis

2.1 Preliminaries and Axioms

The setup is inspired by Anscombe and Aumann (1963). There is a set \mathcal{S} of states of the world s , endowed with an algebra Σ of events E, F , etc.; a set \mathcal{X} of outcomes x ; and a set \mathcal{F} of possible acts f, g , etc. The only restrictions on \mathcal{S}, \mathcal{X} , and Σ are that \mathcal{X} must have at least two elements, that theorem 3 requires the existence of three distinct, nonempty events. An act f is a Σ -measurable, finite step function $f : \mathcal{S} \rightarrow \Delta\mathcal{X}$ that maps states s onto finite outcome distributions $f(s)$. An act is *constant* if f does not depend on s . I embed $\Delta\mathcal{X}$ in \mathcal{F} by writing p^* for the constant act associated with $p \in \Delta\mathcal{X}$. Mixtures of acts are identified with statewise mixtures, i.e. $h \equiv \lambda f + (1 - \lambda)g$ is generated by performing f with probability λ and g otherwise and is characterized by $h(s) = \lambda f(s) + (1 - \lambda)g(s)$. The word “convex” henceforth denotes closure under such mixture. The decision maker can choose from a finite, nonempty menu $M \subseteq \mathcal{F}$. Randomization is not allowed; this will be relaxed later. For

any menu M , $\lambda M + (1 - \lambda)g$ denotes the menu generated by replacing every $f \in M$ with the analog mixture. I do not presume existence of a preference ordering but of a choice correspondence C that maps every M onto some nonempty $C(M) \subseteq M$. (This is a good moment to emphasize that in this paper, \subseteq and \subset are distinct symbols.) I also define the problem of choosing an act after state s has been learned. The according choice correspondence will be labelled C_s . As is standard in the literature, I impose some notion of dynamic consistency by assuming that choice after revelation of s is equivalent to choice from constant acts; more formally, $f \in C_s(M)$ iff $(f(s))^* \in C(\{(g(s))^* : g \in M\})$. I call an act f *strictly potentially optimal* in M if there exists s s.t. $C_s(M) = \{f\}$. Finally, for future use, let $\Delta\mathcal{S}$ denote the set of Σ -measurable distributions on \mathcal{S} .

The following axioms on C are maintained throughout this paper.

Axiom 1 *Nontriviality*

$$\exists M : C(M) \subset M.$$

Axiom 2 *Monotonicity*

If $f \in M$, $g \in C(M)$, and $f \in C_s(\{f, g\})$ for all s , then $f \in C(M)$.

Axiom 2*: *Strict Monotonicity*

If $f \in M$, $g \in M$, and $C_s(\{f, g\}) = \{f\}$ for all s , then $g \notin C(M)$.

Axiom 3 *Independence*

$$C(\lambda M + (1 - \lambda)f) = \lambda C(M) + (1 - \lambda)f.$$

Axiom 4 *Ambiguity Aversion*

$C(M)$ is convex relative to M (the intersection of M with a convex set).

Axiom 5 *Independence of Irrelevant Alternatives (IIA) for Unambiguous Choices*

Let M and N consist of constant acts, then

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Axiom 6 *Independence of Never Strictly Optimal Alternatives (INA)*

Let M and N be such that $C_s(M \cup N) \cap M \neq \emptyset$ for all s . Then

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Axiom 7 Mixture Continuity (Archimedean Property)

Fix any acts f, g, h and menu M s.t. $h \in M \setminus C(M), g \in C(M), C(M \cup \{f\}) = \{f\}$. Then:

$$\begin{aligned} \implies \exists \lambda, \gamma \in (0, 1) : \quad & C(M \cup \{\lambda f + (1 - \lambda)h\}) = \{\lambda f + (1 - \lambda)h\}, \\ & \{\gamma f + (1 - \gamma)h\} \notin C(M \cup \{\gamma f + (1 - \gamma)h\}). \end{aligned}$$

Some remarks on the axioms are in order. Monotonicity states that if the agent would choose f from $\{f, g\}$ in every state of the world, then she cannot revealed prefer g over f in any menu. It is the revealed preference equivalent of the axiom of the same name in Gilboa and Schmeidler (1989) and other references. Its stronger (given continuity) version is only needed for a special purpose in section 3.3 below. Independence requires that choice is invariant under mixing of *entire menus* with some act. One intuition for this comes from the following thought experiment. Suppose an agent chooses from a menu, but then learns that her choice will be actualized only conditional on heads in a previous coin toss; she has no control over what will happen conditional on tails. Then it can be argued that her choice behavior should not be affected.³ The same adaptation of independence was recently used by Eliaz and Ok (2006) and Ortoleva (2008).

Axioms 5 and 6 touch upon a crucial, and controversial, feature of minimax regret, namely that it violates independence of irrelevant alternatives (IIA). This axiom would translate into the present setting as

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}, \forall M, N,$$

meaning that preferences revealed by $C(M \cup N)$ cannot contradict those revealed by $C(M)$: The set chosen from $M \cup N$ is either strictly revealed preferred to $C(M)$, in which case it cannot contain elements of the latter, or it is revealed indifferent to $C(M)$, in which case it contains all of it.⁴

In this paper, IIA will only be imposed on restricted domains. Specifically, axioms 5 and 6, as well as several axioms to come, introduce a leitmotif: Comparison of $C(M)$ and $C(M \cup N)$ may reveal violations of IIA if the ambiguity perceived in a choice problem changes as one expands M to $M \cup N$.

³The leap from the thought experiment to the axiom relies on a hidden assumption of compound roulette lottery reduction. Here as in many other references, that axiom is implicit in the notation for mixture acts; see Seo (2009) for a treatment that makes it explicit.

⁴See Arrow (1959, definition C.4). To further underscore that this is a natural formulation of IIA, compare it to Sen's (1971) consistency requirements for choice correspondences. In the present notation, these can be written as follows:

$$\begin{aligned} f \in C(M \cup N) \cap M &\implies f \in C(M), \\ \{f, g\} \in C(M) &\implies C(M \cup N) \cap \{f, g\} \in \{\{f, g\}, \emptyset\}. \end{aligned}$$

The former (“property α ,” “contraction consistency”) can be traced back at least to Chernoff (1954). It ensures that choice correspondences cannot shrink upon contraction of sets, but allows for them to expand. The second condition (“property β ,” “expansion consistency”) prevents this by imposing that revealed indifferences are not contradicted upon expansion of sets. In this paper’s setting, the conjunction of both is equivalent to IIA.

This cannot be the case if both M and N consist of constant acts, because then there is no ambiguity to begin with; hence axiom 5. Furthermore, axiom 6 (INA) specifies that it does not happen if acts added to menus are not strictly potentially optimal. To see this, and to understand the axiom's name, note that menus M and N fulfil the axiom's hypothesis iff enlarging M to $M \cup N$ does not add any act f that would be strictly potentially optimal in $M \cup \{f\}$. An intuition for this restriction is that the agent's attitude to one and the same outcome in different states may be influenced by what could have been achieved in a state, hence the nature of an act's ambiguity may change with this information. This intuition is obviously related to the concept of regret, and INA accordingly plays a major role in enforcing regret-based choices.⁵

Mixture continuity usually presumes that $f \succ g \succ h$ within some menu and concludes that $\lambda f + (1 - \lambda)h \succ g \succ \gamma f + (1 - \gamma)h$ for some λ, γ . This cannot be done here for reasons explained in the introduction. To see that axiom 7 is a rather literal adaptation, note the hypothesis states that f is revealed preferred to g and g is revealed preferred to h , albeit in different menus. Ambiguity aversion translates an axiom proposed by Schmeidler (1989); see Milnor (1954) for a precursor. It is equivalent to Hayashi's (2008) regret aversion. Note that unlike in most other contexts, the axiom is not a weakening of independence because it would follow from the latter only in conjunction with IIA.

Consider now the following axiom.

Axiom 8 Symmetry

For any menu M , let $E, F \in \Sigma \setminus \{\emptyset\}$ be any two disjoint events s.t. for any $f \in M$, $f(s)$ is constant on E as well as F . Define f' by

$$f'(s) = \begin{cases} f(s)|_{s \in E}, & s \in F \\ f(s)|_{s \in F}, & s \in E \\ f(s) & \text{otherwise} \end{cases} .$$

Let the function $(\cdot)': 2^{\mathcal{A}} \rightarrow 2^{\mathcal{A}}$ map every set of acts $N \subseteq \mathcal{A}$ onto $N' \equiv \{f' : f \in N\}$. Then

$$C(M') = (C(M))'.$$

In words, symmetry states the following. Take any two events such that all acts in a menu are constant on either event, then exchanging the consequences of the events for every act in the menu does not affect choices. This is certainly not an innocuous condition – the two events might be of very different size with respect to some measure on state space. Indeed, symmetry enforces a strong attitude of ignorance regarding elements of \mathcal{S} , specifically a refusal to weight them according to some

⁵Krähmer and Stone (2006) give a similar, informational motivation for regret, although their technical notion of regret is outside this paper's scope. Essentially the same axiom as INA is used in Hayashi (2008), but the original source is Milnor (1954, "special row adjunction").

importance criterion like subjective probability. The idea that prior ignorance about events should be modelled in this way is due to Arrow and Hurwicz (1972), who specify similar axioms for choice correspondences; see also Cohen and Jaffray (1980) for a preference-based formulation that is otherwise analog to the above.

Symmetry is implausible if one has available, and wishes to consider, prior information about states, respectively if one wishes to model agents who have and use such information. On the other hand, if no prior information exists, the axiom is compelling because decisions would otherwise be sensitive to manipulations of the state space, e.g. the relabeling of states or their duplication via conditioning on trivial events (Arrow and Hurwicz (1972)). These observations are as they should be, given that symmetry will turn out to characterize prior-less minimax regret. For the intermediate case of *vague* prior information – not enough to commit to a prior but sufficient to cast doubt on symmetry –, I would point to the characterization of exogenous prior minimax regret below.

I finally consider the following axiom. Say that a menu has a *state independent outcome distributions* if the set $\{f(s) : f \in M\}$ does not vary with s . In words, the set of feasible outcome lotteries is constant across states. Notice that any menu consisting of constant acts induces state independent outcome distributions, but a menu can have this property without containing any constant act. Then one might consider the following.

Axiom 9 C-Betweenness When Outcome Distributions are State Independent

For any act f , constant act p^ , scalar $\lambda \in (0, 1)$, and menu $M \supseteq \{p^*, f, \lambda f + (1 - \lambda)p^*\}$ with state independent outcome distributions, if $p^* \notin C(M)$ and $f \notin C(M)$, then $\lambda f + (1 - \lambda)p^* \notin C(M)$.*

C-betweenness for state independent outcome distributions is related to betweenness (Chew (1983), Dekel (1986)), which states that if two acts are ranked indifferent, then they are also ranked indifferent to any mixture between them. Rewriting the axiom in terms of choice correspondences requires some tweaking. One might want to think of revealed indifference as the absence of revealed strict preference, thus acts in a menu are revealed indifferent if either both or none are chosen. Betweenness would then require that any mixture of the two acts is revealed indifferent as well. If both acts are chosen, this is already implied by ambiguity aversion, so I merely need to impose it for the case that neither option is chosen.⁶

The motivation of c-betweenness is related to the usual motivation of c-independence: It limits the scope of ambiguity aversion, that is, of preferences for mixtures. Intuitively, a decision maker might strictly prefer mixtures because they constitute a hedging of bets across ambiguous states. Both

⁶Intuitively, it may seem a stretch to think of this case as revealed indifference. While preferences over non-chosen acts are not axiomatized, one of those acts might be commonsensically very unattractive. However, when that happens, one would not worry about imposing the axiom's conclusion anyway.

independence and betweenness stipulate that this will not happen in certain cases. C-independence states that mixture with a constant act cannot constitute a hedging of bets, just like purchase of a safe asset does not hedge any risks in a conventional portfolio problem. Axiom 9 further tightens the conditions under which a hedge is denied, thus weakening the axiom. The first tightening is that the menu must have state independent outcome distributions. The idea here is to once again acknowledge that ambiguity can arise not just because outcomes differ across states, but also because the evaluation of outcomes might depend on what could have been achieved. The restriction shuts down the latter channel: Observed outcomes may still be informative about which state occurred, and hence about which act should have been chosen, but if the set of ex post feasible outcome distributions is state independent, then none of this information matters for what could have been achieved. The second tightening is that the axiom does not apply to mixture of f and some other act g with a third, constant act, but only to mixture of f with a constant act that is already ranked indifferent to it. This is why the axiom should be thought of as a weakening of betweenness and not just of independence. Substantively, it sharpens the focus on preferences for or against randomization. By applying for all constant acts from very bad to very good ones, c-independence additionally limits the dependence of ambiguity attitudes upon stakes.

I conclude this section by stating a straightforward result. Specifically, a subset of axioms that will be maintained throughout implies that C extends some expected utility choice correspondence \tilde{C} in the sense of agreeing with it on unambiguous choice problems.

Lemma 1 *A choice correspondence fulfils axioms 1, 3, 5, and 7 iff choices from sets of constant acts are consistent with maximization of von Neumann-Morgenstern expected utility. Thus, there exists a unique (up to affine transformation), nonconstant function $U : \mathcal{X} \mapsto \mathbb{R}$ s.t. the restriction \tilde{C} of C to menus \tilde{M} consisting of constant acts p^* is*

$$\tilde{C}(\tilde{M}) = \arg \max_{p^* \in \tilde{M}} \int U(x) dp^*.$$

2.2 Characterizations of Minimax Regret

This section is devoted to characterizing different versions of minimax regret. The key to doing this is contained in the following lemma.

Lemma 2 *A choice function fulfils axioms 1 through 7 iff it can be represented as*

$$C(M) = \arg \min_{f \in M} I(r \circ (f, M)),$$

where the finite step function $r \circ (f, M) : \mathcal{S} \mapsto \mathbb{R}^+$ is defined by

$$\begin{aligned} (r \circ (f, M))(s) &\equiv \max_{g \in M} u \circ g(s) - u \circ f(s) \\ u \circ f(s) &\equiv \int U(x) df(s) \end{aligned}$$

with U as in lemma 1, and the functional I , which maps functions of type $r \circ (f, M)$ into \mathbb{R}^+ , is quasi-convex, mixture continuous, weakly monotonic ($r \geq r'$ for all s implies $I(r) \geq I(r')$), and homothetic.

The formulation of lemma 2 is modelled on lemma 3.3 in Gilboa and Schmeidler (1989), with notational differences indicating substantive ones. The lemma tightly limits the ways in which ambiguity can affect choices. Specifically, choices from ambiguous menus must reveal a preference ordering \succsim , here represented by value functional I , over objects $r \circ (f, M)$ that one might call *regret acts*.

The proof of lemma 2 contains three crucial steps, the third of which is one of this paper's main insights.

- First, lemma 1, monotonicity, and INA jointly imply that acts can be identified with *utility acts* $u \circ f : \mathcal{S} \mapsto \mathbb{R}$. This idea is standard except for the observation that it requires merely INA instead of IIA.
- Second, independence can be used to restrict attention to the set \mathcal{M}_0 of menus whose join or ex-post utility frontier, $\{\max_{g \in M} u \circ g(s)\}_{s \in \mathcal{S}}$, is everywhere zero. This insight was anticipated more than half a century ago (Chernoff (1954, theorem 2)) but seems to have gone unused since; e.g., it is missing in Milnor (1954) and Borodin and El-Yaniv (1998). An intuitive justification for it is that if there exists an act h with $u \circ h(s) = -\max_{g \in M} u \circ g(s)$, then independence implies that

$$f \in C(M) \iff \frac{1}{2}f + \frac{1}{2}h \in C\left(\frac{1}{2}M + \frac{1}{2}h\right),$$

but by construction, $\max_{g \in \frac{1}{2}M + \frac{1}{2}h} u \circ g(s) = 0$ for every s . Thus, C is determined by its restriction to \mathcal{M}_0 .

- Third, and most importantly, INA implies that one can construct a menu-independent preference ordering \succsim_C which rationalizes the restriction of C to \mathcal{M}_0 . Specifically, call an act choosable if it is chosen from some $M \in \mathcal{M}_0$ and define

$$\begin{aligned} f \succ_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in M \setminus C(M) \\ f \sim_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in C(M). \end{aligned}$$

that is, $f \succ_C g$ if f is strictly revealed preferred to g in some $M \in \mathcal{M}_0$ and $f \sim_C g$ if the two are revealed indifferent. The proof of lemma 2 establishes that \succsim_C is a preference ordering over

choosable acts. Furthermore, it generates the restriction of C to \mathcal{M}_0 as choice correspondence: For all $M \in \mathcal{M}_0$, it is true that $C(M) = \{a \in M : a \succsim_C b, \forall b \in M\}$. Finally, \succsim_C can be extended to an ordering \succsim over all nonpositive utility acts without affecting C .

- For convenience, the lemma finally identifies \succsim with a value functional I and collects some properties of it that are not central to the idea but will be used later.

Returning to the substantive importance of lemma 2, its upshot is as follows. The proofs in Gilboa and Schmeidler (1989), as well as many related papers, initially establish that preferences can be represented by a value functional I operated on utility acts. Lemma 2 is analogous to this, except that I is operated on regret acts which absorb any menu dependence of the agent's evaluation of acts. Substantively, this leads to a separation of risk aversion as well as menu dependence of preferences, both of which are absorbed by $r \circ (f, M)$, and attitude to uncertainty about s , which is reflected in I .⁷ Formally, it re-instates a dualism between choice correspondences and preferences, albeit a more intricate one than is generated by IIA. The practical benefit of this dualism is that existing axiomatic results for preferences can be imported if \succsim_C can be shown to fulfil their if-sides.

This idea will now be exploited to generate different axiomatizations of choice correspondences. The first results are theorems 3 and 4, which characterize minimax regret choice correspondences with no respectively endogenous priors.

Theorem 3 *Prior-less Minimax Regret*

A choice correspondence fulfils axioms 1 through 8 iff it can be represented as

$$C(M) = \arg \min_{f \in M} \max_{s \in S} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2.

Given lemma 2, theorem 3 is established by applying theorem 1(iii) of Stoye (2006) to \succsim_C . Substantively, its message is as follows. The axioms except symmetry are the standard axioms of Bayesianism, with the caveat that independence of irrelevant alternatives has been weakened. If symmetry, i.e. a refusal to make likelihood judgments, were added to the full Bayesian axioms, a contradiction would be encountered.⁸ A known way to avoid this contradiction is to relax independence to c-independence,

⁷This separation is less clear in Gilboa and Schmeidler's (1989) lemma 3.3 because there, the preferences encoded by I may change if u is replaced with a positive affine transformation of itself; since this is considered an equivalence transformation, I is not really identified separately from u . To achieve this separation, one also has to impose c-independence of I (Ghirardato et al. (2005)). The problem does not arise here: A positive affine transformation of u induces a positive *linear* transformation of $r \circ (f, M)$; since \succsim_C is homothetic, the choice correspondence cannot be affected.

⁸This follows immediately from theorem 3 since the minimax regret choice correspondence violates IIA.

leading to prior-less α -maximin utility, i.e. the “Hurwicz criterion,” and to prior-less maximin utility if ambiguity aversion is added (Milnor (1954), Stoye (2006)). But we now see that one can avoid the contradiction while insisting on independence, namely by weakening independence of irrelevant alternatives. Maintaining independence yet avoiding priors is not a contradiction; it leads to a well known decision criterion, albeit at the price of menu dependence.

Endogenous prior minimax regret can be characterized by replacing symmetry with c-betweenness for menus with state independent outcome distributions.

Theorem 4 *Endogenous Prior Minimax Regret*

A choice correspondence fulfils axioms 1 through 7 and 9 iff it can be represented as

$$C(M) = \arg \min_{f \in M} \max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi$$

for some compact, convex $\Gamma \subseteq \Delta\mathcal{S}$. Here, Γ is unique and u is as in lemma 2.

To establish this result, one needs to show that \succsim_C is c-independent, after which Gilboa and Schmeidler (1989) can be invoked. Proof of c-independence has two main ingredients: First, axiom 9 together with ambiguity aversion ensures that \succsim_C fulfils a preference version of c-betweenness, i.e. if f is revealed indifferent to a constant act p^* , then it is also revealed indifferent to any mixture of the two. Intuitively, the upshot is that indifference sets of \succsim_C (in the space of regret acts) are collections of rays emanating from constant acts. Homogeneity of degree zero of \succsim_C ensures that these rays cannot fan out or in, leading to c-independence. Thus, the “stake independence” aspect of c-independence is delivered by independence of C , which drives homogeneity of \succsim_C .

Substantively, theorem 4 resembles a previous finding by Hayashi (2008). The least obviously similar axioms are c-betweenness here and “constant-regret independence of regret premium” there. Also, monotonicity displaces Hayashi’s (2008) admissibility axiom; as a result of this weakening, Γ need not intersect the interior of $\Delta\mathcal{S}$. Furthermore, Hayashi (2008) uses a stronger continuity notion (namely, hemicontinuity of the choice correspondence), as a result of which he can conclude that U is continuous. The same could be concluded here upon strengthening continuity accordingly.⁹ Despite these differences, the contribution of theorem 4 lies less in the substantive result than in its derivation, specifically in exhibiting a tight link to Gilboa and Schmeidler (1989).

Theorems 3 and 4 do not exhaust the potential applications of lemma 2. For example, one can produce variations on theorem 4 by varying the domain of axiom 9. At an extreme, one could drop the requirement that p^* is constant and thereby assume a revealed preference equivalent of betweenness (whenever feasible outcome distributions are state independent). For an intermediate approach, betweenness could be imposed (with the same caveat) for any two acts f and g s.t. the corresponding

⁹It does not follow that the present result strengthens Hayashi (2008), however, because I use simple acts.

regret acts are comonotonic (Schmeidler (1989)), i.e. there exists an ordering of \mathcal{S} that renders both of them nondecreasing.¹⁰ The results of these variations are as follows.

Corollary 5 *If axiom 9 is imposed with p^* replaced by an arbitrary act g , then Γ is reduced to a singleton and the behavioral implications of expected utility theory are recovered.*

Corollary 6 *If axiom 9 is imposed with p^* replaced by an act g s.t. f and g have comonotonic regret profiles, then the choice correspondence can be characterized by minimization of Choquet expected regret, where the Choquet expectation is as in Schmeidler (1989).*

3 Extensions

3.1 Minimax Regret when Agents can Randomize

Here as well as in Hayashi (2008), randomization was excluded so far, allowing for axioms to be asserted over arbitrary finite – and in particular, nonconvex – sets.¹¹ But in many intended applications, agents can randomize, and strictly speaking, choices from finite sets are then not observable. Insofar as a focus on choice correspondences is motivated by a revealed preference program, this is a severe concern. After all, recall that the revealed preference criticism of axiomatizations of minimax regret preferences is that they imply or even begin from statements about unobservables. By the same token, if agents can randomize, one should really define C on convex sets only.

The problem is no less severe if one’s motivation is on the normative side, specifically to provide axioms for statistical decision makers. Statistical decision rules can in general be randomized, and it is well understood that minimax regret rules randomize quite frequently (Manski (2005, 2009); Schlag (2006); Stoye (2009a)). Indeed, the ambiguity aversion axiom maintained throughout enforces a weak revealed preference for randomization. From this perspective, it appears less than satisfying to assume that agents (weakly) want to randomize yet to prevent them from doing so even though the agents in questions (i.e., statisticians) clearly can.

Thus, for this subsection, assume that agents can randomize, meaning that all menus now are convex hulls ΔM of finite menus M . This convexification has severe effects. Most importantly, even a minimal sense of revealed preference between two acts is lost. Without randomization, choice from $\{f, g\}$ at least reveals what can be thought of as preference between them in menu $\{f, g\}$. With randomization, $C(\Delta\{f, g\})$ might (and, in the case of minimax regret, frequently will) contain only proper mixtures of the two, thus revealed preferences are inherently incomplete. What’s more, even

¹⁰I emphasize this variation less because if regret is not a primitive notion, then this definition of comonotonicity may appear artificial.

¹¹I thank a referee for raising the question answered in this section.

an unrestricted independence of irrelevant alternatives assumption only amounts to the weak but not the strong axiom of revealed preference; as soon as it is restricted to convex sets, it fails to imply the latter and, therefore, transitivity of revealed preference.¹² An additional loss of structure occurs with respect to the continuity axiom, which needs to be restated as follows:

Axiom 7* Continuity for Convex Hulls

For any menu M , act f s.t. $C(\Delta(M \cup \{f\})) \cap \Delta M = \emptyset$, and act $h \in \Delta M / C(\Delta M)$, there exist $\lambda, \gamma \in (0, 1)$ s.t. $C(\Delta(M \cup \{\lambda f + (1-\lambda)h\})) \cap \Delta M = \emptyset$ and $\gamma f + (1-\gamma)h \notin C(\Delta(M \cup \{\gamma f + (1-\gamma)h\}))$.

Note in particular that the new continuity axiom's hypothesis can only be fulfilled if f is not in the convex hull of M , rendering the axiom noticeably weaker. Finally, note that the ambiguity aversion axiom now simply imposes that $C(\Delta M)$ is convex.

As a result of these complications, I am not able to recover lemma 2. However, it is possible (with substantial additional effort) to reconstruct theorems 3 and 4.

Theorem 7 Minimax Regret When Agents can Randomize

Consider a setting exactly as in the preceding theorems, but where agents can randomize, thus choices are from convex hulls ΔM of finite menus M . Then theorems 3 and 4 continue to hold as stated.

While restricting attention to convex sets does not seem to affect these theorems, much has changed below the surface. In particular, the revealed preference ordering \succsim_C generated from choice problems in \mathcal{M}_0 is now highly incomplete. By nonemptiness of C , however, it is true that every menu $M \in \mathcal{M}_0$ contains an act that is \succsim_C -preferred to any other act in the same menu. This observation can be used to argue that every completion \succsim of \succsim_C will induce C as choice correspondence on \mathcal{M}_0 , and theorems 3 and 4 can be recovered by specifying a completion \succsim of \succsim_C that fulfils the relevant axioms.

The correct completion to use is to assign to every act the value of a suitably defined certainty equivalent, specifically the best constant act that is not strictly \succsim_C -preferred to it. Of course, one must show both that \succsim extends \succsim_C and that it fulfils the relevant axioms. This is somewhat intricate, partly because revealed \succsim_C -indifference between an act and its certainty equivalent cannot be presumed.¹³ In particular, I am not able to demonstrate from baseline axioms that \succsim extends \succsim_C but need to invoke either c-betweenness or symmetry to achieve this; thus the absence of a generalization of lemma 2. Theorem 7 therefore stands as the only main result in this paper that is not usefully thought of as a corollary of lemma 2.

¹²See Stoye (2009c) for a detailed analysis of revelation of menu-independent preferences by choice from convex sets.

¹³Let there be just two states, i.e. $\mathcal{S} = \{s_1, s_2\}$, identify acts f with utility vectors $(u \circ f(s_1), u \circ f(s_2))$, and assume that the choice correspondence is the priorless one. Then $(0, -1)$ is ranked higher than $(-1, -1)$, but there exists no menu $M \in \mathcal{M}_0$ s.t. $(0, -1) \in C(\Delta M)$.

3.2 A Characterization of Perceived Ambiguity

Lemma 2 can also be used to develop a behavioral notion of perceived ambiguity in the framework of theorem 4. The construction requires one preliminary step:

Definition 1 For any choice correspondence C , define its revealed unambiguous preference \succeq_C by writing that $f \succeq_C g$ iff

$$[\lambda g + (1 - \lambda)p^* \in C(M) \Rightarrow \lambda f + (1 - \lambda)p^* \in C(M)],$$

$$\forall \lambda \in (0, 1), p^* \text{ constant}, M \supseteq \{\lambda f + (1 - \lambda)p^*, \lambda g + (1 - \lambda)p^*\}.$$

If the range of U is unbounded, one can define $f \succeq_C g$ iff

$$[g \in C(M) \Rightarrow f \in C(M)], \forall M \supseteq \{f, g\}.$$

To get an intuition, the reader should inspect the second definition: $f \succeq_C g$ if there exists *no* menu in which g is chosen over f , or in other words, f is weakly revealed preferred over g in every menu containing both. While \succeq_C can be shown to be transitive, it is of course in general incomplete. I relate it to ambiguity to once again emphasize the conceptual link between menu dependence and ambiguity. The idea is that if $f \succeq_C g$, then the comparison between f and g is context independent, hence revealed to be unambiguous.

In contrast, the mixing of both acts with a constant act is a technical detail that becomes necessary if U is bounded. It can then happen that a third act will be chosen from any menu containing both f and g , rendering both $f \succeq_C g$ and $g \succeq_C f$ vacuously true. Mixing the acts of interest with constant ones allows to manipulate their utility range so as to avoid this.

It is, then, also intuitive to define comparative perceived ambiguity as follows.

Definition 2 The choice correspondence C reveals (weakly) more perceived ambiguity than C' if

$$f \succeq_C g \implies f \succeq_{C'} g.$$

Comparative ambiguity perception can be tightly characterized in terms of utility functions and sets of priors.

Theorem 8 Characterization of Comparative Ambiguity Aversion

Assume that theorem 4 applies. The choice correspondence C reveals (weakly) more perceived ambiguity than C' iff both can be represented by the same utility function U in conjunction with sets of priors Γ (to represent C) and Γ' (to represent C') s.t. $\Gamma \supseteq \Gamma'$.

The theorem lends further support to the identification of revealed unambiguous preference with ambiguity perception, because it tightly links perceived ambiguity to sets of priors, which prima facie represent the decision maker’s perception of ambiguity in her environment. It should be kept in mind, however, that this paper is behavioral and hence, Γ is not claimed to map onto any real objects, including true sets of beliefs. One might therefore want to more cautiously interpret the result as a characterization of \succeq_C that illustrates conceptual consistency of this paper’s terminology. (See also the similar discussion in Ghirardato et al. (2004).)

To understand why the theorem is true, recall that the link between C and \succsim_C from lemma 2 is partly established by mapping arbitrary menus into \mathcal{M}_0 via mixing with a “normalizing” act. This link can be used to show that $f \succeq_C g$ iff $\lambda f + (1-\lambda)h \succsim_C \lambda g + (1-\lambda)h$ for any probability $\lambda \in [0, 1)$ and any act h within a specified set of acts. But this means that $f \succeq_C g$ is the independence-abiding, incomplete preference ordering defined by Ghirardato et al. (2004, see also Bewley (2002)) to capture perceived ambiguity. Theorem 8 then follows by importing their results. It implies that comparative ambiguity perception coincides with comparative regret aversion in Hayashi (2008). Once the link between \succsim_C and \succeq_C has been established, this result is expected because Hayashi’s notion of comparative regret aversion adapts Ghirardato and Marinacchi’s (2002) notion of comparative ambiguity aversion, which coincides with the one of Ghirardato et al. (2004) for maxmin utility preferences – but \succsim is just such a preference.¹⁴

An interesting application of theorem 8 is axiomatization of minimax regret where the set of priors is informative but exogenous. Let there exist a compact, convex object $\Gamma^* \subseteq \Delta\mathcal{S}$ and consider linking it axiomatically to the object Γ in the preceding representation. As explained in the introduction, this approach is mainly motivated by statistical decision theory, where exogenous sets of priors are ubiquitous; specifically, corollary 9 below characterizes Γ -minimax regret. Under a revealed preference interpretation, the corollary specifies how Γ would be revealed to equal Γ^* . In either case, the necessary axioms are as follows.¹⁵

¹⁴It is worth reiterating that the analogy to Ghirardato et al.’s (2004) device of mixing with an act h is the manipulation of the menu and not the mixing of f and g with a constant act. To see why mixing with a constant act can be necessary, consider the previous footnote’s example, i.e. priorless minimax regret in a two-state world. Then $f \succeq_C g$ iff f dominates g statewise, so $f = (1, 0)$ and $g = (-1, 1)$ are not comparable. Sure enough, one can easily construct a menu in which g is chosen over f , namely $M = \{f, g, (-2, 3)\}$, whereas it is clear that one could not do so if f dominated g . But if U is bounded above by 1, then f incurs regret of exactly 1 but g incurs regret of at least 2 in every menu containing both, hence the construction fails. Allowing one to mix f and g with constant acts resolves the problem. In the numerical example, note that $C((1/3, 0), (-1/3, 1/3), (-2/3, 1)) = \{(-1/3, 1/3)\}$.

¹⁵I find it most intuitive to state the axioms in terms of u , but of course, existence of this object depends on previous axioms. To avoid this, one can rephrase $\int u \circ f(s)d\pi \geq \int u \circ g(s)d\pi, \forall \pi \in \Gamma^*$ as $(\int f(s)d\pi)^* \in C\{(\int f(s)d\pi)^*, (\int g(s)d\pi)^*\}, \forall \pi \in \Gamma^*$. The statements are equivalent by lemma 1.

Axiom 10 Γ^* -Monotonicity

$$\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^* \implies f \succeq_C g.$$

Axiom 11 Consistency with Γ^* -Ambiguity

$$f \succeq_C g \implies \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

To understand Γ^* -monotonicity, recall that in view of lemma 1, the standard monotonicity axiom can be slightly rewritten: $f \in C_s(\{f, g\}), \forall s$ is equivalent to $u \circ f(s) \geq u \circ g(s), \forall s$. This reveals that Γ^* -monotonicity is intuitively similar to monotonicity, but strengthens it. It imposes that if the comparison of f and g is commonsensically unambiguous to anybody who accepts priors Γ^* , then revealed preferences should indeed not reveal any ambiguity in the sense of menu dependence, and should furthermore have the obvious direction. The same axiom is used, to a similar effect as here, by Gajdos et al. (2004).

Conversely, consistency with Γ^* -ambiguity stipulates that a revealed preference is unambiguous in the sense of menu-independent only if the according comparison of acts is commonsensically unambiguous given Γ^* . Equivalently, if the comparison of two acts under Γ^* is ambiguous in the sense that either act is favored by some elements of Γ^* , then the choice correspondence reflects this ambiguity in the sense of violating IIA. This relates consistency with Γ^* -ambiguity to the aforementioned leitmotif, namely that violations of IIA should be driven by ambiguity.

In short, Γ^* -monotonicity can be thought of as ensuring that the decision maker does not see more ambiguity than is encoded in Γ^* ; consistency with Γ^* -ambiguity can be thought of as ensuring that she does not see less. The axioms' effects accord with these intuitions.

Corollary 9 Exogenous Priors Minimax Regret

Assume that theorem 4 applies. Then:

- (i) *A choice correspondence is consistent with Γ^* -monotonicity iff $\Gamma \subseteq \Gamma^*$.*
- (ii) *A choice correspondence is consistent with Γ^* -ambiguity iff $\Gamma \supseteq \Gamma^*$.*

Corollary 9 is true because axioms 10 and 11 effectively restrict C to reveal less (respectively more) ambiguity than the minimax regret ordering with the same utility function and set of priors Γ^* . It builds on theorem 4, but also connects theorem 3 to theorem 4 because it identifies prior-less minimax regret as minimax regret with maximal perceived ambiguity. This yields an alternative characterization of prior-less minimax regret and again illustrates conceptual consistency of terminology. After all, we would surely think of the absence of any prior information as maximizing ambiguity. I finally note that

corollary 8 is easily re-imported into the setting of Ghirardato et al. (2004) to yield a characterization of exogenous prior (or Γ -) maximin utility.

A limitation of this subsection’s result is that it is not (easily) extended to randomized menus. This is because when choice is from convex menus, then

$$[g \in C(\Delta M) \Rightarrow f \in C(\Delta M)], \forall M \supseteq \{f, g\}$$

cannot any more be interpreted to mean that f is revealed preferred to g , and it does *not* imply that f has a higher value on some criterion function that represents choices. The reason is that the above logical implication will also hold true if some proper mixture of f and g is strictly revealed preferred to either.

3.3 Minimax Regret with Different Benchmarks:

Axiomatizing Hannan Regret

This section considers the possibility that the “regret benchmark” may be generated not by the ex post best acts in M , but by the ex post best acts within a subset of M . The approach may seem unfamiliar to economic theorists, but can be specialized to yield a characterization of Hannan regret, which is frequently used in the statistics as well as computer science literatures. An informal motivation for it is that comparisons with all acts may generate benchmarks that are unreasonably high in some states. It is hard to imagine an axiomatization that would directly capture this concern without presupposing benchmarks, and therefore regret. However, nonstandard benchmarks can be characterized by varying the potential extent of menu dependence. I will first do this at some level of generality and then specialize the discussion to Hannan regret.

Assume there exists a set $\tilde{\mathcal{F}}$ of Σ -measurable *special acts* with the following structure: (i) $\tilde{\mathcal{F}}$ is closed under probabilistic mixture; (ii) it is closed under statewise recombination, i.e. $f, g \in \tilde{\mathcal{F}}$ and $E \in \Sigma$ imply that $f_E g \in \tilde{\mathcal{F}}$, where $f_E g$ is the act that coincides with f on E and with g otherwise; (iii) there exist constant acts $\tilde{p}, \tilde{q} \in \tilde{\mathcal{F}}$ as well as $p, q \in \mathcal{F}$ s.t. $C(\{\tilde{p}, \tilde{q}, q\}) = \{\tilde{p}\}$ and $C(\{\tilde{p}, \tilde{q}, p\}) = \{p\}$.¹⁶ Restrict attention to menus that contain some such act, i.e. menus M with $M \cap \tilde{\mathcal{F}} \neq \emptyset$. Suppose that the ambiguity inherent in a decision problem is not driven by all acts at the decision maker’s disposal, but only by the special ones. In a normative context, one would have to argue why this restriction is compelling; this argument would depend on the application at hand, and along with specification of $\tilde{\mathcal{F}}$, is left to users who wish to employ the model. In a revealed preference interpretation, this section shows how different degrees of menu dependence of choice behavior can be interpreted as revealing

¹⁶These conditions are simple and sufficient, but not necessary for results to go through. What is actually needed is the “outcome range overlap” assumption spelled out in Puppe and Schlag (2008), plus convexity and closure under statewise recombination of the image of \mathcal{F}^* under u .

different ways of forming benchmarks; $\tilde{\mathcal{F}}$ is then a set to be revealed behaviorally. In any case, the idea of equating ambiguity with menu independence leads to imposing axiom INA with respect to elements of $\tilde{\mathcal{F}}$ and full independence of irrelevant alternatives otherwise. Technically, the new axiom would be:

Axiom 12 INA for Special Acts

Let M and N be such that $C_s \left([M \cup N] \cap \tilde{\mathcal{F}} \right) \cap M \neq \emptyset$ for all s . Then

$$C(M \cup N) \cap M \in \{C(M), \emptyset\}.$$

Replacing axiom 6 with axiom 12 affects lemma 2 and theorem 3 as follows.

Theorem 10 *Restrict attention to menus M s.t. $M \cap \tilde{\mathcal{F}} \neq \emptyset$. Then a choice correspondence fulfils axioms 1 through 5, 7, 8, and 12 iff it can be represented as*

$$C(M) = \arg \min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M \cap \tilde{\mathcal{F}}} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2. It fulfils axioms 1, 2*, 3 through 5, 7, 9, and 12 iff it can be represented as

$$C(M) = \arg \min_{f \in M} \max_{\pi \in \Gamma} \int \left(\max_{g \in M \cap \tilde{\mathcal{F}}} u \circ g(s) - u \circ f(s) \right) d\pi$$

with Γ as before.

In words, replacing INA with INA for special acts has the effect that the choice correspondence benchmarks not against $\max_{g \in M} u \circ g(s)$ but against $\max_{g \in M \cap \tilde{\mathcal{F}}} u \circ g(s)$. The ex post utility frontier is evaluated only over special acts. A caveat is that the multiple prior version of this result requires a slight strengthening of monotonicity.

To elaborate the example that motivates this result, fix a finite-state, finite-action decision problem where action $\hat{f} \in \hat{\mathcal{F}}$ generates utility $u \circ \hat{f}(\hat{s})$ in state $\hat{s} \in \hat{\mathcal{S}}$ and consider the N -superproblem generated by replicating this problem N times. Let the state space for the superproblem be $\mathcal{S} \equiv \times_{n=1}^N \hat{\mathcal{S}}$ with typical element $s \equiv (\hat{s}^1, \dots, \hat{s}^N)$; in particular, the true state of the world in the one-shot problem can vary arbitrarily across time periods. Let \hat{M} denote the decision maker's menu in the one-shot problem. A decision rule then specifies a decision maker's act \hat{f}^n in each of the N periods. This act may always be a function of the decision maker's history of play, $(\hat{f}^1, \dots, \hat{f}^{n-1})$ and usually also of the history of states of the world $(\hat{s}^1, \dots, \hat{s}^{n-1})$; indeed, statistical learning is a core applications of the model. Assume that if she knew s , the decision maker would maximize $u \circ f(s) \equiv \sum_n u \circ \hat{f}^n(\hat{s}^n)$. Then a popular criterion by which to judge feasible decision rules is (the negative of)

$$\max_{s \in \mathcal{S}} \left\{ \max_{\hat{g} \in \hat{M}} \sum_n u \circ \hat{g}(\hat{s}^n) - u \circ f(s) \right\},$$

the maximal regret incurred relative to constant decision rules. This criterion function was introduced by Hannan (1957), is accordingly known as *Hannan regret*, and is widely used in the machine learning literature as well as occasionally in economics.¹⁷ Hannan regret has not, to my knowledge, been axiomatized before. Assuming nontriviality of u , its characterization is a special case of theorem 9. To see this, let \mathcal{F} be the set of decision rules as explained, let $\tilde{\mathcal{F}}$ be the closure under statewise recombination of the (risk functions induced by) constant decision rules (noting that $\max_{g \in M \cap \tilde{\mathcal{F}}} u \circ g(s)$ will always be achieved by a constant rule), and use u and s as defined in this paragraph.

There are two caveats to this observation. The less important one, in this author's view, is that natural sets of special acts may not meet the above richness conditions, so one must introduce hypothetical utility acts into play. Note, though, that this and other papers routinely assume the same richness conditions for \mathcal{F} , even though the same remark applies at that level; so this observation is really a caveat about revealed preference analysis in general (and not a new one at that; see Sen (1977)). The more important point is that INA with respect to this special class of acts has a certain degree of artifice. In this author's personal judgment, the contribution is, therefore, to clarify how Hannan regret can be embedded in the previous developments and how one could characterize it behaviorally; with respect to normative foundations for the criterion, the finding may be more thought-provoking than positive.

4 Conclusion

This paper unified some of the recent, axiomatic literature on minimax regret. It adopted choice correspondences as general framework, but demonstrated that even without independence of irrelevant alternatives, there exists a tight link between axiomatizations of preference orderings respectively choice correspondences. Under restrictions shared by many regret-based approaches, results of the latter kind can be generated from existing results of the former kind. I used this insight to provide a number of minimax regret characterizations, including a choice correspondence analog of a result in Stoye (2006), a reconstruction of a result in Hayashi (2008) as implication of Gilboa and Schmeidler (1989), the adaptation of results in Ghirardato et al. (2004) to characterize perceived ambiguity, and the extension of most of these results to choice from convex (through randomization) sets. Finally, I characterized minimax regret with exogenous priors as well as Hannan regret.

The framework is intended to be rather universal and covers all of the applications cited in the introduction as well as applications of Hannan regret in other literatures. Nonetheless, it is impossible

¹⁷See the textbook by Cesa-Bianchi and Lugosi (2006) and references therein. The authors define minimax regret with nonstandard benchmarks in section 2.10. The further specialization to Hannan regret underlies the notion of Hannan consistency (as in the last expression on p. 72), which is also used as optimality criterion by Hart and Mas-Colell (2001).

to unify in one paper every concept that has been labelled “regret.” I therefore conclude by clarifying the relation between minimax regret as formalized here and some other notions in economics and related fields.

Statistical decision theory is one motivation of this paper, and many but not all uses of regret there coincide with formalisms here. To comply with this paper’s notion of minimax regret, statistical decision rules or estimators must be compared to an “oracle estimator” which is best among the feasible ones, given hypothetical knowledge of the true state of the world. Examples include the treatment choice problems in Manski (2004, 2006, 2007), Schlag (2007), and Stoye (2007a, 2007b, 2008), where the oracle treatment rules are no-data rules that respond to true expectations, but also the estimation problems in Droge (1998, 2006), Eldar et al. (2004), and Hansen (2005), where they are the ex post best from certain classes of estimators. An incompatible example, however, is the “predictive entropy regret” approach of Sweeting et al. (2006), which benchmarks against a specific act, thereby is not menu dependent, and in the terminology of (economists’) decision theory, rather looks like maximin utility with a specific, state dependent utility function.

The word “regret” resounds in everyday language, and some readers may accordingly be interested in it from the vantage point of psychology or behavioral economics. From that perspective, one may critically remark that regret here benchmarks against an “omniscient” ex post stage in which the true state of the world has been revealed. This does not correspond to a situation that the decision maker anticipates to actually experience, so one may hesitate to identify this paper’s notion of regret with anticipated feelings. If the latter are a core motivation, one may want to explore regret preferences that benchmark against what the decision maker will, in fact, learn from outcome realizations. This is the motivation of Krähmer and Stone (2006). Interestingly, it can lead to preferences against information: Of two otherwise identical acts, the one whose outcomes are less correlated with, and hence less informative about, other acts’ potential outcomes may be strictly preferred. The approach has not, to my knowledge, been axiomatized.

A well known invocation of regret in economic theory is due to Loomes and Sugden (1982; see also Fishburn (1989), Sugden (1993)). This approach has in common with the current one that regret is evaluated from an omniscient view; some papers partially justify this by imposing independence of outcome lotteries across acts, thus removing one wedge between the “realistic ex post” and the omniscient information stage. Major differences to the present perspective are the imposition of a concave transformation of regret and the use of subjective priors rather than maxmin-operators. Indeed, the dichotomy of uncertainty/risk versus ambiguity/Knightian uncertainty is not emphasized in this approach, and the relevant papers switch between imposing objective probabilities (Loomes and Sugden (1982)) and a Savage environment (Sugden (1993)).

Finally, Sarver (2008) proposed a model of regret that embeds it in the recent literature on menu

dependent preferences. One difference to the present approach is that Sarver’s utility function combines conventional utility with an additive regret penalty. More importantly, he follows Kreps (1979) and Gul and Pesendorfer (2001) in axiomatizing preferences over menus; in the language of utility maximization, the axiomatization is of the value functional. For Gul and Pesendorfer’s (2001) as well as Sarver’s (2008) motivation, this is an ingenious device. For the present paper’s notion of regret, its use would be less obvious because the minimax regret value functional does not have a clean interpretation. If choice problem M causes more minimax regret than problem N , this may mean that learning is more valuable in M , and this intuition may well inform a future axiomatization – but it does not imply that N is more desirable by any commonsensical standard. Indeed, it could easily be the case that any option in M dominates any element of N in utility terms. An instructive axiomatization of this value functional would, accordingly, have to be quite different from the present contribution.

A Proofs

Lemma 1 Follows from standard results after defining $p^* \succsim q^* \Leftrightarrow p^* \in C(\{p^*, q^*\})$ for constant acts.

Lemma 2 In this and the next two proofs, I only show “only if.” Recall that lemma 1 applies. For any act f , define the mapping (“utility act”) $u \circ f : \mathcal{S} \mapsto \mathbb{R}$ by $u \circ f(s) \equiv \int U(x)df(s)$ and use the shorthand notation $\geq [\gg]$ as follows: $f \geq [\gg]g \Leftrightarrow u \circ f(s) \geq [>]u \circ g(s), \forall s$. Observe that in the statement of monotonicity, $[f \in C_s(\{f, g\}), \forall s]$ can now be written as $f \geq g$.

Step 1: No information is lost by identifying every act f with $u \circ f$. To see this, fix any menus M and M' such that there exists a one-to-one mapping $(\cdot) : M \rightarrow M'$ with $u \circ f' = u \circ f$ for every $f \in M$. Then $f \in C(M) \Leftrightarrow f' \in C(M')$. To see this, consider $C(M \cup M')$. By INA, $C(M \cup M') \cap M \in \{C(M), \emptyset\}$ and $C(M \cup M') \cap M' \in \{C(M'), \emptyset\}$. Nonemptiness of C and monotonicity now jointly imply that $C(M \cup M') = C(M) \cup C(M')$, and monotonicity (applied to f and f') then yields the claim. With abuse of notation, I henceforth identify acts with utility acts, that is, I write $f = g$ when I really mean $u \circ f = u \circ g$.

Nonconstancy of U is necessary for monotonicity and nontriviality to be mutually consistent. It implies that after normalization, $U^{-1}(-2)$ and $U^{-1}(1)$ can be assumed to exist. Hence, any finite, Σ -measurable step function $u : \mathcal{S} \rightarrow [-2, 1]$ can be identified with a feasible act f . This specifically includes p_0^* , the constant act with utility value 0. Independence implies that $C(\lambda M + (1 - \lambda)p_0^*) = \lambda C(M) + (1 - \lambda)p_0^* = \lambda C(M)$. Hence, C is homogeneous of degree one: For any menu M and scalar $\lambda \in (0, 1)$, the menu λM exists and $C(\lambda M) = \lambda C(M)$.

Step 2: For any menu M , let \bar{f}_M denote the act with $u \circ \bar{f}_M(s) = \max_{f \in M} u \circ f(s)$. This “oracle act” or join always exists, although it need not be an element of M . Let \mathcal{M}_0 denote the set of menus M s.t. $\bar{f}_M = p_0^*$, i.e. menus whose ex post best possible utility is zero in every state. Call an act $f \in \mathcal{F}_{[-1,0]}$ choosable if there exists $M \in \mathcal{M}_0$ s.t. $f \in C(M)$. Assume that some constant act $p^* \ll p_0^*$ is choosable; the case where this fails is handled separately in step 4. By homogeneity of C , it is w.l.o.g. to set the utility value of p^* to -1 ; as a reminder of this, the act will be labelled p_{-1}^* henceforth.

Let $\mathcal{F}_{[-1,0]}$ denote the set of finite, Σ -measurable mappings from \mathcal{S} into $[-1, 0]$, noting that in view of step 1, this set can be identified with the set of all utility acts with range in $[-1, 0]$. Define the relation \succ_C on $\mathcal{F}_{[-1,0]} \times \mathcal{F}_{[-1,0]}$ as follows:

$$f \succ_C g \iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in M \setminus C(M)$$

$$f \sim_C g \iff \exists M \in \mathcal{M}_0 : f \in C(M), g \in C(M).$$

Let $\widetilde{\mathcal{M}}_0$ collect menus $M \in \mathcal{M}_0$ s.t. $M \subset \mathcal{F}_{[-1,0]}$. The restriction of C to $\widetilde{\mathcal{M}}_0$ is the choice correspondence induced by \succ_C on the same set, i.e. $C(M) = \{f \in M : g \in M \Rightarrow f \succ_C g\}$. To see this, fix $M \in \widetilde{\mathcal{M}}_0$. Then $f \in C(M) \Rightarrow f \succ_C g$ for all $g \in M$, but also $f \in M \setminus C(M) \Rightarrow g \succ_C f$ for some $g \in M$ because $C(M)$ is nonempty. Axiom INA straightforwardly implies that \succ_C is antisymmetric and that \succ_C and \sim_C are disjoint, thus the claim.

Step 3: I now collect some properties of \succ_C . Weak monotonicity follows immediately from the according axiom. To see completeness, fix any $f, g \in \mathcal{F}_{[-1,0]}$ and consider any $M \in \mathcal{M}_0$ from which f is chosen, then by axiom INA, choice from $M \cup \{g\}$ must define \succ_C . Given completeness, homogeneity of degree 1 of C implies homotheticity of \succ_C . To see transitivity ($f \succ_C g, g \succ_C h \Rightarrow f \succ_C h$), consider its contrapositive: $h \succ_C f \Rightarrow (h \succ_C g \vee g \succ_C f), \forall g$. Assume $h \succ_C f$, hence there exists $M \in \mathcal{M}_0$ with $h \in C(M)$ and $f \in M \setminus C(M)$. Then either $g \in C(M \cup \{g\})$, in which case $g \succ_C f$, or $g \notin C(M \cup \{g\})$, in which case $h \succ_C g$.

Mixture continuity implies the analog property of \succ_C . Let $f \succ_C g \succ_C h$, thus there exists $M \in \mathcal{M}_0$, $M \supseteq \{g, h\}$ with $C(M) \cap \{g, h\} = \{g\}$ and $N \in \mathcal{M}_0$, $N \supseteq \{f, g\}$ with $C(N) \cap \{f, g\} = \{f\}$. To derive the then-side of mixture continuity from axiom 7, it suffices to show $C(M \cup \{f\}) = \{f\}$. To see “ \supseteq ,” suppose that $f \notin C(M \cup \{f\})$, thus $C(M \cup \{f\}) \cap M = C(M)$, thus $g \in C(M \cup \{f\})$, thus $g \succ_C f$, a contradiction. To see “ \subseteq ,” consider $C(M \cup N) \cap N = C((M \cup \{f\}) \cup N) \cap N$. Applying INA to the expanded expression, one finds that $\{f\} \in C(M \cup N)$, hence $C(M \cup N) \cap C(N) = C(N)$, hence $g \notin C(M \cup N)$, hence $g \notin C(M \cup \{f\})$, hence $C(M \cup \{f\}) = \{f\}$.

Ambiguity aversion implies that $f \sim_C g \Rightarrow \lambda f + (1 - \lambda)g \succ_C f$ for all $\lambda \in (0, 1)$. To see this, fix any $M \supseteq \{f, g\}$, then $f \sim_C g$ implies that either $\{f, g\} \in C(M)$ or $C(M) \cap \{f, g\} = \emptyset$. In the latter case, M vacuously fulfils the condition defining $\lambda f + (1 - \lambda)g \succ_C f$. In the former case, ambiguity

aversion implies that $\lambda f + (1 - \lambda)g \in C(M \cup \{\lambda f + (1 - \lambda)g\})$, thus the condition is fulfilled as well.

Together, these properties imply that \succsim_C can be represented by a functional $J : \mathcal{F}_{[-1,0]} \rightarrow [-1, 0]$ s.t. $-J$ has the properties claimed in the lemma except for nontriviality, which will be established later.

Step 4: Independence can be used to completely characterize C from \succsim_C . This will be done in two steps. First, let $\mathcal{F}_{[-1,1]}$ collect all utility acts with utility range in $[-1, 1]$ and restrict attention to menus $M \subset \mathcal{F}_{[-1,1]}$. Noting $-\bar{f}_M \in \mathcal{F}_{[-1,1]}$, which specifically means that $-\bar{f}_M$ exists, use independence to write

$$\frac{1}{2} (C(M) + (-\bar{f}_M)) = C \left(\frac{1}{2}M + (-\frac{1}{2}\bar{f}_M) \right),$$

but $\frac{1}{2}M + (-\frac{1}{2}\bar{f}_M) \in \widetilde{\mathcal{M}}_0$, hence $C(\frac{1}{2}M + (-\frac{1}{2}\bar{f}_M)) = \arg \max_{f \in \frac{1}{2}M - \frac{1}{2}\bar{f}_M} J(u \circ f)$. It follows that

$$\begin{aligned} C(M) &= 2C \left(\frac{1}{2}M + (-\frac{1}{2}\bar{f}_M) \right) + \bar{f}_M \\ &= 2 \arg \max_{f \in \frac{1}{2}M - \frac{1}{2}\bar{f}_M} J(u \circ f) + \bar{f}_M \\ &= \arg \min_{f \in M} I(r \circ (f, M)), \end{aligned}$$

where $I \equiv -J$ has the properties asserted in the lemma. In a second step, homogeneity of C implies that the conclusion applies to any menu. Finally, it is now clear that if \succsim_C ranked all acts in $\mathcal{F}_{[-1,0]}$ equally, then C would be trivial, hence \succsim_C is nontrivial.

Step 5: Let \mathcal{C} be the set of choosable acts. Assume that \mathcal{C} contains no act $p^* \ll p_0^*$, then by monotonicity, it contains no act $f \ll p_0^*$. But ambiguity aversion implies that \mathcal{C} is convex, hence there exists a state s^* s.t. $\mathcal{C} \subseteq \{f : u \circ f(s^*) = 0\}$. Let f be s.t. $u \circ f(s^*) = 0$. Let g be s.t. $u \circ g(s^*) = -1$ and $u \circ g(s) = 0$ for every s that is distinct from s^* (i.e. there exists an event $E \in \Sigma$ separating s and s^*). Then $g \notin \mathcal{C}$ and $\{f, g\} \in \mathcal{M}_0$, hence $C(\{f, g\}) = \{f\}$, hence $f \in \mathcal{C}$. It follows that $\mathcal{C} = \{f : u \circ f(s^*) = 0\}$.

$\mathcal{C} \cap \mathcal{F}_{[-1,0]}$ is an indifference set of \succsim_C . To see this, fix any act $f \in \mathcal{C} \cap \mathcal{F}_{[-1,0]}$. Suppose by contradiction that $f \prec_C p_0^*$ and define g as in the preceding paragraph, then axiom INA and $g, p_{-1}^* \notin \mathcal{C}$ imply that $C(\{f, p_0^*, p_{-1}^*, g\}) = \{p_0^*\}$ and that $C(\{f, p_1^*, g\}) = \{f\}$. Now continuity implies existence of $\lambda > 0$ s.t. $C(\{f, \lambda p_{-1}^*, g\}) = \{\lambda p_{-1}^*\}$. Hence $\lambda p_{-1}^* \in \mathcal{C}$, a contradiction.

It follows that $C(M) = \{f : u \circ f(s^*) = 0\}$ for any $M \in \mathcal{M}_0$. This choice correspondence can be represented as maximizing $u \circ f(s^*)$, thus it fulfils the lemma's conclusion.

Theorem 3 Let p_{-2}^* denote the constant act with utility value -2 . Apply lemma 2. The previous proof established that \succsim_C fulfils all axioms used in theorem 1(iii) of Stoye (2006) except for symmetry,

which will be established now. Fix acts $f, g \in \mathcal{F}_{[-1,0]}$ and events $E_1, E_2 \in \Sigma$ s.t. f and g are constant on E_1 and E_2 . Define $E = E_1 \cup E_2$ and $F = \mathcal{S} \setminus E$. Consider $M \equiv \{p_{0E}^* p_{-2}^*, p_{-2E}^* p_0^*, p_{-1}^*\}$, then symmetry and ambiguity aversion jointly imply that $p_{-1}^* \in C(M)$. (As an aside, this implies that p_{-1}^* is choosable.) Consider now $N \equiv M \cup \{f, g\}$. Noting that p_{-1}^* is dominated by both f and g , axiom INA and monotonicity jointly imply that $C(N) \cap \{f, g\} \neq \emptyset$, hence $f \succsim_C g \Leftrightarrow f \in C(N)$. But now Stoye's (2006) symmetry axiom for preferences is implied upon comparing N and N' , the menu generated from N by interchanging the consequences of E_1 and E_2 .

Thus \succsim_C is priorless maximin utility: $f \succsim_C g$ iff $\min_{s \in \mathcal{S}} u \circ f(s) \geq \min_{s \in \mathcal{S}} u \circ g(s)$. Substituting into lemma 2 yields

$$C(M) = \arg \min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

as required. For necessity of three events as well as individual necessity of axioms, see Stoye (2006).

Theorem 4 The choice correspondence discovered in step 5 of the proof of lemma 2 fulfils this theorem's conclusion. Hence, restrict attention to the case where p_{-1}^* is choosable. It needs to be shown that \succsim_C is c-independent. For use in the corollaries, isolate the following lemma.

Lemma 11 Under assumptions maintained in this proof, $f \succsim_C g \Leftrightarrow \lambda f + (1 - \lambda)p^* \succsim_C \lambda g + (1 - \lambda)p^*$ for all acts $f, g, p^* \in \mathcal{F}_{[-1,0]}$ (p^* being constant) and scalars $\lambda \in (0, 1)$. In words, \succsim_C is c-independent.

Proof. Initially assume that \succsim_C fulfils strict monotonicity (axiom 2*), thus $f \gg g$ implies $f \succ_C g$. The case where this fails will be considered later. To begin, axiom 9 implies that there exists no menu $M \in \mathcal{M}_0$ s.t. $f, p^* \in M \setminus C(M)$ and $\lambda f + (1 - \lambda)p^* \in C(M)$ for some (f, p^*, λ) . To see this, assume that M exists. Let $\Sigma_0 \subset \Sigma$ be some partition of \mathcal{S} s.t. every act $f \in M$ is constant on every event $E \in \Sigma_0$; this is feasible because acts and menus are finite. Define the act \underline{f}_M by $u \circ \underline{f}_M(s) = \min_{f \in M} u \circ f(s)$, define $\mathcal{U}_M \equiv \{v \in [-1, 0] : u \circ f(s) = v \text{ for some } f \in M, s \in \mathcal{S}\}$, let p_v^* denote the constant act with utility value v , and let $M^* = M \cup \{p_{vE}^* \underline{f}_M : E \in \Sigma_0 \cup \{\emptyset\}, v \in \mathcal{U}_M\}$. Every element of M^* is weakly dominated by some element of M , hence INA and monotonicity jointly imply that $C(M \cup M^*) \cap M = C(M)$. But M^* has state independent outcome distributions, thus axiom 9 is contradicted.

Next, \succsim_C fulfils what might be called c-betweenness for preferences: $f \sim_C p^* \Leftrightarrow f \sim_C \lambda f + (1 - \lambda)p^*$ for all $\lambda \in (0, 1)$. To see "only if," assume that $f \sim_C p^*$, then $f \succsim_C \lambda f + (1 - \lambda)p^*$ follows from ambiguity aversion. Assume by contradiction that $f \prec_C \lambda f + (1 - \lambda)p^*$, then there exists $M \in \mathcal{M}_0^*$ s.t. $\lambda f + (1 - \lambda)p^* \in C(M)$ but $f \in M \setminus C(M)$. Consider now $M \cup \{p^*\}$, noting that $M \cup \{p^*\} \in \mathcal{M}_0^*$. Two consecutive uses of INA yield $f \notin C(M \cup \{p^*\})$ and then $p^* \notin C(M \cup \{p^*\})$. Nonemptiness of C and INA now jointly imply that $\lambda f + (1 - \lambda)p^* \in C(M \cup \{p^*\})$, contradicting the preceding paragraph's conclusion. Now suppose that $f \sim_C \lambda f + (1 - \lambda)p^*$ for some $\lambda \in (0, 1)$. This implies $f \sim_C p^*$,

establishing “if.” To see this, note that monotonicity and continuity jointly imply existence of a constant act q^* s.t. $f \sim_C q^*$, hence $f \sim_C \lambda f + (1 - \lambda)q^*$ by “only if,” hence $\lambda f + (1 - \lambda)p^* \sim_C \lambda f + (1 - \lambda)q^*$ by transitivity – but this is consistent with strict monotonicity only if $p^* = q^*$.

Next, $f \sim_C g \Leftrightarrow \lambda f + (1 - \lambda)p^* \sim_C \lambda g + (1 - \lambda)p^*$. For this and the following step, assume that $f, g \prec_C p_0^*$, which specifically holds if $f, g \ll p_0^*$; the preceding paragraph’s result can be used to extend indifference sets to boundary acts. If $p^* = p_0^*$, then the claim is immediate from homogeneity of C . Else, suppose $f \sim_C g$, then by monotonicity and continuity, there exists $\gamma > 0$ s.t. $f \sim_C g \sim_C \gamma p^*$. By the preceding paragraph’s result and transitivity, $\rho f + (1 - \rho)\gamma p^* \sim_C \rho g + (1 - \rho)\gamma p^*$ for any $\rho \in (0, 1)$. Let $\delta \equiv (1 - \lambda + \gamma\lambda) / \gamma$ and $\rho \equiv \gamma\lambda / (1 - \lambda + \gamma\lambda)$, then $\lambda f + (1 - \lambda)p^* = \delta[\rho f + (1 - \rho)\gamma p^*]$ and $\lambda g + (1 - \lambda)p^* = \delta[\rho g + (1 - \rho)\gamma p^*]$. Homogeneity of C therefore implies that $\lambda f + (1 - \lambda)p^* \sim_C \lambda g + (1 - \lambda)p^*$. The converse follows from the reverse argument, using the “if”-direction of c-betweenness.

Finally, $f \prec_C g \Leftrightarrow \lambda f + (1 - \lambda)p^* \prec_C \lambda g + (1 - \lambda)p^*$. Suppose $f \prec_C g$, then by strict monotonicity and continuity, there exists $\gamma \in (0, 1)$ s.t. $\gamma f \sim_C g$. The preceding paragraph’s conclusion implies that $\lambda\gamma f + (1 - \lambda)p^* \sim_C \lambda g + (1 - \lambda)p^*$ for all constant acts p^* , which in turn implies $\lambda f + (1 - \lambda)p^* \prec_C \lambda g + (1 - \lambda)p^*$ by strict monotonicity and transitivity. If $f \succ_C g$, use the same argument with the roles of f and g reversed. Finally, assume $\lambda f + (1 - \lambda)p^* \prec_C \lambda g + (1 - \lambda)p^*$, then $f \sim_C g$ would violate the preceding paragraph’s conclusion, and $f \succ_C g$ would violate this paragraph’s preceding conclusion, hence $f \prec_C g$.

Now suppose that strict monotonicity fails, thus there exist $f, g \in \mathcal{F}_{[-1,0]}$ with $f \gg g$ but $g \succsim_C f$. Then there exists $\gamma < 1$ with $f \gg \gamma g$, thus $f \succsim_C \gamma g \succsim_C g$ by monotonicity, thus $g \sim_C \gamma g$ by transitivity. Monotonicity and continuity imply the existence of p^* s.t. $g \sim_C p^*$, where $p^* \ll p_0^*$ by nontriviality, $\gamma g \sim_C \gamma p^*$ by homotheticity, and hence $p^* \sim_C \gamma p^*$ by transitivity. Now homotheticity and transitivity imply that $p^* \sim_C q^*$ for all constant acts $p^*, q^* \ll p_0^*$. By monotonicity, this finally implies that $\{f \in \mathcal{F}_{[-1,0]} : f \ll p_0^*\}$ constitutes an indifference set. Nontriviality implies the existence of a distinct (and maximal) indifference set containing p_0^* . Suppose there exists an act h s.t. $p_0^* \succ_C h \succ_C p_{-1}^*$, then continuity and transitivity imply that $\lambda p_{-1}^* \succ_C p_{-1}^*$ for some $\lambda \in (0, 1)$, a contradiction. Thus \succsim_C has exactly two indifference sets. Homotheticity, monotonicity, and transitivity now imply that the higher indifference set can be written as $\{f \in \mathcal{F}_{[-1,0]} : u \circ f(s) = 0, \forall s \in E\}$, where $E \in \Sigma$ is some nonempty event. Suppose E is an atom of Σ , then the choice correspondence would be the one identified in step 5 of lemma 2 and therefore fulfil this lemma’s conclusion. Suppose $E = \mathcal{S}$, then continuity is violated: Restrict attention to acts that are measurable on $\{F, \mathcal{S} \setminus F\}$ for some $F \in \Sigma$ with $\emptyset \subset F \subset \mathcal{S}$ and identify utility acts with vectors $(u, v) \in \mathbb{R}_-^2$ with obvious interpretation, then $C(\{(0, 0), (1, 1), (1, 0)\}) = \{(1, 1)\}$, $C(\{(0, 0), (1, 0)\}) = \{(1, 0)\}$, yet $C(\{(0, 0), (\lambda, \lambda), (1, 0)\}) = \{(0, 0), (\lambda, \lambda), (1, 0)\}$ for any $\lambda \in (0, 1)$. Suppose the intermediate case where $E \subset \mathcal{S}$ but E is not an atom, then the preceding example is easily adapted by restricting attention to acts whose utility is

zero on $\mathcal{S} \setminus E$. ■

Lemma 11 delivers c-independence of \succsim_C , hence \succsim_C can be represented by $\min_{\pi \in \Gamma} \int v \circ f(s) d\pi$, where $\Gamma \subseteq \Delta\mathcal{S}$ is a unique, convex, compact set of priors, and v is an object entirely analogous to u . Indeed, v can be identified with u through lemma 1 and monotonicity.

Individual necessity of most axioms is established in the working paper version of Hayashi (2008). For necessity of c-betweenness, consider a choice correspondence as in lemma 2, but with $I(r \circ (f, M)) = [f(r \circ (f, M))^2(s) d\pi(s)]^{1/2}$, where $\pi \in \Delta\mathcal{S}$ is a prior.

Corollaries 5 and 6 Follow from the above by adapting lemma 11.

Theorem 7 Recall that for this theorem, all choice sets are convex hulls ΔM of menus M . Lemma 1 continues to hold. This is shown in Stoye (2009d), the argument is repeated here for completeness. Define the relation \succeq_C on constant acts by $p^* \succeq_C q^*$ iff $p^* \in C(\Delta\{p^*, q^*\})$. Then \succeq_C is complete: Assume by contradiction that $C(\Delta\{p^*, q^*\}) \cap \{p^*, q^*\} = \emptyset$, then by nonemptiness of C there exists λ s.t. $\lambda p^* + (1-\lambda)q^* \in C(\Delta\{p^*, q^*\})$, thus $\lambda p^* + (1-\lambda)q^* \in C(\Delta\{p^*, \lambda p^* + (1-\lambda)q^*\})$ by IIA for constant acts, thus $q^* \in C(\Delta\{p^*, q^*\})$ by independence. \succeq_C is also transitive: Suppose by contradiction that $p^* \succeq_C q^* \succeq_C r^* \succ_C p^*$. The latter implies (using independence twice) that $(1-\gamma+\lambda\gamma)r^* + \gamma(1-\lambda)q^* \succ_C (1-\gamma)r^* + \gamma\lambda p^* + \gamma(1-\lambda)q^*$ for all $\gamma, \lambda \in (0, 1]$. IIA for constant acts then yields $C(\Delta\{p^*, q^*, r^*\}) \subseteq \Delta\{q^*, r^*\}$, hence $\lambda r^* + (1-\lambda)q^* \in C(\Delta\{p^*, q^*, r^*\})$ for some $\lambda \in (0, 1)$. Independence and IIA then imply $q^* \in C(\Delta\{p^*, q^*, r^*\})$, after which IIA implies that $p^* \in C(\Delta\{p^*, q^*, r^*\})$, a contradiction. Since \succeq_C is complete and transitive, it induces choice correspondence C (Arrow (1959)). Furthermore, \succeq_C is von Neumann-Morgenstern utility by Herstein and Milnor (1953). That it fulfils independence is obvious; to see that it fulfils continuity, let $p^* \succ_C q^* \succ_C r^*$. As p^* is not in the convex hull of $\{q^*, r^*\}$, there exists $\lambda \in (0, 1)$ s.t. $\lambda p^* + (1-\lambda)r^* \succ_C q^*$ by continuity of C . Continuity of C also implies the existence of $\gamma \in (0, 1)$ s.t. $\gamma p^* + (1-\gamma)r^* \notin C(\Delta\{q^*, \gamma p^* + (1-\gamma)r^*\})$, but then completeness of \succeq_C implies $q^* \succ_C \gamma p^* + (1-\gamma)r^*$.

Step 1 of lemma 2 goes through unchanged for convex sets, hence acts can again be identified with utility acts. In contrast to before, \succsim_C is defined for any acts with nonpositive utility range, thus

$$\begin{aligned} f \succ_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(\Delta M), g \in M \setminus C(\Delta M) \\ f \sim_C g &\iff \exists M \in \mathcal{M}_0 : f \in C(\Delta M), g \in C(\Delta M) \end{aligned}$$

for any $f, g \in \mathcal{F}_{(-\infty, 0]}$.

The relation \succsim_C is in general incomplete for two reasons: f and/or g might lie outside the range of U , in which case the menus posited above cannot exist. More intricately, neither f nor g might be chosen from any menu containing both, namely if the decision maker reveals strict preference for a

mixture of the two. However, every relation \succsim on $\mathcal{F}_{(-\infty,0]} \times \mathcal{F}_{(-\infty,0]}$ that completes \succsim_C induces the restriction of C to menus $M \in \mathcal{M}_0$ as choice correspondence. To see this, fix $M \in \mathcal{M}_0$. Let $f \in C(M)$, then $f \succsim_C g$ for all $g \in M$, hence $f \succsim g$ for all $f \in M$. Let $h \in M \setminus C(M)$, then by nonemptiness of C , there exists $f \in M$ with $f \succ_C h$, hence $f \succ h$.

The theorems can be recovered by appropriately completing \succsim_C . This extension occurs in two steps that correspond to the aforementioned sources of incompleteness. The completion \succsim will be shown to be priorless maximin respectively maximin expected utility, after which the last step of the proof is exactly as before.

The first step to completing \succsim_C is to define the homothetic extension

$$f \succsim_C^* g \iff \exists \lambda \in (0, 1] : \lambda f \succsim_C \lambda g.$$

That this extends \succsim_C (and never contradicts it) follows from independence (used with p_0^* as mixture act) and INA. Note also that it coincides with \succsim_C if U is unbounded from below. The use of homothetic extension is the following: Some arguments below construct menus $M \in \mathcal{M}_0$ that may contain acts whose utility range exceeds the one of U . Homothetic extension allows one to ignore this concern because for any menu $M \in \mathcal{M}_0$, $\lambda M \subset \mathcal{F}_{[-1,0]}$ for λ small enough. For brevity of arguments, I will henceforth skip this step and simply ignore the possibility that U is bounded from below (hence, $\succsim_C^* = \succsim_C$).

To define the final extension, introduce the following shorthand notation: If v^* is a constant act, then the scalar v is its utility value, and the act $u_E^* v^*$ is more simply denoted $u_E v$. Now, for any $f \in \mathcal{F}_{(-\infty,0]}$, define its certainty equivalent $c(f)$ as the constant act v^* , where $v = \inf\{u : u^* \succ_C f\}$ and $v = 0$ if $\{u : u^* \succ_C f\} = \emptyset$. This object is well-defined because if $\underline{f} = \min_{s \in \mathcal{S}} u \circ f(s)$, then $\underline{f}^* \succ_C f$ would violate monotonicity. Define $f \succsim g \iff c(f) \geq c(g)$. Monotonicity and INA easily imply $f \geq g \Rightarrow f \succsim g$, and homotheticity implies $c(\lambda f) = \lambda c(f)$. Together with an additional use of monotonicity and INA, these two findings also yield that if $f \gg g$, then either $c(f) = p_0^*$ or $f \succ g$. Also, assume $f \sim_C p_0^*$ for some $f \ll p_0^*$, then homotheticity and monotonicity jointly imply that $f \sim_C p_0^*$ for *any* $f \ll p_0^*$. This is consistent with all subsequent steps of this proof but would ultimately imply triviality of C . I therefore henceforth presume that $f \sim_C p_0^*$ for no $f \ll p_0^*$, in which case monotonicity implies $f \prec_C p_0^*$ for all $f \ll p_0^*$.

From here, the proofs diverge.

Recovering Theorem 3

Step 1: $c(f) \ll p_0^*$ for any $f \neq p_0^*$. To see this, assume that $c(f) = p_0^*$ for some constant $f \ll p_0^*$, then $f \sim_C p_0^*$ by monotonicity, but now it would easily follow that C is trivial. For f not

constant, let $\underline{f} = \min_{s \in \mathcal{S}} u \circ f(s)$ and let $E \in \Sigma$ be the event on which $u \circ f(s) = \underline{f}$. Consider choice from $M = \{\underline{f}_E 0, 0_E \underline{f}, p_0^*\}$. Suppose that $\underline{f}_E 0 \in C(\Delta M)$, then $0_E \underline{f} \in C(\Delta M)$ by symmetry, thus $(\underline{f}/2)^* \in C(\Delta M)$ by convexity, thus $(\underline{f}/2)^* \sim_C p_0^*$, contradicting the conclusion for constant acts. Thus, $\underline{f}_E 0 \notin C(\Delta M)$. Continuity now implies that $\underline{f}_E 0 \notin C(\Delta\{\underline{f}_E 0, 0_E \underline{f}, q^*\})$ for some $q^* \ll p_0^*$, hence $c(\underline{f}_E 0) \ll p_0^*$, hence $c(f) \ll p_0^*$. A corollary is strict monotonicity, i.e. $f \gg g \Rightarrow c(f) \gg c(g)$.

Step 2: Fix any events $E, F \in \Sigma \setminus \{\emptyset, \mathcal{S}\}$ and scalars $u, v \leq 0$ in the range of U , then $c(u_E v) = c(u_F v)$. To see this, let $u > v$ w.l.o.g., initially assume that E and F are disjoint, and suppose $p^* \succ_C u_E v$. If $u + v < 2p$, the conclusion follows from observing that $C(\Delta\{(2p)_E 0, 0_E(2p)\}) = \{p^*\}$ by symmetry, convexity, and strict monotonicity and then interchanging the consequences of E and F . Else, let the acts $0_E x$ and $x_E 0$ be s.t. p^* is a convex combination of $u_E v$ and $x_E 0$. This implies $x < 2p$, thus symmetry, convexity, and strict monotonicity yield that $C(\Delta\{(0_E x, u_E v, x_E 0)\}) \subseteq \Delta\{0_E x, u_E v\} \cup \Delta\{u_E v, p^*\}$. Suppose $C(\Delta\{(0_E x, u_E v, x_E 0)\}) \subseteq \Delta\{0_E x, u_E v\}$, then there exists $f \in \Delta\{0_E x, u_E v\}$ s.t. $f \succ_C p^*$. On the other hand, let q^* be s.t. $u_E v$ is a convex combination of $0_E x$ and q^* , then application of symmetry etc. to $C(\Delta\{0_E x, q^*, x_E 0\})$ yields $q^* \succ_C f$. Consider the function ϕ from $[0, 1]$ into the power set of R^2 defined by $\phi(\lambda) = C(\Delta\{0_E x, \lambda p^* + (1 - \lambda)q^*, x_E 0\})$. Continuity implies that the graph of this function is continuous; also, we discovered $q^* \in \phi(0)$, $f \in \phi(1)$. These findings are compatible only if there exists $\lambda' \in [0, 1]$ for which $\phi(\lambda)$ contains an act $g \ll f$. It follows that $g \succ \lambda' p^* + (1 - \lambda')q^*$, but homotheticity and INA now easily lead to a contradiction with $p^* \succ_C u_E v$. Thus, $C(\Delta\{(0_E x, u_E v, x_E 0)\}) \subseteq \Delta\{u_E v, p^*\}$, but then $p^* \succ_C u_E v$ implies that $C(\Delta\{(0_E x, u_E v, x_E 0)\}) \cap \{u_E v, p^*\} = \{p^*\}$. The conclusion now follows by applying symmetry to $C(\Delta\{(0_E x, u_E v, x_E 0)\})$ and $C(\Delta\{(0_F x, u_F v, x_F 0)\})$.

If E and F are non-nested but overlap, use symmetry to interchange the consequences of $E \cap F^c$ and $F \cap E^c$. If they are nested, say $E \subset F$, then use symmetry twice, using $p^* \succ_C u_F v$ as intermediate step. The argument works in both directions, hence $p^* \succ_C u_E v \Leftrightarrow p^* \succ_C u_F v$, hence the claim.

Step 3: For this and the following step, consider a partition of \mathcal{S} into three nonempty events $\{E_1, E_2, E_3\}$ and restrict attention to acts that are measurable with respect to this partition; these acts will be identified with utility vectors (u, v, w) .

This step's substantial claim is that $c((u + v)/2, (u + v)/2, v) \geq c(u, v, v)$ for any $u > v$. To see this, it suffices to show that $c(u, v, v) \succ_C ((u + v)/2, (u + v)/2, v)$ does *not* hold. Suppose by contradiction that it does, then $C(\Delta M) \cap \{c(u, v, v), ((u + v)/2, (u + v)/2, v)\} = \{c(u, v, v)\}$ for some $M \in \mathcal{M}_0$. Let $M' = \{(0, 0, x), ((u + v)/2, (u + v)/2, v), c(u, v, v), (x, x, 0)\}$, then arguments very similar to step 2 yield that $C(\Delta M') \cap \{c(u, v, v), ((u + v)/2, (u + v)/2, v)\} = \{c(u, v, v)\}$ for a suitably low choice of x . Now consider $M'' = M' \cup \{(u, v, v), (v, u, v)\}$. M'' is invariant under exchange of the

consequences of the first two events and $\Delta M'$ contains all fixed points of such an exchange, hence symmetry and convexity jointly imply that $C(\Delta M'') \cap \Delta M' \neq \emptyset$, hence INA yields $C(\Delta M'') \cap \Delta M' = C(\Delta M')$, hence $C(\Delta M'') \cap \{c(u, v, v), ((u+v)/2, (u+v)/2, v)\} = \{c(u, v, v)\}$. Noting that $M'' \in \mathcal{M}_0$, $(u, v, v) \notin C(\Delta M'')$ would imply $c(u, v, v) \succ_C (u, v, v)$, a contradiction; hence $(u, v, v) \in C(\Delta M'')$, hence $(v, u, v) \in C(\Delta M'')$ by symmetry, hence $((u+v)/2, (u+v)/2, v) \in C(\Delta M'')$ by convexity, a contradiction.

Step 4: Step 2 implies that $c(u, v, v) = c(u, u, v)$ and that $c((u+v)/2, (u+v)/2, v) = c((u+v)/2, v, v)$; step 4 implies that $c((u+v)/2, (u+v)/2, v) \geq c(u, v, v)$. Recall that $c(u, u, v) \geq c((u+v)/2, (u+v)/2, v)$ by monotonicity. Together, these findings imply that $c(u, v, v) = c((u+v)/2, v, v)$. Iterating this argument and using monotonicity, one finds that $c(u, v, v) = c(w, v, v)$ for any $w \in (v, 0)$. If $c(u, v, v) \gg v^*$, then $c(u, v, v) \succ_C c(w, v, v)$ for some $w > v$ by strict monotonicity. On the other hand, $c(u, v, v) \geq v^*$ by monotonicity, hence $c(u, v, v) = v^*$.

This finding can be extended to general acts: Fix any act $f \in \mathcal{F}_{[-1,0]}$, define $\underline{f}[\bar{f}] = \min_{s \in \mathcal{S}} [\max_{s \in \mathcal{S}} u \circ f(s)]$, and let E be the event on which $u \circ f(s) = \underline{f}$. Then $\underline{f}_E \bar{f} \geq f$, hence $c(f) \leq c(\underline{f}_E \bar{f}) = \underline{f}^*$, but also $f \geq \underline{f}^* \Rightarrow c(f) \geq \underline{f}^*$. It follows that $c(f) = \underline{f}^*$, thus \succsim is priorless maximin.

Step 5: To see that \succsim extends \succsim_C , suppose $f \succ_C g$ and let M be the menu in which this preference is observed. Consider choice from $N \equiv M \cup \{c(f), c(g)\}$. Step 4 implies that $f \geq c(f)$ and $g \geq c(g)$, hence repeated uses of INA yield $f \in C(\Delta N) \Rightarrow c(f) \in C(\Delta N)$ and $g \notin C(\Delta N) \Rightarrow c(g) \notin C(\Delta N)$, hence $c(f) \gg c(g)$. The argument for $f \sim_C g \Rightarrow f \sim g$ is similar.

Recovering Theorem 4

Step 1: Call an act $f \in \mathcal{F}_{[-1,0]}$ *uniquely* choosable if there exists a menu $M \in \mathcal{M}_0$ s.t. $C(\Delta M) = \{f\}$. Assume first that p_0^* is not uniquely choosable, thus there exists no $M \in \mathcal{M}_0$ s.t. $C(\Delta M) = \{p_0^*\}$. INA and monotonicity then imply that there exists a set $I_0 \subseteq \mathcal{F}_{(-\infty,0]}$ of acts s.t. $M \in \mathcal{M}_0 \Rightarrow M \cap I_0 \neq \emptyset$ and where $f \sim_C p_0^*$ for any $f \in I_0$. If I_0 contained an act $f \ll p_0^*$, one could easily conclude that C is trivial. Furthermore, ambiguity aversion implies that I_0 is convex, and independence (used with p_0^* as mixture act) implies that it is a cone. All these findings are consistent only if $I_0 = \{f : u \circ f(s^*) = 0\}$ for some $s^* \in \mathcal{S}$. The choice correspondence is then the one from step 5 of the proof of lemma 2 and thereby fulfils this theorem's conclusion. For the remainder of this proof, assume therefore that p_0^* is uniquely choosable. By continuity, this implies that some constant act $p^* \ll p_0^*$ is choosable, hence (by homotheticity) all constant acts are, a fact that will be used liberally.

Suppose by contradiction that $f \succ_C c(f)$, then continuity implies $f \succ_C p^*$ for some $p^* \gg c(f)$, then monotonicity implies $f \succ_C q^*$ for all $q^* \ll p^*$, contradicting the definition of $c(f)$. Suppose

by contradiction that $f \prec_C c(f)$, then continuity implies that $f \prec_C p^*$ for some $p^* \ll c(f)$, again contradicting the definition of $c(f)$. It follows that $f \sim_C c(f)$ or $f \bowtie_C c(f)$.

If f is choosable, then $f \sim_C c(f)$. To see this, fix f and let M be s.t. $f \in C(\Delta M)$. Suppose that f and $c(f)$ are noncomparable, then $f \notin C(\Delta(M \cup \{c(f)\}))$ but also $c(f) \notin C(\Delta(M \cup \{c(f)\}))$. The former implies that $C(\Delta(M \cup \{c(f)\})) \cap \Delta M = \emptyset$, but c-betweenness then implies that $C(\Delta(M \cup \{c(f)\})) = \emptyset$.

Step 2: If f is not choosable, then $c(f) = \underline{f}^*$. To see this, fix a non-choosable act f and any constant act $p^* \gg \underline{f}^*$. The plan is to construct a menu in which $p^* \succ_C f$. Let \tilde{f} be an act on the boundary of $F_{(-\infty, 0]}$, i.e. $\max_{s \in \mathcal{S}} u \circ \tilde{f}(s) = 0$, s.t. f is a convex combination of p^* and \tilde{f} . Let $M \in \mathcal{M}_0$ be s.t. $C(\Delta M) = \{p^*\}$. If $\tilde{f} \in M$ or \tilde{f} is dominated by an element of M , the argument is complete. Else, let $\{g^1, g^2, \dots\}$ collect the boundary acts in M . Let E denote the event on which $u \circ \tilde{f}(s) = 0$ and replace every g^i with $q_E^* g^i$, where the utility value of q^* equals $\min_{s \in \mathcal{S}, g \in M} u \circ g(s)$ (thus all acts become worse). Finally, multiply all elements of $\{g^1, g^2, \dots\}$ by $\lambda > 1$, where λ is large enough s.t. for any i , some convex combination of \tilde{f} and g^i is strictly dominated by p^* . (This is possible because the previous step insured that $u \circ \tilde{f}(s) = u \circ g^i(s) = 0$ for no state s .)

Now consider choice from $\{p^*, \tilde{f}, g^1, g^2, \dots\} \in \mathcal{M}_0$. The act f is contained in the convex hull of this menu but is not choosable, hence it suffices to show that p^* is chosen. Suppose otherwise. By c-betweenness, one then has $C(\Delta(\{p^*, \tilde{f}, g^1, g^2, \dots\})) \subseteq \Delta(\{\tilde{f}, g^1, g^2, \dots\})$. By construction, some convex combination of \tilde{f} and g^i is strictly dominated by p^* for any i , hence any proper mixture of \tilde{f} and g^i is strictly dominated by some mixture of either \tilde{f} and p^* or g^i and p^* , hence $C(\Delta(\{p^*, \tilde{f}, g^1, g^2, \dots\}))$ does not contain any such mixture or a mixture containing such a mixture, hence $C(\Delta(\{p^*, \tilde{f}, g^1, g^2, \dots\})) \subseteq [\tilde{f}] \cup \Delta(\{g^1, g^2, \dots\})$. $C(\Delta(\{p^*, \tilde{f}, g^1, g^2, \dots\})) \cap \Delta(\{g^1, g^2, \dots\}) = \emptyset$ by construction (any element of $\Delta(\{g^1, g^2, \dots\})$ is dominated by an element of M), hence $C(\Delta(\{p^*, \tilde{f}, g^1, g^2, \dots\})) = \tilde{f}$. This implies that \tilde{f} is choosable, but now homotheticity and the fact that $\tilde{f} \sim_C c(\tilde{f})$ easily imply that f is choosable, a contradiction.

Step 3: To see that \succsim completes \succsim_C , assume first that $f \succ_C g$. Fix M s.t. $C(\Delta M) \cap \{f, g\} = \{f\}$. Consider choice from ΔN , where $N \equiv M \cup \{c(g)\}$. If $c(g)$ were chosen, g would have to be chosen by INA, a contradiction. Thus $c(g)$ is not chosen. Now if $C(\Delta N) \cap \Delta M = \emptyset$, c-betweenness would imply $C(\Delta N) = \emptyset$; thus $C(\Delta N) \cap \Delta M = C(\Delta M)$, in particular $C(\Delta N) \cap \{f, g\} = \{f\}$. Now suppose by contradiction that $c(f) \leq c(g)$, then choice from $\Delta(N \cup \{c(f)\})$ would have to contradict either INA or monotonicity (recalling that $f \succ_C g$ implies f choosable, hence $f \sim_C c(f)$). Thus, $c(f) \gg c(g)$ and hence $f \succ g$. A similar argument establishes that $f \sim_C g \Rightarrow f \sim g$ (again, recall that $f \sim_C g$ implies f, g choosable, hence $f \sim_C c(f), g \sim_C c(g)$).

Step 4: It remains to verify that \succsim fulfils Gilboa and Schmeidler's (1989) axioms. The first two steps and convexity jointly imply that indifference curves in $\mathcal{F}_{[-1,0]}$ are collections of rays emanating from the constant acts, thus \succsim fulfils c-independence. Let the function $\phi(f) = I(u \circ c(f))$ map acts f onto the utility values of their certainty equivalents, then I represents \succsim . Furthermore, steps 1 and 2 and continuity of C easily imply that I is continuous on $\Delta\{f, g\}$ for any acts f, g . This firstly implies that \succsim is mixture continuous. To see ambiguity aversion, fix any acts f, g with $f \sim g$ and assume by contradiction that $\min_{\lambda \in [0,1]} I(\lambda f + (1-\lambda)g) < I(f)$. Let the minimum be attained at λ^* , then it now follows that one can choose $\underline{\lambda} < \lambda^* < \bar{\lambda}$ s.t. $I(\underline{\lambda}f + (1-\underline{\lambda})g) = I(\bar{\lambda}f + (1-\bar{\lambda})g) = \max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} I(\lambda f + (1-\lambda)g)$. More generally, it follows that if ambiguity aversion fails, then there exist acts f and g s.t. $I(f) = I(g) = \max_{h \in \Delta\{f,g\}} I(h)$. Thus, to complete the argument, pick f and g s.t. this latter condition holds.

Now assume that both f and g are choosable, thus $f \sim_C c(f)$ and $g \sim_C c(g) = c(f)$. A construction very similar to the one in step 2 can be used to find $M \in \mathcal{M}_0$ s.t. $C(\Delta M) \cap \Delta\{f, g\} \neq \emptyset$. As ϕ represents \succsim , it follows that $C(\Delta M)$ is not convex. If at least one of f and g (say g) is not choosable, then $c(f) = \underline{g}^*$. Recalling that certainty equivalents respect monotonicity, one has $I(\lambda f + (1-\lambda)g) \geq I(\lambda f + (1-\lambda)\underline{g}^*) = I(\lambda f + (1-\lambda)c(f)) = I(c(f))$ by c-independence.

Theorem 9 Recall that theorem 4 applies. Define the incomplete relation \succeq on $\mathcal{F}_{[-1,0]} \times \mathcal{F}_{[-1,0]}$ by

$$f \succeq g \iff \lambda f + (1-\lambda)h \succsim_C \lambda g + (1-\lambda)h, \forall \lambda \in (0, 1], h \in \mathcal{F}_{[-1,0]}.$$

Then by proposition 5 in Ghirardato et al. (2004; see also Bewley (2002)), there exists a unique, compact, convex, set of probabilities $\tilde{\Gamma}$ s.t.

$$f \succeq g \iff \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \tilde{\Gamma}.$$

Now, proposition 19 of the same paper (used with $\alpha = 1$) implies that $\tilde{\Gamma} = \Gamma$.

For any acts f and g , define the utility act $f \ominus g$ by $u \circ (f \ominus g)(s) = u \circ f(s) - u \circ g(s)$. Recall that for any menu M , \bar{f}_M is the act with $u \circ \bar{f}_M(s) = \max_{f \in M} u \circ f(s)$. Then it easily follows from INA and independence that for any $f, g \in \mathcal{F}_{[-1,0]}$, $f \succeq_C g$ iff

$$\gamma(f \ominus \bar{f}_M) \succsim_C \gamma(g \ominus \bar{f}_M)$$

for any menu $M \supseteq \{f, g\}$ and any γ small enough s.t. $\gamma(f \ominus \bar{f}_M), \gamma(g \ominus \bar{f}_M) \in \mathcal{F}_{[-1,0]}$ (implying that \succsim_C is defined).

The claim is that $\succeq_C = \succeq$. This is so because the following two statements are equivalent:

$$\begin{aligned} \exists \lambda \in (0, 1], h \in \mathcal{F}_{[-1,0]} : \lambda f + (1-\lambda)h \prec_C \lambda g + (1-\lambda)h \\ \exists \gamma \in (0, 1], M \supseteq \{f, g\} : \gamma(f \ominus \bar{f}_M) \prec_C \gamma(g \ominus \bar{f}_M). \end{aligned}$$

To see this, use independence and the utility act representation repeatedly to write

$$\begin{aligned}
& \gamma(f \ominus \bar{f}_M) \prec_C \gamma(g \ominus \bar{f}_M) \\
& \iff \frac{\gamma}{2}f - \frac{\gamma}{2}\bar{f}_M \prec_C \frac{\gamma}{2}g - \frac{\gamma}{2}\bar{f}_M \\
& \iff \underbrace{(\gamma\rho/2)f}_{=: \lambda} + (1 - \gamma\rho/2) \underbrace{\left[\frac{1 - \rho}{1 - \gamma\rho/2} p_{-1/2}^* - \frac{\gamma\rho/2}{1 - \gamma\rho/2} \bar{f}_M \right]}_{=: h} \prec_C \lambda g + (1 - \lambda)h,
\end{aligned}$$

assuming that ρ is chosen small enough s.t. $h \in \mathcal{F}_{[-1,0]}$. It follows that if $\gamma(f \ominus \bar{f}_M) \prec_C \gamma(g \ominus \bar{f}_M)$ for some (γ, M) , then $\lambda f + (1 - \lambda)h \prec \lambda g + (1 - \lambda)h$ for (λ, h) as just defined. The converse holds because the construction is reversible.

Corollary 10 Define \succeq^* by

$$f \succeq^* g \iff \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

Then axiom 10 states that $f \succeq^* g \Rightarrow f \succeq g$, whereas axiom 11 states that $f \succeq g \Rightarrow f \succeq^* g$. The claim then follows from theorem 9.

Theorem 11 The proof of lemma 1 goes through unchanged. For any menu M , redefine the ‘‘oracle act’’ \bar{f}_M by $u \circ \bar{f}_M(s) \equiv \max_{f \in M \cap \tilde{\mathcal{F}}} u \circ f(s)$. Conditions on $\tilde{\mathcal{F}}$ are s.t. after appropriate normalization of U , any finite, Σ -measurable step function $u \circ f : \mathcal{S} \rightarrow [-1, 1]$ can be identified both with an act $f \in \mathcal{F}$ and an act $\tilde{f} \in \tilde{\mathcal{F}}$. Then the proof of lemma 2 goes through unchanged.

Subsequent proofs go through essentially unaltered; the fact that one can add (nonspecial) acts to menus without altering \bar{f}_M actually leads to some simplification. The main caveat to this assessment concerns the proof of lemma 11. Here one need not worry about the possibility that strict monotonicity fails, however c-independence can only be established separately for the strict upper and lower contour set of p_0^* . Monotonicity ensures that these sets cannot intersect, and strict monotonicity ensures that the intersection of their complements cannot contain two acts ordered by strict dominance, which together implies c-independence. In contrast, the axioms from theorem 4 are now consistent with a multiple prior minimax regret choice correspondence where the set of priors discontinuously expands as one moves from the lower to the upper contour set of p_0^* , leading to a discontinuous ‘‘fanning in’’ of indifference sets at p_0^* and thereby implying that the indifference set containing p_0^* is thick.

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