

# Statistical Decisions under Ambiguity

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## Abstract

This paper provides axiomatic foundations for maximin criteria in statistical decision theory. Specifically, consider a decision maker who faces a number of possible models of the world. Every model generates objective probabilities, but no probabilities of models are given. This is the classic problem captured by Wald's (1950) device of risk functions.

I characterize a number of decision rules including Bayesianism (as a backdrop), maximin utility, the Hurwicz criterion, and especially several variations of minimax regret. The main contributions are the unified axiomatization of these rules in a framework tailored to statistical decision making, an axiomatic system that relaxes transitivity as well as menu-independence of preferences, and the introduction of new, regret-based decision criteria. Interestingly, the axiom that picks regret-based rules over maximin utility is independence.

**Keywords:** statistical decisions, risk functions, ambiguity, maximin utility, minimax regret.

**JEL classification codes:** C44, D81.

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# 1 Introduction

“[T]here are two types of uncertainty: one as to the hypothesis, which is expressed by saying that the hypothesis is known to belong to a certain class or model, and one as to the future events or observations given the hypothesis, which is expressed by a probability distribution.” (Arrow 1951, p. 418)

This paper provides axiomatic foundations for minimax criteria in statistical decision theory. It therefore extends a literature that began and peaked in the 1950’s.<sup>1</sup> The meantime saw dramatic progress in decision theory, but much of it was in a form geared toward economic theorists rather than statisticians or decision makers. The present paper is informed by these developments, but differs both in the framework used and in the type of results generated.

As to the framework, I assume that a decision maker simultaneously faces nonprobabilistic ambiguity or “Knightian uncertainty” (about the true model) and probabilistic uncertainty or risk (given the true model). This idea is expressed in the preceding quotation. I formalize it by using an Anscombe/Aumann (1963) setting not just for simplicity, but as the appropriate model. The decision maker is presumed to resolve the uncertainty by expected utility, but may react differently to ambiguity. This captures what statisticians do when they write down *risk functions*. Different criteria in statistical decision theory differ in the way these risk functions are evaluated. I characterize Bayesian and maximin utility decision rules, the Hurwicz criterion, minimax regret, and orderings I call “pairwise minimax regret” and “pairwise  $\alpha$ -minimax regret.” Minimax regret is a historically old (Savage 1951) criterion that has recently attracted renewed attention (Brock 2006; Chamberlain 2000; Manski 2004, 2006, 2007, 2008; Schlag 2003, 2006); the other regret-based criteria are essentially new to the literature. To capture their specific properties, I relax some of the most standard axioms and allow preference orderings to be potentially intransitive as well as dependent on the choice set.

A very quick summary of results goes as follows:

- Imposing the classic axioms of Bayesian rationality on top of expected utility for constant acts leads to a characterization of Bayesianism. Adding an axiom that reflects ignorance with respect to states of the world then generates a contradiction.
- This contradiction can be resolved by relaxing some of the Bayesian axioms. Relaxing independence leads to characterizations of the Hurwicz criterion as well as maximin utility. Relaxing independence of irrelevant alternatives leads to a characterization of minimax regret. Relaxing

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<sup>1</sup>Relevant references include the books by Luce and Raiffa (1957), Savage (1954), von Neumann and Morgenstern (1944), and Wald (1950), as well as articles by Arrow (1951), Arrow and Hurwicz (1972), Chernoff (1954), Milnor (1954), and Savage (1951).

transitivity leads to characterizations of the other regret-based orderings. It also leads to a conflict between generating decisive choice criteria, that is, criteria which generate strict preferences in nontrivial comparisons, and avoiding preference cycles.

This paper is a companion paper to Stoye (2007b). The papers are non-nested in scope: I here axiomatize numerous decision rules other than minimax regret; Stoye (2007b) focuses on minimax regret but provides several extensions. A more fundamental difference is that Stoye (2007b) is firmly rooted in conventional decision theory, including the “revealed preference” paradigm. In contrast, the present paper is a normative analysis of statistical decision making, and as such has some unconventional features. This will be elaborated in section 2.3.

The remainder of this paper is structured as follows. Section 2 is devoted to explaining the setup and notation and to further elaborating its relation to statistical decisions as well as differences to the existing literature. Section 3 contains the axiomatic treatment and section 4 concludes. The appendices collect all technical arguments.

## 2 The Decision Theoretic Framework

### 2.1 Setup and Notation

Consider a set  $\mathcal{S}$  of states of the world  $s$ , endowed with an algebra  $\Sigma$  of events  $E, F$ , etc.; a set  $\mathcal{Z}$  of outcomes  $z$ ; and a set  $\mathcal{A}$  of possible acts (e.g. treatments or policy choices)  $a, b$ , etc. There must be at least three nonempty events and two outcomes; other than that, the objects  $\mathcal{S}$ ,  $\mathcal{Z}$ , and  $\Sigma$  are not restricted. An act  $a$  is a  $\Sigma$ -measurable, initially finite step function from states  $s$  onto probability measures  $P(a, s) \in \Delta\mathcal{Z}$ ;  $\Delta(\cdot)$  denotes the set of finite probability mixtures over the argument. I embed  $\Delta\mathcal{Z}$  in  $\mathcal{A}$  by writing  $P^*$  for the constant act defined by  $P(a, s) = P$  for every  $s$ . Acts that are not constant in this sense are called *ambiguous*. Mixtures between acts are written in the usual way and are identified with statewise mixtures, i.e.  $c = pa + (1 - p)b$  is the act generated by performing  $a$  with probability  $p$  and  $b$  otherwise and is characterized by  $P(c, s) = pP(a, s) + (1 - p)P(b, s)$  for every  $s$ .

The decision maker can choose from a nonempty, initially finite menu  $M \subseteq \mathcal{A}$ . She can randomize, thus her choice set is really  $\Delta M$ .<sup>2</sup> The notation  $pM + (1 - p)b$  denotes the menu generated by replacing every element  $a$  of  $M$  with the analog mixture. The object to be axiomatized is a family of preference relations  $\succsim_M$ , where  $\succsim_M$  is a binary relation on  $\Delta M \times \Delta M$  from which  $\succ_M$  and  $\sim_M$  are derived in the usual manner. Thus, preferences over acts may depend on the choice context. All in all, the setting is

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<sup>2</sup>This assumption is natural given the intended applications – statistical decision rules can, and sometimes should, randomize –, but does not drive any result. Notice, however, the implicit assumption that the decision maker integrates probabilities generated by her randomization device with objective probabilities generated by the models.

just as in Gilboa and Schmeidler (1989) as well as many other references, with the existence of menus being made explicit.

Maintained axioms will insure that von Neumann-Morgenstern expected utility applies to constant acts  $P^*$ , thus there exists  $U : \mathcal{Z} \rightarrow \mathbb{R}$  s.t. those acts are ranked according to  $\int U(z)dP^*$ . Indeed, I will generally use the notational shorthand  $u(a, s) \equiv \int U(z)dP(a, s)$ . The range of  $U$  can be bounded or unbounded; in particular, best and worst possible outcomes may or may not exist.<sup>3</sup> An act is *admissible* if it is not dominated by another act with respect to  $u(a, s)$ . It is (*strictly*) *potentially optimal* within  $M$  if it (*strictly*) maximizes  $u(a, s)$  over  $M$  for some  $s \in \mathcal{S}$ . A potentially optimal act is always (weakly) admissible, but not vice versa.

This paper is motivated by statistical decision problems, hence it may be of interest to clarify the relation to statistical terminology. Statisticians typically use notation along the lines of Berger (1985), whose primitives are a parameter  $\theta \in \mathbb{R}^j$ , a sample space  $\mathcal{X}$  with typical element  $x$ , a decision rule  $\delta : \mathcal{X} \rightarrow \mathbb{R}^j$ , and a loss function  $L : \mathbb{R}^{2j} \rightarrow \mathbb{R}$ . To translate to the present setup, identify decision rules  $\delta$  with acts  $a$ , loss  $L$  with (the negative of) utility  $U$ , and states  $s$  with couplets  $(\theta, P(x)) \in \mathbb{R}^j \times \Delta\mathcal{X}$ , then the *risk function* of a decision rule is a mapping  $r : \mathcal{S} \rightarrow \mathbb{R}$  that maps states  $s$  onto risks  $r(\delta, s) \equiv \int L(\theta, \delta)dP(x)$  and that corresponds to  $u$  here. The Bayesian and maximin utility criteria are used throughout statistics. Minimax regret has seen occasional use, especially in recent work in econometrics: Droge (1998, 2006) uses it to choose between “selection” and “shrinkage” estimators, Eldar et al. (2004) to solve a modified OLS estimation problem, DasGupta and Studden (1991) employ it in regression design, Manski (2004), Hirano and Porter (2005), Schlag (2006), and Stoye (2006, 2007a, 2007c) apply it to treatment choice, and Schlag (2003) brings it to the closely related problem of playing a two-armed bandit. Also related is the technique of “scenario planning” in Operations Research and business economics. See the textbook by Kouvelis and Yu (1997) for explorations of minimax regret in this context and Loulou and Kanudia (1999) for an application.

## 2.2 The Contenders

This section exhibits the decision rules to be axiomatized. An initial benchmark is the one ordering whose strict part should be uncontroversial.

**Definition 1** *Strict Statewise Dominance (SSD)*

$$\begin{aligned} a \succ_M b &\iff u(a, s) > u(b, s), \forall s \in \mathcal{S} \\ a \sim_M b &\iff a \not\succeq b \wedge b \not\succeq a. \end{aligned}$$

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<sup>3</sup>Puppe and Schlag (2008) relax an assumption made here, namely that the outcome space is state independent. The weakest structure actually needed to generate results is identified in their paper. A somewhat stronger, but simple and sufficient condition is that there exists constant acts  $P^*$  and  $Q^*$  with  $P^* \succ_{\{P^*, Q^*\}} Q^*$ .

SSD only ranks acts whose comparison is trivial and prescribes indifference over the set of weakly admissible acts. This is just how far one can go without committing to some attitude about ambiguity. A polar attitude is given by Bayesianism:

**Definition 2 Bayesianism (“Subjective Expected Utility”)**

There exists a probability measure  $\pi$  over  $\mathcal{S}$  (the “prior”) s.t.

$$a \succsim_M b \iff \int u(a, s) d\pi \geq \int u(b, s) d\pi.$$

The Bayesian model is well known not only to theorists, but also in statistical applications, where it corresponds to the ranking of decision rules by Bayes risk.

The focus of this paper is on criteria that avoid probabilistic treatment of ambiguity and are rather concerned with uniformity (in some sense) of performance across states. The best-known among these are the following:

**Definition 3 Maximin Utility (MU)**

$$a \succsim_M b \iff \min_{s \in \mathcal{S}} u(a, s) \geq \min_{s \in \mathcal{S}} u(b, s).$$

**Definition 4  $\alpha$ -Maximin Utility ( $\alpha$ -MU, “Hurwicz Criterion”)**

There exists  $\alpha \in [0, 1]$  s.t.

$$a \succsim_M b \iff \alpha \max_{s \in \mathcal{S}} u(a, s) + (1 - \alpha) \min_{s \in \mathcal{S}} u(a, s) \geq \alpha \max_{s \in \mathcal{S}} u(b, s) + (1 - \alpha) \min_{s \in \mathcal{S}} u(b, s).$$

Maximin utility was introduced to statistical decision theory by Wald (1950) and received a famous philosophical endorsement in Rawls’ “Theory of Justice” (1999). Its criticisms, which must date back almost as far, usually focus on examples like Rawls’ own (p. 136):

**Rawls’ Example Against Maximin Utility** *Let there be two states of the world  $s_1$  and  $s_2$  and acts  $a$  and  $b$  inducing the following utilities (for some  $n > 1$ ):*

	$s_1$	$s_2$
$u(a, s)$	0	$n$
$u(b, s)$	$1/n$	1

In this example, maximin utility always picks action  $b$ . Rawls concedes that this becomes implausible as  $n \rightarrow \infty$ . In addition, the decision maker would reverse her preference upon learning that whatever she chooses, she will receive a large enough payment in state  $s_1$  only. This latter observation informs Berger’s (1985, p. 372) argument against maximin utility.

If these features are seen as problems, the issue may be that attention is focused on the worst possible state of nature, rather than the one in which one’s action matters. This informs the idea to

minimize maximum regret, where regret is defined as the loss incurred by not having chosen the ex post optimal act.

**Definition 5 *Minimax Regret (MR)***

$$a \succsim_M b \iff \max_{s \in \mathcal{S}} \{ \max_{a^* \in M} u(a^*, s) - u(a, s) \} \leq \max_{s \in \mathcal{S}} \{ \max_{b^* \in M} u(b^*, s) - u(a, s) \}.$$

Minimax regret was first suggested in Savage’s (1951) reading of Wald (1950). Numerous authors consider minimax regret so obviously superior over maximin utility that they think of it as the “real” maximin utility (Savage 1951, 1954, Berger 1985, Manski 2004). However, minimax regret preference between  $a$  and  $b$  may depend on the menu from which the two can be chosen. In particular, it can happen that  $a$  is chosen from the menu  $\{a, b\}$ , yet  $b$  is chosen from  $\{a, b, c\}$ .<sup>4</sup> This is avoided by the following criterion.

**Definition 6 *Pairwise Minimax Regret (PMR)***

$$a \succsim_M b \iff \max_{s \in \mathcal{S}} \{ u(b, s) - u(a, s) \} \leq \max_{s \in \mathcal{S}} \{ u(a, s) - u(b, s) \}.$$

Pairwise minimax regret says that  $a$  is preferred over  $b$  if the worst-case regret from having chosen  $b$  over  $a$  exceeds the worst-case regret from having chosen  $a$  over  $b$ . This idea relates it to regret theories as proposed by Loomes and Sugden (1982) and Fishburn (1989), and it has in common with them that it is not, in general, transitive. This latter feature led to its immediate rejection by Luce and Raiffa (1957), who mention it briefly after discussing minimax regret; to my knowledge, this is the criterion’s only appearance in the literature so far.

Pairwise minimax regret can be generalized as follows.

**Definition 7 *Pairwise  $\alpha$ -Minimax Regret ( $\alpha$ -PMR)***

There exists  $\alpha \in [0, 1]$  s.t.

$$\begin{aligned} a \succ_M b &\iff \max_{s \in \mathcal{S}} \{ u(b, s) - u(a, s) \} < \alpha \max_{s \in \mathcal{S}} \{ u(a, s) - u(b, s) \} \\ a \sim_M b &\iff a \not\succeq b \wedge b \not\succeq a. \end{aligned}$$

With pairwise  $\alpha$ -minimax regret,  $a$  will be preferred over  $b$  only if the worst-case regret from choosing  $b$  over  $a$  exceeds the worst-case regret in the other direction by sufficiently much. As a result, the ordering exhibits thick indifference sets. In fact, pairwise  $\alpha$ -minimax regret can be seen as a parameterized compromise between pairwise minimax regret and strict statewise dominance: It equals the former for  $\alpha = 1$ , the latter for  $\alpha = 0$ , and generates a smooth transition for intermediate  $\alpha$ .

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<sup>4</sup>For a striking example, assume there are three states  $\{s_1, s_2, s_3\}$ , identify acts with vectors  $(u(\cdot, s_1), u(\cdot, s_2), u(\cdot, s_3))$ , and consider the acts  $a \equiv (1, 2, 3)$  and  $b \equiv (3, 4, 2)$ . It can be verified that  $b \succ_{\{a, b\}} a$ , that the MR-act is to choose  $b$  with probability  $2/3$  and that the pure act  $a$  is the least preferred choice in  $\Delta(\{a, b\})$ . If one adds  $c \equiv (-10, -10, 5)$  to the picture, then the ranking is  $a \succ_{\{a, b, c\}} b \succ_{\{a, b, c\}} c$ , and the MR-act is to choose  $a$  with certainty, i.e. an act that was feasible yet *worst* absent  $c$ . Arrow (1951), Chernoff (1954), and Milnor (1954) provide further examples.

## 2.3 Comparison to Related Literature

The minimax criteria proposed here minimize utility, or maximize regret, over the entire state space. This contrasts with much other work in decision theory, which uses sets of priors. Recall that Gilboa and Schmeidler (1989) axiomatize the preference ordering represented by  $U(a) \equiv \min_{\pi \in C} \int u(a, s) d\pi$ , where  $C$  can be thought of as a set of priors. No claims are made regarding the psychological accuracy of the model, and  $C$  should not be interpreted as describing the agent’s actual beliefs. In cases where some set of priors  $C^*$  is externally specified, Gilboa and Schmeidler’s representation result is silent on the relation between  $C$  and  $C^*$ .<sup>5</sup>

This is all as it should be from a revealed preference point of view. In most descriptive applications, agent’s beliefs are unobservable except insofar as they are revealed through choice behavior, and one would certainly not want to impose that they encompass the whole state space. But there are at least three reasons why it is not an accurate model of statistical decision theory, and the model proposed here is motivated by those concerns.

(i) Statisticians routinely specify utility (respectively loss) functions and accept expected utility as criterion conditional on any given state. Indeed, the very device of risk functions makes no sense otherwise. The question of how to deal with ambiguity is raised only after these steps, and it is where different approaches diverge. Hence, statisticians conceptually differentiate between uncertainty given a model (“estimation uncertainty,” which is always seen as probabilistic) and uncertainty about models (“model uncertainty,” which may not) in the exact manner suggested by the Arrow quote, and the distinction is practically relevant for all non-Bayesian statisticians. The Anscombe/Aumann model, together with the assumption that von Neumann-Morgenstern utility applies to roulette acts, captures this mindset.<sup>6</sup>

(ii) Statistician’s beliefs are certainly observable to themselves and usually specified explicitly before actions are contemplated. A clean example of this is the progression from Manski’s (2000) description of the ambiguity inherent in a decision problem to his use of minimax regret (Manski 2007, 2008) to solve that problem. This is certainly not captured by presuming that statisticians infer their beliefs

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<sup>5</sup>Ghirardato et al. (2004) may appear to identify  $C$  with an agent’s perception of ambiguity. But in their discussion (p. 137-138), they clarify that “ambiguity perception” is only a label for the special role that  $C$  plays in their approach; no relation between  $C$  and an agent’s objective information and/or subjective (set of) beliefs is claimed. Stoye (2007b, corollary 8) shows how Ghirardato et al. (2004) can be used to axiomatize a link between  $C$  and an exogenous object  $C^*$ .

<sup>6</sup>The one type of statistician to voice dissent here would be a strict subjectivist who denies any meaningful difference between objective and subjective probability, maintaining that all probability is the latter and should arise from restrictions on behavior as in Savage (1954). Notice, however, that being Bayesian per se does not force one to agree with this argument’s premises that (i) probabilities on coin tosses are epistemically equivalent to probabilities on elections and (ii) actions are primary and beliefs are secondary. In fact, both of these are controversial.

from observing their own actions. A revealed preference interpretation of  $C$  is, therefore, not warranted.

(iii) If maximin-type decision rules are advocated, they tend to compute some worst-case contingency over the entire state space rather than over a set of priors (e.g., Wald 1950, Manski 2004). This is just what is modelled here; indeed, it is pretty much unavoidable for a frequentist.

By accommodating these concerns, the present model resurrects some features of the classical decision theoretic literature in a modern framework. Doing so comes at a cost because it requires more axioms and renders the specification of  $\mathcal{S}$  a more sensitive matter:  $\Sigma$  is really the algebra of plausible, and not merely conceivable, events, where a characterization of “plausible” exceeds the scope of this paper. The benefit is that the present axioms, if believable, provide a justification for what statistical decision makers actually do.<sup>7</sup>

In the specific case of minimax regret, framing the discussion in terms of preferences sacrifices an “as if”-perspective in yet another sense: The minimax regret preference ordering, and hence any axiomatization of it, entails statements that are vacuous in terms of observable choices, like the second part of  $a \succ_{\{a,b,c\}} b \succ_{\{a,b,c\}} c$ . A revealed preference characterization should, therefore, focus on choice correspondences as in Hayashi (2008). This difference does not much affect substantive results: Stoye (2007b, theorem 2) shows that the present characterization of minimax regret has a close analog in the language of choice correspondences. I here stick with preferences for three reasons: First, I suspect that intuitions are more often about preferences than about choice correspondences, so axioms about the former lend themselves more easily to normative evaluation. Second, intransitive preference orderings cannot be thought of in terms of choice correspondences. Third, it is natural to assume, as I do here, that statistical decision makers can randomize, but this possibility causes delicate problems with respect to choice correspondences (Stoye 2007b, section 2.2).

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<sup>7</sup>The idea of taking sets of priors to be non-behavioral has recently surfaced in other decision theoretic work. Gajdos et al. (2004) take as given a set of priors that contains a reference prior and then axiomatize maximin utility with respect to a behavioral  $C$  that is constrained by the exogenous objects. In Gajdos et al. (2008),  $C$  is furthermore centered on an endogenously identified reference prior. Ahn (2008) and Olszewski (2007) take the sets of lotteries induced by acts as primitives, thus avoiding state spaces altogether. Neither  $C$  nor  $\mathcal{S}$  explicitly appear in their approach, but a non-behavioral take on beliefs is implied; indeed, Olszewski (2007) uses the phrase “objective ambiguity.” Their approach does not permit a discussion of regret-based criteria, however, because these use state-space information. Finally, in research subsequent to earlier versions of this paper, both Puppe and Schlag (2008) and Stoye (2007b, theorem 3) follow the present approach.

### 3 Axiomatic Treatment

#### 3.1 Axioms and a Representation Result

This section is devoted to introducing the axioms and stating the representation theorem. Recall that attention will be restricted to decision rules which extend von Neumann-Morgenstern expected utility; that is, they agree with it over constant acts. To limit the number of explicit axioms, I do this informally: Assume that preferences over constant acts fulfil the usual axioms of von Neumann-Morgenstern independence, independence of irrelevant alternatives (see axiom 6 below), and transitivity. In conjunction with other maintained axioms (nontriviality, completeness, and continuity), this implies that the according restriction of  $\succsim_M$ , labelled  $\succsim^*$  henceforth, is menu independent and can be represented by  $\int U(z)dP^*$ , where  $U : \mathcal{Z} \mapsto \mathbb{R}$  is nonconstant and unique up to positive affine transformation.

Natural axioms for preferences over general acts include the following.

**Axiom 1 *Monotonicity***

$$P^*(a, s) \succsim^* P^*(b, s), \forall s \in \mathcal{S} \implies a \succsim_M b.$$

**Axiom 2 *Nontriviality***

$$\exists a, b, M : a \succ_M b.$$

**Axiom 3 *Completeness***

$$a \succsim_M b \vee b \succsim_M a.$$

**Axiom 4 *Mixture Continuity (Archimedean Property)***

$$a \succ_M b \succ_M c \implies \exists \gamma, \gamma' \in (0, 1) : \gamma a + (1 - \gamma)c \succ_M b \succ_M \gamma' a + (1 - \gamma')c.$$

**Axiom 5 *Transitivity***

$$a \succsim_M b \succsim_M c \implies a \succsim_M c.$$

**Axiom 6 *Independence of Irrelevant Alternatives (IIA)***

$$a \succsim_M b \iff a \succsim_N b$$

for all menus  $M, N$ .

**Axiom 7 *Independence***

$$a \succsim_M b \iff pa + (1 - p)c \succsim_{pM+(1-p)c} pb + (1 - p)c, \forall p \in (0, 1).$$

Mixture continuity can be contrasted with sequential continuity, i.e. the requirement that  $a_n \succsim_M b, \forall n$  and  $a_n \rightarrow a$  implies  $a \succsim_M b$ . Apart from not requiring a notion of convergence, the Archimedean property imposed here is weaker. As usual, strengthening it to sequential continuity would insure continuity of  $U$ . Regarding independence, note that the third act  $c$  is mixed into the entire menu. To see why, recall that independence is frequently motivated by the following thought experiment. Imagine an agent prefers  $a$  over  $b$ , but then she is told that her choice will be actualized only if heads occur in a previous coin toss; otherwise,  $c$  will occur whatever her intentions. Then it can be argued that this information should not reverse her preferences, hence these should obey independence. But in the thought experiment,  $c$  would be mixed into all options, so absent IIA, the argument really supports axiom 7.

Controversial discussions of Bayesianism show that some decision makers will want to avoid prior probabilistic judgment about states. Axioms that exclude such judgment were first proposed by Arrow and Hurwicz (1972, written 20 years prior) and have been formalized for a context similar to the present one, but with a finite state space, by Milnor (1954). The below formulation reflects their further adaptation to the state space considered here and compares to Cohen and Jaffray (1980).

**Axiom 8 *Symmetry***

*For any menu  $M$ , let  $E, F \in \Sigma \setminus \{\emptyset\}$  be disjoint events s.t. for any  $a \in M$ ,  $P(a, s)$  is constant on  $E$  as well as  $F$ . Define  $a'$  by*

$$P(a', s) = \begin{cases} P(a, s)|_{s \in E}, & s \in F \\ P(a, s)|_{s \in F}, & s \in E \\ P(a, s) & \text{otherwise} \end{cases} .$$

*Let  $M'$  be the menu generated by replacing every act  $a \in M$  with  $a'$ . Then*

$$a \succsim_M b \iff a' \succsim_{M'} b' .$$

The idea behind symmetry is that a preference ordering should not impose prior beliefs by implicitly assigning different likelihoods to different events. This is implausible if one has available, and wishes to consider, sharp prior information about states. Indeed, a Bayesian analysis would then seem appropriate. If no prior information about states exists, the restriction makes sense since in its absence, a decision criterion would be sensitive to arbitrary manipulations of the state space, either by relabeling states or by duplicating some via conditioning on trivial events. These considerations should be especially clear to frequentists since they are closely related to standard objections against “noninformative” priors.

It turns out that axioms 1 through 8 generate a contradiction, hence some of them must be weakened. I will now present a number of standard, but also some more novel axioms that can be used to do so.

**Axiom 9 *C-Independence***

*Let  $c$  be constant, then*

$$a \succsim_M b \iff pa + (1-p)c \succsim_{pM+(1-p)c} pb + (1-p)c, \forall p \in (0, 1].$$

**Axiom 10 *Ambiguity Aversion***

$$a \sim_M b \implies pa + (1-p)b \succsim_M b, \forall p \in (0, 1].$$

C-independence requires that the ranking of acts is not affected by mixing the menu with some constant act. The intuition is that violations of independence should be due only to ambiguity, i.e. they occur when mixing with  $c$  constitutes a hedging of bets across states with respect to  $a$  but not, or less so, with respect to  $b$ . This effect cannot occur when  $c$  is constant.

Under ambiguity aversion, a randomization between two equally good treatments must be weakly preferred to either of them, intuitively because of the aforementioned hedge. The axiom is first found in Milnor (1954) and was prominently advocated by Schmeidler (1989).

The next axiom weakens IIA.<sup>8</sup>

**Axiom 11 *Independence of Never Strictly Optimal Alternatives (INA)***

*Let  $c$  be not strictly potentially optimal in  $M \cup \{c\}$ . Then*

$$a \succsim_M b \iff a \succsim_{M \cup \{c\}} b.$$

The arguments in favor of IIA are well known. Why would one want to argue against it, and if so, advocate INA in its place? Sen (1993) cites the phenomena of positional choice (not wanting to take the largest slice of cake), choosing something mainly to display rejection of something else (as in fasting versus starving), and situations where the menu has epistemic value (as when items on a restaurant’s menu signal quality). The first two clearly fail to apply here. The last one is relevant in some situations where minimax-type criteria are employed. For example, Borodin and El-Yaniv (1998) use it to argue for minimax regret (in the guise of “competitive ratio”) in computer science, where the arrival of a new algorithm that performs well for certain request sequences is informative about the

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<sup>8</sup>Borodin and El-Yaniv (1998) label axiom 11 “independence of dominated alternatives.” I avoid this because  $c$  need not be dominated by any feasible option but only by the ex-post utility frontier. Milnor (1954) uses the term “special row adjunction.”

difficulty of a problem. This intuition neatly supports the weakening of IIA to INA since under INA, the addition of an act to a menu can reverse rankings between other options only if it affects the utility frontier. On the other hand, if it were possible to infer from the menu to properties or probabilities of states, then this should ideally be modelled explicitly and not informally via twists on some axiom.

Finally, consider the possibility of relaxing transitivity. I take for granted that other things equal, transitivity would be desirable; but it conflicts with other axioms for which one can claim the same, hence a closer look is required. On such reconsideration, transitivity is overwhelmingly compelling only if one perceives the statement “ $a \succsim b$ ” as comparing an intrinsic, one-dimensional property (“goodness”) of  $a$  and  $b$ , a perspective called the “Maximization Thesis” by Schwartz (1972). If the statement “ $a \succsim b$ ” rather describes a property of the pair  $\{a, b\}$ , then transitivity is not obvious and will be generically violated, whether by pairwise minimax regret or by its aforementioned relatives.<sup>9</sup>

Notice, however, that this argument can be taken further because in the present formulation of transitivity,  $a$ ’s and  $b$ ’s “goodness” may depend on  $M$ , making it not so intrinsic after all. In other words, if one really adopts the perspective just outlined, then one should also be inclined to embrace IIA – thus minimax regret, which was already seen to be menu-dependent, may not be favored by the arguments that support transitivity.<sup>10</sup>

I now propose some axioms designed to preserve transitivity’s most compelling aspects. To formalize the first one, let  $a_E b$  denote the act that performs  $a$  in any  $s \in E$  and  $b$  otherwise.

**Axiom 12 *Transitive Extension of Monotonicity***

Let  $a \geq b$  denote that  $P^*(a, s) \succsim P^*(b, s), \forall s$ . Then

$$a \geq b \succsim_M c \implies a \succsim_M c.$$

For brevity, this axiom will also be called transitive monotonicity below. It reinstates transitivity if one of the orderings on the axiom’s if-side is due to statewise dominance. After all, which aspect of the comparison between  $b$  and  $c$  could be invalidated by unambiguously upgrading  $b$ ?

Transitive monotonicity is fulfilled by the strict statewise dominance ordering, so it cannot guarantee resolution of any nontrivial decision situation. To enforce a more decisive attitude, one could impose the following:

$$P^*(a, s) \succ P^*(b, s), \forall s \implies a \succ_M b.$$

Interestingly, this property is implied by maintained axioms. An axiom with significant cutting power can, however, be generated by further strengthening it.

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<sup>9</sup>See Fishburn and LaValle (1988) and Sugden (1985) for more detailed critiques of transitivity by economists and Schwartz (1972) for a philosopher’s take.

<sup>10</sup>The same caveat holds for Dutch book arguments: If one may manipulate the menu faced by a decision maker, then one can easily construct Dutch books against minimax regret.

**Axiom 13 *Transitive Extension of Strict Monotonicity***

Let  $a \gg b$  denote that  $P^*(a, s) \succ P^*(b, s), \forall s$ . Then

$$a \gg b \succsim_M c \implies a \succ_M c.$$

The basic intuition of this matches the one of transitive monotonicity, namely that dominance relations should make for comparisons that have some degree of transitivity. However, transitive extension of strict monotonicity “feels” stronger and therefore requires more faith in this intuition.<sup>11</sup> It essentially imposes that orderings must be “sharp”: If I am indifferent between  $b$  and  $c$ , then an arbitrarily small, certain utility gain will induce me to trade one for the other. This makes much sense if indifference is taken very literally and may be acceptable if decisiveness is called for, but is dubious if indifference at least partly stands in for noncomparability.

Consider finally the following axiom:

**Axiom 14 *Acyclicity***

There exists no strict preference cycle, that is, no  $M$  and finite set  $\{a, b, \dots\} \subseteq M$  such that

$$a \succ_M b \succ_M \dots \succ_M a.$$

Substantively, this is perhaps the weakening of transitivity that retains most of its spirit. Since it does not invoke dominance, its plausibility stands or falls with the aforementioned idea that rankings reflect degrees of “goodness” – as Schwartz (1972) writes, “to accept [...] Noncircularity is to accept the guts of the Maximization thesis.” The axiom will receive a pragmatic motivation later.

A final remark on the axioms’ mutual relation is in order. I presented them as weakening independence, IIA, and transitivity, respectively, but they are strict weakenings of these only in some cases. For example, c-independence is implied by, and hence weakens, independence, but ambiguity aversion is implied by independence only jointly with IIA, and transitive extension of strict monotonicity can be derived from different subsets of axioms 1 through 7. It is easy, however, to verify the following:

**Lemma 1** *Axioms 1 through 7 jointly imply axioms 9 through 14.*

Hence, if one takes axioms 1 through 7 but replaces independence with c-independence and ambiguity aversion, then there is a clear sense in which one has weakened independence. It is in this sense that terms like “weakening transitivity” are used here.

I will now state this paper’s main result, which consists of a characterization of all decision rules that were introduced above. The result will then be related to existing findings and discussed in some more depth.

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<sup>11</sup>The axioms are individually independent, but given continuity, axiom 13 implies axiom 12.

### Theorem 1 *Characterization of Preference Orderings*

- (i) A preference ordering fulfils axioms 1, 2, 3, 4, 5, 6, and 7 iff it is Bayesian.
- (ii) A preference ordering fulfils axioms 1, 2, 3, 4, 5, 6, 8, and 9 iff it is  $\alpha$ -maximin utility.
- (iii) A preference ordering fulfils axioms 1, 2, 3, 4, 5, 6, 8, and 10 iff it is maximin utility.
- (iv) A preference ordering fulfils axioms 1, 2, 3, 4, 5, 7, 8, 10, and 11 iff it is minimax regret.
- (v) A preference ordering fulfils axioms 2, 3, 4, 6, 7, 8, and 12 iff it is pairwise  $\alpha$ -minimax regret.
- (vi) A preference ordering fulfils axioms 2, 3, 4, 6, 7, 8, and 13 iff it is pairwise minimax regret.
- (vii) Let  $\#\Sigma$  denote the number of atoms in  $\Sigma$ . Then a preference ordering fulfils axioms 2, 3, 4, 6, 7, 8, 12, and 14 iff it is pairwise  $\alpha$ -minimax regret with  $\alpha \leq 1/(\#\Sigma - 1)$ . If  $\Sigma$  is infinite, a preference ordering fulfils these axioms iff it is strict statewise dominance.
- (viii) There exists no preference ordering that fulfils axioms 1 through 8.

**Proof.** See appendix A. Tightness of axioms is established in appendix B. ■

Part (i) of this result is due to Anscombe and Aumann (1963) and is provided as a backdrop. Parts (ii) and (iii) relate to findings in Gilboa and Schmeidler (1989), Ghirardato et al. (2004), and Milnor (1954). The proof uses some of the latter's ideas; innovations include the embedding in an Anscombe-Aumann setup and the substantial weakening of several axioms as well as restrictions on the environment. An important benefit of this is alignment: The environment as well as some axioms mirror Gilboa and Schmeidler (1989), a fact that will be useful in the discussion.

Part (iv) is again related to Milnor (1954). An additional adjustment concerns the fact that minimax regret takes differences between expected utilities, thus seemingly presuming cardinally measurable utility. In Milnor (1954), this feature is due to an axiom that uses utility differences and therefore transparently introduces this presumption. Here, it is generated by independence, a trick that was anticipated by Chernoff (1954) but is missing in the subsequent literature (Milnor 1954, Borodin and El-Yaniv 1998). To my knowledge, there are no precursors for results (v) through (viii).

## 3.2 Extension to Measurable Acts and Infinite Menus

Before turning to a substantive discussion of the theorem, I formally state its extension to a more general domain.<sup>12</sup> Specifically, I drop the restriction to finite acts and menus. Finite *acts* are quite standard in the literature, and the extension of results to measurable acts (e.g. lemma 4.1 in Gilboa and Schmeidler 1989) is usually routine. The reader might find finite *menus* more restrictive. Relaxing this requires some more work for two reasons. First, once menus are potentially infinite, nothing is to be gained from assuming finite acts. The reason is that with infinite menus, the ex post utility frontier

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<sup>12</sup>This section can be skipped without loss of continuity.

$u(a^*, s) \equiv \max_{a \in M} u(a, s)$  will, in general, correspond to an infinite “oracle act”  $a^*$ . What’s more, maximin-type orderings over infinite acts may violate strict monotonicity: An act  $a$  that dominates  $b$  strictly but not uniformly so (in utility terms) may be ranked indifferent to it. This fact necessitates adjustment of an axiom.

To extend results, assume that acts are  $\Sigma$ -measurable, meaning that the inverse image of any distribution  $P(a, s)$  is an element of  $\Sigma$ , and also that they are bounded, meaning that for every act  $a$ , there exist constant acts  $\underline{P}_a$  and  $\overline{P}_a$  s.t.  $\underline{P}_a \preceq_M a \preceq_M \overline{P}_a$  for any  $M \supseteq \{a, \underline{P}_a, \overline{P}_a\}$ . For the characterization of minimax regret, the latter condition has to be strengthened to boundedness of menus, that is, for every menu  $M$ , there exist constant acts  $\underline{P}_M$  and  $\overline{P}_M$  s.t.  $\underline{P}_M \preceq_{M \cup \{\underline{P}_M, \overline{P}_M\}} a \preceq_{M \cup \{\underline{P}_M, \overline{P}_M\}} \overline{P}_M$  for all  $a \in M$ . A sufficient but not necessary condition for this is that acts are uniformly bounded, e.g. because  $U$  is. Regarding the decision rules, I will avoid their formal restatement but remind the reader that the familiar max and min operators must be replaced with sup and inf. Then the following is true.

**Proposition 1** *Consider the extended domain defined in the preceding paragraph. Fix any constant acts  $\overline{P}^*, \underline{P}^*$  s.t.  $\overline{P}^* \succ \underline{P}^*$ . In the statement of axiom 13, let  $a \gg b$  denote that there exists  $\alpha \in (0, 1)$  s.t.  $\alpha P^*(a, s) + (1 - \alpha)\underline{P}^* \succ \alpha P^*(b, s) + (1 - \alpha)\overline{P}^*, \forall s$ . Then theorem 1 continues to hold.*

**Proof.** See appendix A. ■

The adjustment to axiom 13 is stated in terms of preferences, but is most easily understood in utility language. Specifically,  $u(a, s)$  must dominate  $u(b, s)$  uniformly over  $\mathcal{S}$ .<sup>13</sup>

### 3.3 Discussion

Substantive discussion of theorem 1 is facilitated by table 1, in which + denotes compliance, – non-compliance, and  $\oplus$  indicates axiomatic characterization. (Notice also that in the columns labelled  $\alpha$ -MU and  $\alpha$ -PMR,  $\alpha \in (0, 1)$  is presumed, and that the final column presumes an infinite state space.)

The core trade-offs are between axioms 5 through 8 and can be seen by inspecting the according rows. Once again, symmetry cannot be reconciled with the full list of Bayesian axioms, so if it is to be embraced, some other axiom has to be weakened. Probably the most familiar way of doing so is by relaxing independence. This is here achieved in two steps: relaxing independence to c-independence leads to a characterization of  $\alpha$ -maximin utility, from which the imposition of ambiguity aversion picks  $\alpha = 0$ , i.e. maximin utility. In the latter case, c-independence turns out to be redundant, i.e. it is fulfilled but can be dropped from the characterization.

<sup>13</sup>The two versions of axiom 13 are equivalent for finite acts; the simpler version was used for exposition.

	Bayes	$\alpha$ -MU	MU	MR	$\alpha$ -PMR	PMR	SSD
(1) <b>monotone</b>	$\oplus$	$\oplus$	$\oplus$	$\oplus$	+	+	+
(2) <b>nontrivial</b>	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
(3) <b>complete</b>	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
(4) <b>continuous</b>	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
(5) <b>transitive</b>	$\oplus$	$\oplus$	$\oplus$	$\oplus$	-	-	-
(6) <b>IIA</b>	$\oplus$	$\oplus$	$\oplus$	-	$\oplus$	$\oplus$	$\oplus$
(7) <b>independent</b>	$\oplus$	-	-	$\oplus$	$\oplus$	$\oplus$	$\oplus$
(8) <b>symmetric</b>	-	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
(9) <b>c-independent</b>	+	$\oplus$	+	+	+	+	+
(10) <b>ambiguity averse</b>	+*	-	$\oplus$	$\oplus$	+*	+*	+*
(11) <b>INA</b>	+	+	+	$\oplus$	+	+	+
(12) <b>monotone (trans. ext.)</b>	+	+	+	+	$\oplus$	+	$\oplus$
(13) <b>strictly monotone (trans. ext.)</b>	+	+	+	+	-	$\oplus$	-
(14) <b>acyclic</b>	+	+	+	+	-	-	$\oplus$

\*weakly so, i.e. with “ $\sim_M$ ” in the conclusion

Table 1: An illustration of theorem 1.

This result contrasts sharply to Gilboa and Schmeidler (1989) because the axioms used are exactly theirs, plus symmetry (and less c-independence). The result, on the other hand, differs from theirs exactly by identifying the set of priors with  $\mathcal{S}$ . This feature can, therefore, be attributed to symmetry. The characterization of  $\alpha$ -maximin utility is somewhat similar to results in Ghirardato et al. (2004), but the effect of symmetry cannot be isolated quite as neatly. Using only c-independence, Ghirardato et al. (2004, theorem 4) characterize  $\alpha$ -maximin utility with a behavioral set of priors and the feature that  $\alpha$  can vary between acts in complex, although not unconstrained, ways. This property is also encountered in an axiomatization by Arrow and Hurwicz (1972), but is alien to the criterion’s original (Hurwicz 1951) and more common definition. To achieve constant  $\alpha$ , Ghirardato et al. (2004) propose an additional axiom that seems to introduce a vestige of symmetry: It requires that an act’s evaluation only depends on the range of its image in utility space, and it follows from the present axioms only after symmetry has been added.<sup>14</sup>

Parts (iv) through (vii) establish a core finding: Independence does not enforce Bayesianism and

<sup>14</sup>See Eichberger et al. (2007), however, for some caveats that limit the comparability between Ghirardato et al. (2004) and the present result. In particular, the set of priors in Ghirardato et al. (2004) is not identified separately from the (non-constant)  $\alpha$ , and the axiom enforcing constant  $\alpha$  is stronger than might appear. Olszewski (2007) provides a very different treatment that achieves constant  $\alpha$ .

can be reconciled with symmetry. However, it then leads one to pick regret-based orderings, a first example being the characterization of minimax regret. Interestingly, even the observation of mere consistency between minimax regret and independence is somewhat orthogonal to verbal discussions of minimax regret versus maximin utility, which tend to revolve around examples like Rawls'. One reason for this might be that the present finding depends on a strict separation of independence from IIA as proposed in this paper.

The final cluster of characterizations investigates the effect of a third approach, namely dropping transitivity. A major finding here is that transitive monotonicity already generates quite some structure – it single-handedly (among the nonstandard axioms) enforces pairwise  $\alpha$ -minimax regret. A further sharpening of transitivity-like conditions immediately leads to very specific rankings: Transitive extension of strict monotonicity enforces pairwise minimax regret, and acyclic  $\succ$  enforces a value of  $\alpha$  that rapidly converges to zero as the algebra of events expands. Indeed, although I have not imposed it, an infinite  $\mathcal{S}$  (and  $\Sigma$ ) might be seen as the generic case, and in its presence, acyclicity enforces the ranking by strict statewise dominance.

The trade-off revealed in these last results is further illuminated by some additional observations. Acyclic strict preference can be motivated by a need for well-defined policy prescriptions. To see this point, define the choice correspondence

$$C(M) \equiv \{a \in M : b \in M \Rightarrow a \succsim_M b\},$$

i.e. the set of best options in a menu. Policy prescriptions are well-defined whenever this set is nonempty, although they might be vague if it is large. Yet it is known that under assumptions maintained in this paper, acyclicity of preferences is sufficient for choice correspondences to be nonempty (Bergstrom 1975).<sup>15</sup> Parts (vi) and (vii) of the theorem therefore illustrate a fundamental tension between a desire to have decisive rankings and one to avoid preference cycles, or in other words, the difficulty of finding a middle ground between too large and empty choice sets.

## 4 Summary and Outlook

This paper investigated the foundations of statistical decision theory for situations of simultaneous (“model”) ambiguity and (“estimation”) uncertainty. The purpose was to explore the theoretical foundations of decision criteria that treat uncertainty but not ambiguity in a probabilistic fashion. The ax-

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<sup>15</sup>If agents were not allowed to randomize, acyclicity would also be necessary. If they can randomize, then there exist cyclical preference orderings that generate nonempty choice functions (see Fishburn and LaValle 1988), although  $\alpha$ -PMR is not among them.

Schwartz (1972) proposes a generalized choice correspondence that coincides with the conventional one in well-behaved cases but is never empty. But this correspondence will be uninformative if preference cycles are pervasive.

iomatic discussion differs from previous contributions by being more applied, using the full structure of a real-world problem to give results that are tightly specified for this problem. An Anscombe/Aumann setup is used because its distinction between probabilistic and nonprobabilistic uncertainty captures an important feature of the intended applications. The characterizations link all objects in the decision rule to the decision maker’s environment and therefore apply to maximin-criteria as they are actually used in many applications. Furthermore, I introduced a number of new criteria and, to fully explore the properties of regret-based criteria, examined an axiomatic system that relaxes both transitivity and IIA.

To repeat the main findings, a contradiction arises when one wishes to combine the axioms of Bayesian rationality with Arrow-Hurwicz ignorance. Numerous possibility results are obtained by relaxing this set of axioms in different directions. A weakening of independence leads to a characterization of maximin utility rankings, with c-independence characterizing the Hurwicz criterion and ambiguity aversion maximin utility. Relaxing menu independence, together with again imposing ambiguity aversion, leads to minimax regret. Some plausible weakenings of transitivity characterize  $\alpha$ -pairwise minimax regret respectively pairwise minimax regret, but also reveal a fundamental conflict between avoiding preference cycles and having a criterion that excludes any admissible acts.

The examination of decision rules of this type offers rich opportunities for further research, some of which were recently exploited. Revealed preference axiomatizations of the minimax regret choice correspondence were provided by Hayashi (2007) and Stoye (2007b). For the record, Stoye’s (2007b) axiomatization of multiple prior minimax regret (i.e., along the lines of Gilboa and Schmeidler 1989), of so-called Hannan regret, and of comparative ambiguity perception can easily be imported into the present setting. Puppe and Schlag (2008) investigate possible relaxations of state independence of the outcome space. A further variation would be to think of regret as  $u(a - a^*)$  rather than  $(u(a) - u(a^*))$ , i.e. utility of efficiency loss rather than efficiency loss in utility. This approach is closer in spirit to Loomes and Sugden (1982) or Fishburn (1989), but requires a different setup because the state space underlying objective probabilities must be explicated. Finally, it would be of obvious interest – if challenging – to connect research on non-Bayesian statistical decision rules to analyses of ambiguity aversion in dynamic settings.

## A Proofs

**Lemma 1** By theorem 1(i) below, axioms 1 through 7 imply that the preference ordering is Bayesian, which in turn implies that axioms 9 through 14 are fulfilled.

### Theorem 1

**Preliminaries** I leave the verification of “if”-statements to the reader. Denote by  $\mathcal{U} \subseteq \mathbb{R}$  the convex hull of the range of  $U$ . The first observation is that under axioms maintained henceforth, any act  $a$  is fully described by the mapping  $u \circ a : \mathcal{S} \mapsto \mathcal{U}$  defined by  $(u \circ a)(s) \equiv u(a, s)$ .

**Lemma 2** Let INA and either transitive monotonicity or transitivity hold. Let  $a, \bar{a}, b$ , and  $\bar{b}$  be s.t.  $u(a, s) = u(\bar{a}, s), \forall s \in \mathcal{S}$  as well as  $u(b, s) = u(\bar{b}, s), \forall s \in \mathcal{S}$ . Then

$$a \succsim_{M \cup \{a, b\}} b \iff \bar{a} \succsim_{M \cup \{\bar{a}, \bar{b}\}} \bar{b}, \forall M.$$

**Proof.** Fix any  $a, b, \bar{a}$ , and  $\bar{b}$  that satisfy the hypothesis and define  $M' \equiv M \cup \{a, b, \bar{a}, \bar{b}\}$ . No element of  $\{a, b, \bar{a}, \bar{b}\}$  is strictly potentially optimal in  $M'$ . By axiom INA, it follows that  $a \succsim_{M \cup \{a, b\}} b \iff a \succsim_{M'} b$  and  $\bar{a} \succsim_{M \cup \{\bar{a}, \bar{b}\}} \bar{b} \iff \bar{a} \succsim_{M'} \bar{b}$ , thus it suffices to show  $a \succsim_{M'} b \iff \bar{a} \succsim_{M'} \bar{b}$ . Monotonicity immediately implies  $a \sim_{M'} b$  and thus the result if transitivity is imposed. Otherwise, both  $a \succsim_{M'} b \Rightarrow \bar{a} \succsim_{M'} \bar{b}$  and  $\bar{a} \succsim_{M'} \bar{b} \Rightarrow a \succsim_{M'} b$  follow from the transitive extension of monotonicity. ■

Acts will henceforth be identified with their utility profiles  $u \circ a$ ; to keep notation similar to the main text, I continue to write  $u(a, s)$  rather than  $(u \circ a)(s)$ .

Fix any  $M$ . By finiteness of lotteries  $P$ ,  $u(a, s)$  is finite. By finiteness of acts,  $\max / \min_{s \in \mathcal{S}} \{u(a, s)\}$  exist and are finite as well. Recalling that  $U$  is nonconstant, assume w.l.o.g. that  $[-1, 1] \subseteq \mathcal{U}$ . By defining appropriate mappings from  $s$  to distributions over  $\mathcal{Z}$ , one can generate acts that correspond to any pre-assigned, finite,  $\Sigma$ -measurable mapping from  $\mathcal{S}$  to  $\mathcal{U}$ . In particular, there is an act  $a_0$  s.t.  $u(a_0, s) = 0$  for all  $s$ . Finally, I will use the following notation throughout: The symbols  $\geq [\gg]$  denote weak [strict] statewise dominance, i.e.  $a \geq b$  if  $u(a, s) \geq u(b, s)$  for all  $s \in \mathcal{S}$  and  $a \gg b$  if  $u(a, s) > u(b, s)$  for all  $s \in \mathcal{S}$ ; for any act  $a$  and scalar  $\gamma$ ,  $\gamma a$  denotes the act with  $u(\gamma a, s) = \gamma u(a, s)$  if this act exists.

(i) A proof within (essentially) the present setting is given by Kreps (1988, theorem 7.17).

**Preliminaries to (ii) and (iii)** Since IIA is imposed, the menu subscript on the preference ordering can be dropped. Fix a partition of  $\mathcal{S}$  into three nonempty events  $E^*, F^*, G^* \in \Sigma$ . Fix any act  $a$  and define  $(\underline{a}, \bar{a}) \equiv (\min_{s \in \mathcal{S}} u(a, s), \max_{s \in \mathcal{S}} u(a, s))$ . For scalars  $u, v$  and events  $E \in \Sigma$ , let  $u_E v$

denote the act that achieves utility  $u$  on event  $E$  and utility  $v$  otherwise. Define  $a^* \equiv \underline{a}_{E^*}\bar{a}$ . I will now show that  $a^* \sim a$ .

Consider any two events  $E, F \in \Sigma \setminus \{\emptyset\}$  and scalars  $u, v \in \mathcal{U}$ , then  $u_{E^*}v \sim u_Fv$ . To see this, assume first that  $E \cap F \neq \emptyset$  but also  $E^c \cap F \neq \emptyset$ . In this case, symmetry – used by identifying the events  $E$  and  $F$  in the axiom with  $E \cap F$  respectively  $E^c \cap F$  here – implies that  $u_{E^*}v \succsim u_Fv$  iff  $u_Fv \succsim u_{E^*}v$ , hence the two are indifferent by completeness. Now assume  $E \subset F$ . Then the claim can be similarly established by first exchanging the consequences of  $E$  and  $F^c$ , then the consequences of  $F$  and  $F^c$ . If  $E \subset F^c$ , then one can exchange the consequences of  $E$  and  $F$ . Finally, the claim is immediate if  $E = F$ .

Let  $\underline{E}[\bar{E}] \in \Sigma$  be the event on which  $u(a, s) = \underline{a}[\bar{a}]$ . Then monotonicity implies that  $\underline{a}_{\underline{E}}\bar{a} \succsim a \succsim \bar{a}_{\bar{E}}\underline{a}$ . But  $\underline{a}_{\underline{E}}\bar{a} \sim \bar{a}_{\bar{E}}\underline{a}$  by the previous paragraph's conclusion, hence  $a \sim \underline{a}_{\underline{E}}\bar{a}$  by transitivity. Using the previous paragraph's conclusion and transitivity again, one finds  $\underline{a}_{\underline{E}}\bar{a} \sim a^*$  and finally  $a \sim a^*$ .

It therefore suffices to characterize preferences over acts of the form  $a^*$ ; these acts will be called *standardized*. Standardized acts can be described by vectors  $(\underline{a}, \bar{a})$  that summarize their outcomes over  $E^*$  respectively  $\{F^*, G^*\}$ . This two-dimensional notation will be used whenever sufficient.

From here, the proofs take different directions.

(ii) C-independence implies that preferences are homogeneous of degree zero:  $a \succsim_M b \Leftrightarrow pa + (1-p)a_0 \succsim_{pM+(1-p)a_0} pb + (1-p)a_0$ , but  $pa + (1-p)a_0 = pa$  (in the sense of scalar-act multiplication defined in the preliminaries). Preferences can, therefore, be generated by extending preferences over acts s.t.  $-1 < \underline{a} \leq \bar{a} < 1$ .

Let  $\alpha \equiv \inf\{a : (a, a) \succsim (0, 1)\}$ , then monotonicity implies that  $\alpha \leq 1$ . Suppose by contradiction that  $\alpha < 0$ , then  $\frac{1}{2}(\alpha, \alpha) \succsim (0, 1)$ , implying  $\frac{1}{2}(\alpha, \alpha) \succsim (0, 0)$  by monotonicity and transitivity, but this contradicts the expected utility representation for constant acts. Hence,  $\alpha \geq 0$ . Now suppose by contradiction that  $(\alpha, \alpha) \succ (0, 1)$ . As  $(0, 1) \succsim (0, 0)$  by monotonicity and  $(0, 0) \succ (-1, -1)$  by expected utility for constant acts, transitivity yields  $(0, 1) \succ (-1, -1)$ . By continuity, there then exists  $\gamma < 1$  s.t.  $(\gamma\alpha - (1-\gamma), \gamma\alpha - (1-\gamma)) \succ (0, 1)$ , contradicting the definition of  $\alpha$ . It follows that  $(\alpha, \alpha) \sim (0, 1)$ . By using c-independence, where  $c$  is identified with  $(\alpha, \alpha)$ , this conclusion can be extended to every point on the ray  $R_\alpha \equiv \{\gamma(0, 1) + (1-\gamma)(\alpha, \alpha) : \gamma \geq 0\}$ , hence this ray is part of an indifference set.

Consider now any  $u \in (-1, 1)$  and define  $R_u \equiv R_\alpha + (u, u) - (\alpha, \alpha)$ , the ray through  $(u, u)$  that is parallel to  $R_\alpha$ .  $R_u$  is contained in an indifference set. To see this, suppose  $u < \alpha$ , then there exists  $v$  s.t.  $-1 < v < u$ . The claim then follows from c-independence, where  $c$  is identified with  $(v, v)$  and  $p$  is identified with  $\frac{\alpha-u}{\alpha-v}$ . A similar argument applies if  $u > \alpha$ .

Since the collection  $\{R_u\}_{u \in (-1, 1)}$  partitions the preference domain, every vector in that domain has been mapped onto exactly one ray. By the expected utility representation for constant acts, any two

different rays constitute strictly ordered indifference sets. It is now easily verified that the ordering is  $\alpha$ -MU with  $\alpha$  as defined in this proof.

(iii) For this paragraph only, consider acts that are measurable on  $\{E^*, F^*, G^*\}$  but may not be constant over  $\{F^*, G^*\}$ ; denote these acts by triples  $(u, v, w)$  with the obvious interpretation. Previous arguments imply that  $(\underline{a}, \bar{a}, \underline{a}) \sim (\underline{a}, \underline{a}, \bar{a}) \sim (\underline{a}, \bar{a}, \bar{a})$ , thus ambiguity aversion yields  $(\underline{a}, \frac{\underline{a}+\bar{a}}{2}, \frac{\underline{a}+\bar{a}}{2}) \succsim (\underline{a}, \underline{a}, \bar{a}) \sim (\underline{a}, \bar{a}, \bar{a})$  and monotonicity then  $(\underline{a}, \frac{\underline{a}+\bar{a}}{2}, \frac{\underline{a}+\bar{a}}{2}) \sim (\underline{a}, \bar{a}, \bar{a})$ . By induction over  $n$ , the argument can be extended to show  $(\underline{a}, \underline{a} + \frac{\bar{a}-\underline{a}}{2^n}, \underline{a} + \frac{\bar{a}-\underline{a}}{2^n}) \sim (\underline{a}, \bar{a}, \bar{a})$  for any natural number  $n$ . As the sequence  $\{2^{-n}\}$  is dense at 0, monotonicity and transitivity then jointly imply that  $(\underline{a}, \underline{a} + \gamma(\bar{a}-\underline{a}), \underline{a} + \gamma(\bar{a}-\underline{a})) \sim (\underline{a}, \bar{a}, \bar{a})$  for any  $\gamma \in (0, 1]$ .

Now return attention to standardized acts. The acts in the previous paragraph's conclusion are standardized, hence  $(\underline{a}, \underline{a} + \gamma(\bar{a}-\underline{a})) \sim (\underline{a}, \bar{a})$  for any  $\gamma \in (0, 1]$ . It remains to extend this conclusion to  $\gamma = 0$ . Suppose by contradiction that  $(\underline{a}, \bar{a}) \succ (\underline{a}, \underline{a})$ . Assume first that  $\underline{a}$  is not the minimal element of  $\mathcal{U}$ . Then there exists an act  $(u, u)$  with  $u < \underline{a}$ , and the expected utility representation for constant acts implies that  $(\underline{a}, \underline{a}) \succ (u, u)$ . By continuity, there must then exist  $\delta \in (0, 1)$  s.t.  $(\delta \underline{a} + (1-\delta)u, \delta \bar{a} + (1-\delta)u) \succ (\underline{a}, \underline{a})$ . The previous paragraph's result implies that

$$(\delta \underline{a} + (1-\delta)u, \gamma \delta \bar{a} + (1-\gamma)\delta \underline{a} + (1-\delta)u) \sim (\delta \underline{a} + (1-\delta)u, \delta \bar{a} + (1-\delta)u), \forall \gamma > 0,$$

thus the left side of the above indifference is strictly preferred to  $(\underline{a}, \underline{a})$  for any  $\gamma > 0$ . But as  $\gamma \rightarrow 0$ , one finds that

$$(\delta \underline{a} + (1-\delta)u, \gamma \delta \bar{a} + (1-\gamma)\delta \underline{a} + (1-\delta)u) \rightarrow (\delta \underline{a} + (1-\delta)u, \delta \underline{a} + (1-\delta)u) \ll (\underline{a}, \underline{a}),$$

so monotonicity is eventually contradicted. It follows that  $(\underline{a}, \bar{a}) \precsim (\underline{a}, \underline{a})$  and, by monotonicity,  $(\underline{a}, \bar{a}) \sim (\underline{a}, \underline{a})$ .

Assume now that  $\underline{a}$  is the minimal element of  $\mathcal{U}$ , then  $\underline{a} \leq -1$  by the range normalization of  $U$ . Suppose by contradiction that  $(\underline{a}, 0) \succ (\underline{a}, \underline{a})$ . Monotonicity implies that  $(0, 0) \succ (\underline{a}, 0)$  and the expected utility representation for constant acts yields  $(1, 1) \succ (0, 0)$ , hence  $(1, 1) \succ (\underline{a}, 0)$  by transitivity. By continuity, there then exists  $\delta \in (0, 1)$  s.t.  $(\underline{a}, 0) \succ (\delta + (1-\delta)\underline{a}, \delta + (1-\delta)\underline{a})$ . This is consistent with monotonicity only if  $\delta + (1-\delta)\underline{a} < 0$ . Now,  $\delta + (1-\delta)\underline{a}$  is not a minimal element of  $\mathcal{U}$ , hence the previous paragraph's finding and transitivity jointly imply  $(\underline{a}, 0) \succ (\delta + (1-\delta)\underline{a}, 0)$ , contradicting monotonicity. Hence  $a \succsim b \Leftrightarrow \underline{a} \geq \underline{b}$  as required.

Existence of three events, which has been exploited in this section, is necessary: If  $\Sigma$  has only two atoms,  $\alpha$ -maximin utility with  $\alpha = 0.5$  is ambiguity averse.

(iv) Fix any acts  $a$  and  $b$  in menu  $M$ . Define  $a'$  by

$$u(a', s) = \frac{1}{2} \left( u(a, s) - \max_{a^* \in M} \{u(a^*, s)\} \right)$$

and  $b'$  similarly. I claim that  $a \succsim_M b \Leftrightarrow a' \succsim_{\{a', b', a_0\}} b'$ . To see this, define act  $c$  by  $u(c, s) = -\max_{a^* \in M} \{u(a^*, s)\}$ . Independence, used with  $c$  as just defined and  $p = 1/2$ , then implies that  $a \succsim_M b \Leftrightarrow a' \succsim_{M'} b'$ , where  $M'$  is generated from  $M$  by replacing every  $a \in M$  with  $a'$  as defined above. Observe now that by construction,  $\max_{a \in M'} \{u(a, s)\} = 0$  for every  $s$ . Hence, INA implies that  $a' \succsim_{M'} b' \Leftrightarrow a' \succsim_{\{a', b', a_0\}} b'$ .

It therefore suffices to characterize the menu-independent preference ordering  $\succeq$ , defined by  $a \succeq b \Leftrightarrow a \succsim_{\{a, b, a_0\}} b$ , over  $\Sigma$ -measurable, bounded acts whose utility range is nonpositive. One can straightforwardly establish that the axioms imposed on  $\succsim_M$  restrict  $\succeq$  to be ambiguity averse, monotone, complete, transitive, nontrivial, and symmetric. Hence, part (iii) of this proof implies that  $\succeq$  is the maximin utility ordering. It follows that

$$\begin{aligned} a \succsim_M b &\iff a' \succeq b' \\ &\iff \min_{s \in \mathcal{S}} u(a', s) \geq \min_{s \in \mathcal{S}} u(b', s) \\ &\iff \min_{s \in \mathcal{S}} \left\{ \frac{1}{2} (u(a, s) - \max_{a^* \in M} \{u(a^*, s)\}) \right\} \geq \min_{s \in \mathcal{S}} \left\{ \frac{1}{2} (u(b, s) - \max_{a^* \in M} \{u(a^*, s)\}) \right\} \\ &\iff \min_{s \in \mathcal{S}} \left\{ \max_{a^* \in M} \{u(a^*, s)\} - u(a, s) \right\} \leq \min_{s \in \mathcal{S}} \left\{ \max_{a^* \in M} \{u(a^*, s)\} - u(b, s) \right\}, \end{aligned}$$

i.e. the preference ordering is minimax regret.

**Preliminaries to (v) through (vii)** IIA is imposed, so the reference to a menu can be dropped from notation. Also, independence implies c-independence and therefore homogeneity of degree zero of  $\succsim$  by arguments from (ii). Thus, it suffices to characterize preferences over acts whose utility range is contained in  $(-1, 1)$ . Fix any menu  $M$  and pair of acts  $a$  and  $b$ . Let the act  $b \ominus a$  be characterized by  $u(b \ominus a, s) = \frac{1}{2} (u(b, s) - u(a, s))$ ; recall from the preliminaries that this act exists. Then  $a \succsim b \Leftrightarrow a_0 \succsim b \ominus a$ . To see this, apply independence, where the act  $c$  is characterized by  $u(c, s) = -u(a, s)$  and  $p = 1/2$ .

It suffices, therefore, to characterize preferences relative to  $a_0$ . Similar to previous definitions, let act  $u_E v$  achieve utility  $u$  on event  $E$  and  $v$  otherwise, define  $(\underline{a}, \bar{a}) = (\min_{s \in \mathcal{S}} \{u(a, s)\}, \max_{s \in \mathcal{S}} \{u(a, s)\})$ , and let  $\underline{E}[\bar{a}] \in \Sigma$  be the event on which  $\underline{a}[\bar{a}]$  is achieved. Then  $\underline{a}_E \bar{a} \geq a \geq \bar{a}_{\underline{E}} a$ , and repeated uses of transitive monotonicity yield

$$\begin{aligned} \bar{a}_{\underline{E}} a &\succsim a_0 \implies a \succsim a_0 \\ a &\succsim a_0 \implies \underline{a}_{\underline{E}} \bar{a} \succsim a_0. \end{aligned}$$

Symmetry implies that preferences between  $\underline{a}_E \bar{a}$  and  $a_0$  cannot depend on the identity of event  $E$  (see the preliminaries of (ii) and (iii) for a detailed proof). In particular

$$\bar{a}_E \underline{a} \succsim a_0 \iff \underline{a}_E \bar{a} \succsim a_0 \iff \underline{a}_E \bar{a} \succsim a_0,$$

where  $E \in \Sigma \setminus \{\emptyset\}$  is some pre-assigned event. Taken together, these findings imply that

$$a \succsim a_0 \iff \underline{a}_E \bar{a} \succsim a_0.$$

Hence, it suffices to characterize preferences between  $a_0$  and acts of the form  $\underline{a}_E \bar{a}$ , where  $-1 < \underline{a} \leq \bar{a} < 1$ . The latter acts will henceforth be identified with vectors  $(\underline{a}, \bar{a})$  and  $a_0$  analogously with  $(0, 0)$ .

Define the decision function

$$\delta(\underline{a}, \bar{a}) \equiv \begin{cases} 1, & (\underline{a}, \bar{a}) \succ (0, 0) \\ 0, & (\underline{a}, \bar{a}) \sim (0, 0) \\ -1, & (\underline{a}, \bar{a}) \prec (0, 0) \end{cases}.$$

This function is well-defined due to completeness. The proof is completed by examining its isoquants  $\delta^{-1}(-1)$ ,  $\delta^{-1}(0)$ , and  $\delta^{-1}(1)$  of  $\delta$  in  $(-1, 1)^2$  above the increasing  $45^\circ$  line. First of all,  $\delta(\underline{a}, \bar{a}) = -\delta(-\bar{a}, -\underline{a})$  because

$$\begin{aligned} \delta(\underline{a}, \bar{a}) = 1 &\iff (\underline{a}, \bar{a}) \succ (0, 0) \iff (0, 0) \succ \frac{1}{2}(-\underline{a}, -\bar{a}) \\ &\iff (0, 0) \succ (-\underline{a}, -\bar{a}) \iff (0, 0) \succ (-\bar{a}, -\underline{a}) \iff \delta(-\bar{a}, -\underline{a}) = -1, \end{aligned}$$

where the second step uses this section's first paragraph, the third one uses homogeneity of degree zero, and the fourth one uses symmetry. Thus  $\delta^{-1}(-1)$  is the reflection of  $\delta^{-1}(1)$  about  $\{(\underline{a}, \bar{a}) : \underline{a} = -\bar{a}\}$ , the decreasing  $45^\circ$  line. Since  $\delta^{-1}(1)$  and  $\delta^{-1}(-1)$  are disjoint, the fixed points of said reflection are in neither set, hence  $\{(-a, a) : a \geq 0\} \subseteq \delta^{-1}(0)$ .

Suppose now by contradiction that there exists  $(\underline{a}, \bar{a}) \ll (0, 0)$  with  $(\underline{a}, \bar{a}) \succsim (0, 0)$ . Then transitive monotonicity implies that  $(\bar{a}, \bar{a}) \succsim (0, 0)$ , contradicting the expected utility representation for constant acts. It follows that  $\delta^{-1}(-1)$  contains the interior of the third quadrant and, by the preceding paragraph's symmetry result, that  $\delta^{-1}(1)$  contains the interior of the first one (above the  $45^\circ$  line).

From here, the proofs take different directions:

(v) As  $a \succsim a_0 \iff \gamma a \succsim a_0$  for any scalar  $\gamma > 0$ , the relative interior of any origin ray is contained in an isoquant of  $\delta$ . Consider now two distinct origin rays,  $A$  and  $B$  say, within the second (=northwestern) quadrant. Clearly  $A$  and  $B$  do not intersect, thus assume w.l.o.g. that  $A$  lies above  $B$ . Transitive monotonicity then implies that  $B \subseteq \delta^{-1}(0) \Rightarrow A \subseteq [\delta^{-1}(0) \cup \delta^{-1}(1)]$ . Now suppose by contradiction that  $B \subseteq \delta^{-1}(1)$  yet  $A \subseteq \delta^{-1}(0)$ . Let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  with respect to the decreasing  $45^\circ$  line. Then  $B'$  lies above  $A'$ , hence  $A' \subseteq \delta^{-1}(0) \Rightarrow B' \subseteq [\delta^{-1}(0) \cup \delta^{-1}(1)]$ .

Yet the fact that  $\delta(\underline{a}, \bar{a}) = 1 \Leftrightarrow \delta(-\bar{a}, -\underline{a}) = -1$  implies that  $A' \subseteq \delta^{-1}(0)$  and  $B' \subseteq \delta^{-1}(-1)$ . It follows that  $B \subseteq \delta^{-1}(1) \Rightarrow A \subseteq \delta^{-1}(1)$ .

Thus the intersections of  $\delta^{-1}(-1)$ ,  $\delta^{-1}(0)$ , and  $\delta^{-1}(1)$  with the second quadrant are ordered as follows: tracing the quadrant with origin rays in positive (counterclockwise) direction, one first traces its intersection with  $\delta^{-1}(1)$ , then  $\delta^{-1}(0)$ , then  $\delta^{-1}(-1)$ . As  $\delta^{-1}(1)$  and  $\delta^{-1}(-1)$  also contain the first respectively third quadrant, it follows that  $\delta^{-1}(-1)$ ,  $\delta^{-1}(0)$ , and  $\delta^{-1}(1)$  are convex cones with the same ordering.

Now suppose by contradiction that  $\delta^{-1}(0)$  is open. By considering some acts  $a$  respectively  $b$  on the boundary of  $\delta^{-1}(-1)$  respectively  $\delta^{-1}(1)$ , this is seen to contradict continuity. Thus  $\delta^{-1}(0)$  is closed. It now follows that  $\delta^{-1}(-1)$  is the half-open cone below a downward sloping origin ray with absolute slope  $\alpha$ , where  $\alpha \geq 0$  since  $\delta^{-1}(-1)$  contains the interior third quadrant and  $\alpha \leq 1$  because of the symmetry between  $\delta^{-1}(-1)$  and  $\delta^{-1}(1)$ . Also using symmetry between  $\delta^{-1}(-1)$  and  $\delta^{-1}(1)$ , there exists  $\alpha \in [0, 1]$  s.t.

$$\begin{aligned}\delta^{-1}(-1) &= \{(\underline{a}, \bar{a}) : \underline{a} + \bar{a}\alpha < 0\} \\ \delta^{-1}(0) &= [\delta^{-1}(-1) \cup \delta^{-1}(1)]^c \\ \delta^{-1}(1) &= \{(\underline{a}, \bar{a}) : \alpha\underline{a} + \bar{a} > 0\}.\end{aligned}$$

To summarize, there exists  $\alpha \in [0, 1]$  s.t.

$$\begin{aligned}a \succ b &\iff a_0 \succ b \ominus a \iff (0, 0) \succ \frac{1}{2} \left( \min_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\}, \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} \right) \\ &\iff \alpha \min_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} + \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} > 0 \\ &\iff \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} < \alpha \max_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\}\end{aligned}$$

and similarly for  $a \prec b$ . But this is the definition of pairwise  $\alpha$ -minimax regret.

**(vi)** Assume first that transitive monotonicity holds. Then (v) applies, so the preference ordering is pairwise  $\alpha$ -minimax regret. Recall that  $\{(-u, u) : u \geq 0\} \subseteq \delta^{-1}(0)$ . Consider any  $(\underline{a}, \bar{a})$  with  $\underline{a} > -\bar{a}$ , then there exists  $u \geq 0$  s.t.  $(\underline{a}, \bar{a}) \gg (-u, u) \sim (0, 0)$ . Transitive strict monotonicity then implies that  $(\underline{a}, \bar{a}) \succ (0, 0)$ . Comparing to the characterization of  $\delta$  in (v), it follows that  $\alpha = 1$  as required.

It remains to establish that transitive strict monotonicity and continuity jointly imply transitive monotonicity. Thus, let  $a \geq b \succsim c$  and assume by contradiction that  $c \succ a$ . Note that  $(\gamma, \gamma) + (1 - \gamma)a \gg a$ , hence  $(\gamma, \gamma) + (1 - \gamma)a \succ c$  by transitive strict monotonicity, for any  $\gamma \in (0, 1)$ . Continuity then implies that  $(\delta, \delta) + (1 - \delta)a \prec c$  for some  $\delta \in (0, \gamma)$ , contradicting transitive strict monotonicity.

**(vii)** By part (v), the preference ordering is  $\alpha$ -pairwise minimax regret; it remains to restrict  $\alpha$ . To see the “only if”-direction, let  $\Sigma$  be finite, let  $\alpha > 1/(\#\Sigma - 1)$ , and consider the menu characterized

as follows:

	$s \in \omega_1$	$s \in \omega_2$	$\dots$	$s \in \omega_{\#\Sigma-1}$	$s \in \omega_{\#\Sigma}$
$\mathbf{u}(\mathbf{a}_1, \mathbf{s})$	$\frac{1}{\#\Sigma}$	$\frac{2}{\#\Sigma}$	$\dots$	$\frac{\#\Sigma-1}{\#\Sigma}$	1
$\mathbf{u}(\mathbf{a}_2, \mathbf{s})$	$\frac{2}{\#\Sigma}$	$\frac{3}{\#\Sigma}$	$\dots$	1	$\frac{1}{\#\Sigma}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\mathbf{u}(\mathbf{a}_{\#\Sigma}, \mathbf{s})$	1	$\frac{1}{\#\Sigma}$	$\dots$	$\frac{\#\Sigma-2}{\#\Sigma}$	$\frac{\#\Sigma-1}{\#\Sigma}$

Here,  $(\omega_i)_{i=1}^{\#\Sigma}$  is an arbitrary ordering of the atoms of  $\Sigma$ , and the  $(a_i, \omega_j)$ -cell of the table displays  $u(a_i, \omega_j)$ . It is easily verified that  $a_1 \succ a_2 \succ \dots \succ a_{\#\Sigma} \succ a_1$ . If  $\Sigma$  is infinite, the cycle can be constructed for any  $\alpha > 0$  by using a sufficiently fine partition of  $\mathcal{S}$ .

“If” is trivial if  $\Sigma$  is infinite. To prove it for finite  $\Sigma$ , let  $0 < \alpha \leq 1/(\#\Sigma - 1)$ . Define  $u_i(a) \equiv u(a, s) \Big|_{s \in \omega_i}$ , identify acts  $a$  with the vectors  $(u_i(a))_{i=1}^{\#\Sigma}$ , and let  $|a| \equiv \sum_{i=1}^{\#\Sigma} u_i(a)$ . Then  $a \succ b$  implies that

$$\max_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\} > \frac{\max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\}}{\alpha} \geq (\#\Sigma - 1) \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\}$$

and hence that

$$\begin{aligned} |a| - |b| &= \sum_{i=1}^{\#\Sigma} (u_i(a) - u_i(b)) \geq \max_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\} + (\#\Sigma - 1) \min_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\} \\ &= \max_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\} - (\#\Sigma - 1) \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} > 0. \end{aligned}$$

Since  $|\cdot|$  induces a transitive ordering of acts, it follows that  $\succ$  is acyclic.

(viii) From (i), axioms 1 through 7 imply that the preference ordering is Bayesian. Partition  $\mathcal{S}$  into three events  $E, F, G \in \Sigma \setminus \{\emptyset\}$  and consider four acts that generate the following utilities:

	$s \in \mathbf{E}$	$s \in \mathbf{F}$	$s \in \mathbf{G}$
$\mathbf{u}(\mathbf{a}, \mathbf{s})$	0	1	1
$\mathbf{u}(\mathbf{b}, \mathbf{s})$	1	0	0
$\mathbf{u}(\mathbf{c}, \mathbf{s})$	0	0	1
$\mathbf{u}(\mathbf{d}, \mathbf{s})$	1	1	0

Symmetry (used with  $E$  and  $F \cup G$  as conditioning events) implies that  $a \succsim b$  iff  $b \succsim a$  and hence (in conjunction with completeness) that  $a \sim b$ . It similarly follows that  $c \sim d$ . IIA and monotonicity now imply that all four acts are indifferent. Hence the prior assigns zero probability to  $F$ . Analog constructions show that the prior also assigns zero probability to  $E$  and  $G$ .

The use of three events is necessary: If  $\Sigma$  has only two atoms, all axioms are fulfilled by Bayesianism with a uniform prior, the Hurwicz criterion with  $\alpha = 0.5$ , and by pairwise minimax regret (all of which also coincide).

**Proposition 1** Denote the preference ordering over  $\mathcal{A}^*$  by  $\succsim^*$  and its restriction to  $\mathcal{A}$  by  $\succsim$ . Lemma 2 applies to  $\succsim^*$  because its proof did not use finiteness of acts or menus. Hence, it suffices to characterize preferences over  $\Sigma$ -measurable, bounded mappings from  $\mathcal{S}$  to  $\mathcal{U}$ . Axioms imposed on  $\succsim^*$  restrict  $\succsim$  via the “only if”-direction of theorem 1. In all cases of theorem 1, the monotonic extension of this ordering to  $\Sigma$ -measurable acts is unique. This implies “only if,” the verification of “if” is again left to the reader.

To see the necessity of adjusting axiom 13, observe that two acts with utility range  $(-u, u)$  (and hence indifferent to  $a_0$ ) may be ordered by strict statewise dominance. For example, let  $\mathcal{S} = [0, 1]$  with  $\Sigma$  the Borel algebra, let  $u(a, s) = s - 1/2$  except that  $u(a, 0) = 0$ , and let  $u(b, s) = s^{1/2} - 1/2$  except that  $u(b, 1) = 0$ .

## B Tightness of Theorem 1

This section establishes that axioms are individually necessary. The examples given are of preference orderings that fulfil all axioms of a certain axiomatization except the one whose necessity is to be shown.

**For All Orderings** Necessity of nontriviality: Consider  $a \sim b, \forall a, b$ .

**(i) Bayesianism** Necessity of completeness: Let  $a \succ^* b$  iff  $\int u(a, s)d\Pi > \int u(b, s)d\Pi$  for all  $\Pi \in C$ , where  $C \subseteq \Delta\mathcal{S}$  is closed and convex.

Necessity of continuity: Let  $\Pi \in \Delta\mathcal{S}$  be a prior that has not full support on  $\mathcal{S}$ , and let  $s^*$  be some state outside the support of  $\Pi$ . Consider the Bayesian criterion, except that indifferences are broken lexicographically according to  $u(a, s^*)$ .

Necessity of transitivity: Consider pairwise minimax regret.

Necessity of IIA: Consider minimax regret.

Necessity of independence: Consider maximin utility.

**For All Maximin Orderings** Necessity of completeness: For any ordering  $\succsim$ , let  $\succsim^*$  be the incomplete ordering that agrees with  $\succsim$  except that if  $a \sim b$  and  $(a, b)$  is not ordered by weak statewise dominance, then  $a$  and  $b$  are not comparable under  $\succsim^*$ . (Notice the example violates sequential continuity but not axiom 5.)

**(ii)  $\alpha$ -Maximin Utility** Necessity of monotonicity: Consider

$$a \succ b \iff \min_{s \in \mathcal{S}} u(a, s) - \frac{1}{2} \max_{s \in \mathcal{S}} u(a, s) \geq \min_{s \in \mathcal{S}} u(b, s) - \frac{1}{2} \max_{s \in \mathcal{S}} u(b, s).$$

Necessity of continuity: For any acts  $a$  and  $b$ , let  $\{E_i\}_{i=1}^I \subset \Sigma$  be a partition of  $\mathcal{S}$  that renders both  $a$  and  $b$  measurable. Let  $u_1 \leq u_2 \leq \dots \leq u_I$  be a nondecreasing ordering of  $(u(a, s) : s \in E_i)_{i=1}^I$  and let  $v_1 \leq v_2 \leq \dots \leq v_N$  be defined analogously but with respect to  $b$ . Define  $a \succ b \Leftrightarrow \exists i^* \leq I : u_i = v_i, i < i^*, u_{i^*} > v_{i^*}$ , the “leximin” criterion. (Clearly the criterion does not depend on choice of  $\{E_i\}_{i=1}^I$ .)

Necessity of transitivity: Consider pairwise minimax regret.

Necessity of IIA: Consider minimax regret.

Necessity of symmetry: Consider maximin utility with subjective priors as in Gilboa and Schmeidler (1989).

Necessity of c-independence: Consider  $\alpha$ -maximin utility with act-dependent  $\alpha$ :  $\alpha = f(\bar{a})$ , and  $f : \mathbb{R} \mapsto [0, 1]$  is continuous and nondecreasing.

**(iii) Maximin Utility** Necessity of ambiguity aversion: Consider  $\alpha$ -maximin utility.

For all other axioms, see (i).

**(iv) Minimax Regret** Necessity of monotonicity: Consider

$$\begin{aligned} a \succsim b &\iff \max_{s \in \mathcal{S}} \left\{ \max_{a^* \in M} u(a^*, s) - u(a, s) \right\} - \frac{1}{2} \min_{s \in \mathcal{S}} \left\{ \max_{a^* \in M} u(a^*, s) - u(a, s) \right\} \\ &\leq \max_{s \in \mathcal{S}} \left\{ \max_{b^* \in M} u(b^*, s) - u(b, s) \right\} - \frac{1}{2} \min_{s \in \mathcal{S}} \left\{ \max_{b^* \in M} u(b^*, s) - u(b, s) \right\}. \end{aligned}$$

Necessity of transitivity: Consider pairwise minimax regret.

Necessity of symmetry: Consider minimax regret with endogenous priors in analogy to Gilboa and Schmeidler (1989; see Hayashi 2008 or Stoye 2007b).

Necessity of independence: Consider maximin utility.

Necessity of ambiguity aversion: Consider “minimin regret,” i.e.

$$a \succsim b \iff \min_{s \in \mathcal{S}} \left\{ \max_{a^* \in M} u(a^*, s) - u(a, s) \right\} \leq \min_{s \in \mathcal{S}} \left\{ \max_{b^* \in M} u(b^*, s) - u(b, s) \right\}.$$

Necessity of INA: Consider “maximin joy” (found in earlier versions of Hayashi 2008), i.e.

$$a \succsim b \iff \min_{s \in \mathcal{S}} \left\{ u(a, s) - \min_{a^* \in M} u(a^*, s) \right\} \geq \min_{s \in \mathcal{S}} \left\{ u(b, s) - \min_{b^* \in M} u(b^*, s) \right\}.$$

Necessity of continuity: Let the ordering  $\succeq$  as defined in section (iv) of the proof be the leximin ordering defined in part (ii) of this appendix.

**(v)  $\alpha$ -Pairwise Minimax Regret** Necessity of IIA: Consider minimax regret.

Necessity of independence: Consider maximin utility.

Necessity of symmetry: Consider pairwise minimax regret with subjective priors, i.e.

$$a \succsim b \Leftrightarrow \max_{\Pi \in C} \left\{ \int u(a, s) d\Pi - \int u(b, s) d\Pi \right\} \geq \max_{\Pi \in C} \left\{ \int u(b, s) d\Pi - \int u(a, s) d\Pi \right\},$$

where  $C \subset \Delta\mathcal{S}$  is closed and convex.

Necessity of continuity: There exists  $\alpha \in (0, 1]$  s.t.

$$\begin{aligned} a \succ b &\iff \max_{s \in \mathcal{S}} \{u(b, s) - u(a, s)\} \leq \alpha \max_{s \in \mathcal{S}} \{u(a, s) - u(b, s)\} \\ a \sim b &\iff a \not\prec_{\alpha\text{-PMR}} b \wedge b \not\prec_{\alpha\text{-PMR}} a. \end{aligned}$$

Necessity of transitive extension of monotonicity: Let  $a \succ b$  iff  $a \ominus b$  (as defined in the proof) is constant with  $u(a \ominus b, s) > 0$ .

**(vi) Pairwise Minimax Regret** Necessity of transitive extension of strict monotonicity: Consider  $\alpha$ -PMR.

For all other axioms, see (v).

**(vii) Strict Statewise Dominance** Necessity of acyclicity: Consider  $\alpha$ -PMR.

Necessity of continuity: Consider  $a \succ b \Leftrightarrow [a \geq b, a \neq b]$ .

For all other axioms, see (v).

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