Efficient mechanism design

1 Preliminaries

In the previous lectures we studied the problem of a mechanism designer whose objective was to maximize his expected profits (revenue). We now switch our attention to a mechanism designer whose utility function is the sum of players utilities. We shall focus on the case in which the planner is interested in allocating a single unit of an indivisible good among n individuals indexed by i.

Let $k_i \in \{0,1\}$ denote the number of units allocated to agent i and let t_i denote a monetary transfer paid by i. We assume that each agent has quasi-linear preferences over pairs (k_i, t_i) :

$$\theta_i k_i - t_i$$

where $\theta_i \in [0,1]$ is the agent's private type. The agent's types are independently drawn from the same c.d.f. F. We let $\theta \equiv (\theta_i, \theta_{-i}) \in [0,1]^n$.

Let X denote the set of all possible allocations of the single unit among the n individuals. A direct mechanism is a pair (q,t) where $q : [0,1]^n \to X$ is an allocation rule and $t : [0,1]^n \to \mathbb{R}^n$ is a payment rule, such that $q_i(\theta)$ and $t_i(\theta)$ are the number of units allocated to i and payment assigned to i, when the profile of announcements is θ. Note that $\sum_i q_i(\theta) = 1$. Define $Q_i(\theta_i|(q,t))$ to be the number of units that agent i expects to receive in the mechanism (q,t) when all other agents report truthfully, conditional on his type being θ_i :

$$Q_i(\theta_i|(q,t)) \equiv E_{\hat{\theta}_{-i}}[q_i(\hat{\theta}_i, \hat{\theta}_{-i})|\hat{\theta}_i = \theta_i]$$

Similarly,

$$T_i(\theta_i|(q,t)) \equiv E_{\hat{\theta}_{-i}}[t_i(\hat{\theta}_i, \hat{\theta}_{-i})|\hat{\theta}_i = \theta_i]$$

For notational convenience we shall simply write $Q_i(\theta_i)$ and $T_i(\theta_i)$. Thus, the expected payoff of agent i of type θ_i who announces a type ϕ_i is given by

$$U_i(\phi_i, \theta_i) \equiv \theta_i Q_i(\phi_i) - T_i(\phi_i)$$

Lemma 1 *(Revenue Equivalence) Let (q,t) and (q',t') be two incentive compatible (direct) mechanisms that satisfy*

$$Q_i(\theta_i) = Q'_i(\theta_i)$$

for all i and θ_i. Then for each i there exists a constant c_i such that

$$T_i(\theta_i) - T'_i(\theta_i) = c_i$$

for all θ_i.

Since agents' payoffs are quasi-linear, an allocation rule is efficient iff it maximizes the sum of expected payoffs subject to the constraint that exactly one unit is allocated.

Definition 1 *An allocation rule* $q^* : [0,1]^n \to X$ *is (ex-post) efficient if*

$$\sum_i \theta_i q_i^*(\theta) \geq \sum_i \theta_i q_i(\theta_i)$$

for all $\theta \in [0,1]^n$ *and for all allocation rules* $q : [0,1]^n \to X$

An *efficient mechanism* is any (direct) mechanism with an efficient allocation rule.

$$\begin{aligned} S(\theta) &\equiv \sum_i \theta_i q_i^*(\theta) \\ S_{-i}(\theta) &\equiv \sum_{j \neq i} \theta_i q_i^*(\theta) \end{aligned}$$

Since the probability that any two agents have the same exact value is zero, if q' and q are two efficient allocation rules, then $Q_i(\theta_i) = Q_i'(\theta_i) = Q_i^*(\theta_i)$ for all i and θ_i.

Let $IR_i : [0,1] \to \mathbb{R}$ be a continuous function that assigns each type of agent i a reservation utility.

Definition 2 *A mechanism* (q,t) *is (interim) IR if for all* i *and* θ_i,

$$U_i(\theta_i, \theta_i) \geq IR_i(\theta_i)$$

2 The generalized VCG mechanism

Define

$$\underline{\theta}_i \in \arg\min_{\theta_i \in [0,1]} E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - IR_i(\theta_i)$$

Type $\underline{\theta}_i$ is the lowest type in the sense that his gain from participating in an efficient mechanism is the least among all types.

The *generalized VCG mechanism with basis* $\underline{\theta}$ ($VCG(\underline{\theta})$)is an efficient mechanism with the following payment function:

$$t_i^V(\theta) = [S(\underline{\theta}_i, \theta_{-i}) - S_{-i}(\theta_i, \theta_{-i})] - IR_i(\underline{\theta}_i)$$

Type θ_i's expected VCG payoff when he reports a type ϕ_i, while others report truthfully is

$$U_i^V(\phi_i, \theta_i) = E_{\theta_{-i}}[S(\phi_i, \theta_{-i}) - S(\underline{\theta}_i, \theta_{-i})] + IR_i(\underline{\theta}_i)$$

Note that $U_i^V(\theta_i, \theta_i)$ is the difference in social surplus that would result if i were to report $\underline{\theta}_i$ rather than θ_i, plus the reservation value of the lowest type.

Claim 1 *The $VCG(\underline{\theta})$ is IC*

Proof. Since $t_i^V(\theta)$ is independent of i's report, truth-telling is weakly dominant. ■

Claim 2 *The $VCG(\underline{\theta})$ is IR*

Proof. First, note that

$$U_i^V(\underline{\theta}_i, \underline{\theta}_i) = IR_i(\underline{\theta}_i)$$

Second, by the definition of $\underline{\theta}_i$,

$$E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - IR_i(\theta_i) \geq E_{\theta_{-i}}[S(\underline{\theta}_i, \theta_{-i})] - IR_i(\underline{\theta}_i)$$

for all i and θ_i. Rearranging this inequality we obtain

$$E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - E_{\theta_{-i}}[S(\underline{\theta}_i, \theta_{-i})] + IR_i(\underline{\theta}_i) \geq IR_i(\theta_i)$$

But the LHS of this inequality is precisely $U_i^V(\theta_i, \theta_i)$. ■

Theorem 1 *Among all mechanisms that are efficient, IC and IR, the $VCG(\underline{\theta})$ maximizes the expected payment of each agent.*

Proof. Let $U_i^{(q,t)}(\theta_i, \theta_{-i})$ be the expected payoff of agent i of type θ_i under some efficient, IC and IR mechanism (q, t). Since both (q, t) and $VCG(\underline{\theta})$ are efficient,

$$U_i^{(q,t)}(\theta_i, \theta_i) - U_i^V(\theta_i, \theta_i) = U_i^{(q,t)}(\underline{\theta}_i, \underline{\theta}_i) - U_i^V(\underline{\theta}_i, \underline{\theta}_i)$$

Since (q, t) is IR,

$$U_i^{(q,t)}(\underline{\theta}_i, \underline{\theta}_i) \geq IR_i(\underline{\theta}_i)$$

By design,

$$U_i^V(\underline{\theta}_i, \underline{\theta}_i) = IR_i(\underline{\theta}_i)$$

Hence,

$$U_i^{(q,t)}(\theta_i, \theta_i) \geq U_i^V(\theta_i, \theta_i)$$

But this implies, by the efficiency of (q, t) and $VCG(\underline{\theta})$, that

$$T_i^{(q,t)}(\theta_i, \theta_i) \leq T_i^V(\theta_i, \theta_i)$$

■

Remark 1 *Theorem 1 implies that of all efficient IC and IR Bayesian mechanisms, the revenue maximizing mechanism has a DSE*

3 Budget balance

Definition 3 *The AGV mechanism is an efficient mechanism with the following transfer function*

$$t_i^A(\theta) = \frac{\sum_{j \neq i} E_{\theta_{-j}}[S_{-j}(\theta_j, \theta_{-j})]}{n-1} - E_{\theta_{-i}}[S_{-i}(\theta_i, \theta_{-i})]$$

Claim 3 *The AGV mechanism is budget-balanced (BB)*

Proof. Let $e_j \equiv \sum_{j \neq i} E_{\theta_{-j}}[S_{-j}(\theta_j, \theta_{-j})]$. Note that e_j only depends on θ_j. Then for all θ,

$$\begin{aligned}
t_1^A(\theta) &= \frac{0 + e_2 + e_3 + \cdots + e_n}{n-1} - e_1 \\
t_2^A(\theta) &= \frac{e_1 + 0 + e_3 + \cdots + e_n}{n-1} - e_2 \\
&\vdots \\
t_n^A(\theta) &= \frac{e_1 + e_2 + e_3 + \cdots + 0}{n-1} - e_n \\
\sum_i t_i^A(\theta) &= \frac{(n-1)e_1 + \cdots + (n-1)e_n}{n-1} - \sum_i e_i = 0
\end{aligned}$$

■

Claim 4 *The AGV mechanism is IC*

Proof.

$$\begin{aligned}
U_i^A(\theta_i, \theta_i) &= \theta_i Q_i(\theta_i) + E_{\theta_{-i}}[S_{-i}(\theta_i, \theta_{-i})] - \frac{\sum_{j \neq i} e_j}{n-1} \\
&= E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - \frac{\sum_{j \neq i} e_j}{n-1} \\
&\geq E_{\theta_{-i}}[S(\theta_i', \theta_{-i})] - \frac{\sum_{j \neq i} e_j}{n-1} \\
&= U_i(\theta_i', \theta_{-i})
\end{aligned}$$

■

Note that the AGV mechanism may not satisfy the IR constraints for all θ

Theorem 2 *There exists an efficient, IC and IR mechanism that balances the budget iff the* $VCG(\underline{\theta})$ *results in an expected surplus, i.e., iff*

$$E_\theta[\sum_i t_i^V(\theta)] \geq 0 \quad (1)$$

Proof. To see that (1) is necessary, note that if it did not hold, then by Theorem 1 all other efficient, IC and IR mechanisms would also run a deficit.

To prove that (1) is sufficient, assume it holds. Then

$$E_\theta[\sum_i t_i^V(\theta)] \geq E_\theta[\sum_i t_i^A(\theta)] \tag{2}$$

Let

$$\begin{aligned} c_i^A &\equiv \frac{\sum_{j\neq i} e_j}{n-1} \\ c_i^V &\equiv E_{\theta_{-i}}[S(\underline{\theta}_i, \theta_{-i})] - IR_i(\underline{\theta}_i) \end{aligned}$$

Then by (2),

$$\sum_i c_i^V \geq \sum_i c_i^A \tag{3}$$

For all $i > 1$, define

$$d_i \equiv c_i^A - c_i^V$$

and let

$$d_1 \equiv -\sum_{i\geq 2} d_i$$

Consider an efficient mechanism with the transfer function

$$t_i^*(\theta) = t_i^A(\theta) - d_i$$

Clearly, this mechanism is BB and IC (since the AGV mechanism is IC and the above transfer is the AGV transfer plus a constant). It remains to show that his mechanism is also IR.

First, note that for all i,

$$\begin{aligned} U_i^V(\theta_i, \theta_i) &= E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - c^V \\ U_i^A(\theta_i, \theta_i) &= E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - c^A \end{aligned}$$

Hence, for all $i > 1$,

$$\begin{aligned} U_i^*(\theta_i, \theta_i) &= U_i^A(\theta_i, \theta_i) + d_i \\ &= U_i^A(\theta_i, \theta_i) + c_i^A - c_i^V \\ &= U_i^V(\theta_i, \theta_i) \\ &\geq IR_i(\theta_i) \end{aligned}$$

To show that $IR_1(\theta_1)$, note first that

$$d_1 = -\sum_{i>2} d_i = \sum_{i>2} \left(c_i^V - c_i^A\right) \geq c_1^A - c_1^V$$

where the last inequality follows from (3). It follows that

$$\begin{aligned} U_1^*(\theta_1,\theta_1) &= U_1^A(\theta_1,\theta_1) + d_1 \\ &\geq U_1^A(\theta_1,\theta_1) + c_1^A - c_1^V \\ &= U_1^V(\theta_1,\theta_1) \\ &\geq IR_1(\theta_1) \end{aligned}$$

■

4 Applications

4.1 Bilateral trade

- $n = 2$ (agent 1 is a "seller" and agent 2 is a "buyer")
- $K = 1$
- $IR_1(\theta_1) = \theta_1$, $IR_2(\theta_2) = 0$
- $$S(\theta) = \begin{cases} \theta_1 & if \quad \theta_1 \geq \theta_2 \\ \theta_2 & if \quad \theta_1 < \theta_2 \end{cases}$$
- $$\begin{aligned} \underline{\theta}_1 &\in \arg\min_{\theta_1}[\theta_1 F(\theta_1) + \int_{\theta_1}^1 \theta_2 dF(\theta_2) - \theta_1] = \{1\} \\ \underline{\theta}_2 &\in \arg\min_{\theta_2}[\int_{\theta_2}^1 \theta_1 dF(\theta_1) + \theta_2 F(\theta_2)] = \{0\} \end{aligned}$$

Proposition 1 *There does not exist an efficient, IC and IR mechanism that also balances the budget.*

Proof. Recall that

$$t_i^V(\theta) = [S(\underline{\theta}_i, \theta_{-i}) - S_{-i}(\theta_i, \theta_{-i})] - IR_i(\underline{\theta}_i)$$

Hence, if $\theta_1 \geq \theta_2$, then

$$\begin{aligned} t_1^V(\theta) &= [S(1,\theta_2) - S_2(\theta_1,\theta_2)] - IR_1(1) \\ &= [1 - 0] - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{array}{rcl} t_2^V(\theta) & = & [S(\theta_1, 0) - S_1(\theta_1, \theta_2)] - IR_2(0) \\ & = & [\theta_1 - \theta_1] \\ & = & 0 \end{array}$$

So that in this case BB is satisfied:

$$t_1^V(\theta) + t_2^V(\theta) = 0$$

However, if $\theta_1 < \theta_2$, then

$$\begin{array}{rcl} t_1^V(\theta) & = & [1 - \theta_2] - 1 \\ & = & -\theta_2 \end{array}$$

and

$$\begin{array}{rcl} t_2^V(\theta) & = & [\theta_1 - 0] \\ & = & \theta_1 \end{array}$$

Hence,

$$t_1^V(\theta) + t_2^V(\theta) = \theta_1 - \theta_2 < 0$$

■

4.2 Efficient dissolution of an equal-share partnership

- $n = 2$
- $K = 1$
- $IR_1(\theta_1) = \frac{1}{2}\theta_1$, $IR_2(\theta_2) = \frac{1}{2}\theta_2$
- $$S(\theta) = \begin{cases} \theta_1 & if \quad \theta_1 \geq \theta_2 \\ \theta_2 & if \quad \theta_1 < \theta_2 \end{cases}$$
- $$\begin{array}{rcl} \underline{\theta}_1 & \in & \arg\min\limits_{\theta_1} [\theta_1 F(\theta_1) + \int_{\theta_1}^{1} \theta_2 dF(\theta_2) - \frac{1}{2}\theta_1] \\ \underline{\theta}_2 & \in & \arg\min\limits_{\theta_2} [\int_{\theta_2}^{1} \theta_1 dF(\theta_1) + \theta_2 F(\theta_2) - \frac{1}{2}\theta_2] \end{array}$$

  Hence,

  $$\underline{\theta}_1 = \underline{\theta}_2 \equiv \underline{\theta}$$

Proposition 2 *There exists an IR, IC and BB mechanism that efficiently dissolves the equal-share partnership.*

Proof. Recall that

$$t_i^V(\theta) = [S(\underline{\theta}_i, \theta_{-i}) - S_{-i}(\theta_i, \theta_{-i})] - IR_i(\underline{\theta}_i)$$

$$E_{\theta_{-i}}[S(\underline{\theta}_i, \theta_{-i})] - IR_i(\underline{\theta}_i) \leq E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - IR_i(\theta_i)$$

Hence, if $\theta_1 \geq \theta_2$, then

$$\begin{aligned} t_1^V(\theta) &= [S(\underline{\theta}_1, \theta_2) - S_2(\theta_1, \theta_2)] - IR_1(\underline{\theta}_1) \\ &= S(\underline{\theta}, \theta_2) - \frac{1}{2}\underline{\theta} \end{aligned}$$

and

$$\begin{aligned} t_2^V(\theta) &= [S(\theta_1, \underline{\theta}_2) - S_1(\theta_1, \theta_2)] - IR_2(\underline{\theta}_2) \\ &= [S(\theta_1, \underline{\theta}) - \theta_1] - \frac{1}{2}\underline{\theta} \end{aligned}$$

If $\theta_1 < \theta_2$, then

$$\begin{aligned} t_1^V(\theta) &= S(\underline{\theta}, \theta_2) - \theta_2 - \frac{1}{2}\underline{\theta} \\ t_2^V(\theta) &= S(\theta_1, \underline{\theta}) - \frac{1}{2}\underline{\theta} \end{aligned}$$

There are 6 cases to consider:

	$t_1^V(\theta)$	$t_2^V(\theta)$	$\sum_i t_i^V(\theta)$
$\theta_1 \geq \theta_2 \geq \underline{\theta}$	$\theta_2 - \frac{1}{2}\underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	$\theta_2 - \underline{\theta} \geq 0$
$\theta_1 \geq \underline{\theta} > \theta_2$	$\frac{1}{2}\underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	0
$\underline{\theta} > \theta_1 \geq \theta_2$	$\frac{1}{2}\underline{\theta}$	$\frac{1}{2}\underline{\theta} - \theta_1$	$\underline{\theta} - \theta_1 \geq 0$
$\theta_2 > \theta_1 \geq \underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	$\theta_1 - \frac{1}{2}\underline{\theta}$	$\theta_1 - \underline{\theta} \geq 0$
$\theta_2 \geq \underline{\theta} > \theta_1$	$-\frac{1}{2}\underline{\theta}$	$\frac{1}{2}\underline{\theta}$	0
$\underline{\theta} > \theta_2 > \theta_1$	$\frac{1}{2}\underline{\theta} - \theta_2$	$\frac{1}{2}\underline{\theta}$	$\underline{\theta} - \theta_2 \geq 0$

Therefore, $E_\theta[\sum_i t_i^V(\theta)] \geq 0$. ■