

Solution to Problem Set 4

1. First, note that $IR_1(\theta_1) = \frac{1}{2}\theta_1$, $IR_2(\theta_2) = \frac{1}{2}\theta_2$ and

$$S(\theta) = \begin{cases} \theta_1 & \text{if } \theta_1 \geq \theta_2 \\ \theta_2 & \text{if } \theta_1 < \theta_2 \end{cases}$$

It follows that

$$\begin{aligned} \underline{\theta}_1 &\in \arg \min_{\theta_1} [\theta_1 F(\theta_1) + \int_{\theta_1}^1 \theta_2 dF(\theta_2) - \frac{1}{2}\theta_1] \\ \underline{\theta}_2 &\in \arg \min_{\theta_2} [\int_{\theta_2}^1 \theta_1 dF(\theta_1) + \theta_2 F(\theta_2) - \frac{1}{2}\theta_2] \end{aligned}$$

Hence,

$$\underline{\theta}_1 = \underline{\theta}_2 \equiv \underline{\theta}$$

Recall that

$$\begin{aligned} t_i^V(\theta) &= [S(\underline{\theta}_i, \theta_{-i}) - S_{-i}(\theta_i, \theta_{-i})] - IR_i(\underline{\theta}_i) \\ E_{\theta_{-i}}[S(\underline{\theta}_i, \theta_{-i})] - IR_i(\underline{\theta}_i) &\leq E_{\theta_{-i}}[S(\theta_i, \theta_{-i})] - IR_i(\theta_i) \end{aligned}$$

Hence, if $\theta_1 \geq \theta_2$, then

$$\begin{aligned} t_1^V(\theta) &= [S(\underline{\theta}_1, \theta_2) - S_2(\theta_1, \theta_2)] - IR_1(\underline{\theta}_1) \\ &= S(\underline{\theta}, \theta_2) - \frac{1}{2}\underline{\theta} \end{aligned}$$

and

$$\begin{aligned} t_2^V(\theta) &= [S(\theta_1, \underline{\theta}_2) - S_1(\theta_1, \theta_2)] - IR_2(\underline{\theta}_2) \\ &= [S(\theta_1, \underline{\theta}) - \theta_1] - \frac{1}{2}\underline{\theta} \end{aligned}$$

If, on the other hand, $\theta_1 < \theta_2$, then

$$\begin{aligned} t_1^V(\theta) &= S(\underline{\theta}, \theta_2) - \theta_2 - \frac{1}{2}\underline{\theta} \\ t_2^V(\theta) &= S(\theta_1, \underline{\theta}) - \frac{1}{2}\underline{\theta} \end{aligned}$$

There are 6 cases to consider:

	$t_1^V(\theta)$	$t_2^V(\theta)$	$\sum_i t_i^V(\theta)$
$\theta_1 \geq \theta_2 \geq \underline{\theta}$	$\theta_2 - \frac{1}{2}\underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	$\theta_2 - \underline{\theta} \geq 0$
$\theta_1 \geq \underline{\theta} > \theta_2$	$\frac{1}{2}\underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	0
$\underline{\theta} > \theta_1 \geq \theta_2$	$\frac{1}{2}\underline{\theta}$	$\frac{1}{2}\underline{\theta} - \theta_1$	$\underline{\theta} - \theta_1 \geq 0$
$\theta_2 > \theta_1 \geq \underline{\theta}$	$-\frac{1}{2}\underline{\theta}$	$\theta_1 - \frac{1}{2}\underline{\theta}$	$\theta_1 - \underline{\theta} \geq 0$
$\theta_2 \geq \underline{\theta} > \theta_1$	$-\frac{1}{2}\underline{\theta}$	$\frac{1}{2}\underline{\theta}$	0
$\underline{\theta} > \theta_2 > \theta_1$	$\frac{1}{2}\underline{\theta} - \theta_2$	$\frac{1}{2}\underline{\theta}$	$\underline{\theta} - \theta_2 \geq 0$

Therefore, $E_{\theta}[\sum_i t_i^V(\theta)] \geq 0$.

2. Note that $\psi(\theta) = 2\theta - 1$, hence, $\underline{\theta} = \frac{1}{2}$. It follows that the relaxed problem is:

$$\max \sum_{i \in I} \theta_i a_i \text{ subject to } \sum_{i \in I} a_i = 1$$

whose solution is given by

$$a_i = \begin{cases} 1 & \text{if } \theta_i > \{\frac{1}{2}, \max_{j \neq i} \theta_j\} \\ 0 & \text{if } \text{other} \end{cases}$$

Therefore,

$$V(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \leq \frac{1}{2} \\ \theta^{n-1} & \text{if } \theta_i > \frac{1}{2} \end{cases}$$

$$\begin{aligned} T(\theta_i) &= \theta_i^n - \int_{\frac{1}{2}}^{\theta_i} x^{n-1} dx \\ &= \theta_i^n - \left[\frac{x^n}{n} \right]_{\frac{1}{2}}^{\theta_i} \\ &= \theta_i^n - \frac{\theta_i^n}{n} + \frac{1}{n} \left(\frac{1}{2} \right)^n \\ &= \frac{n-1}{n} \theta_i^n + \frac{1}{n} \left(\frac{1}{2} \right)^n \end{aligned}$$

Notice that $T(\theta_i)$ is exactly the expected payment of a bidder with value $\theta_i > \frac{1}{2}$ in a second-price auction with a reserve price of $\frac{1}{2}$:

$$T(\theta_i) = \left(\frac{1}{2} \right)^n + \int_{\frac{1}{2}}^{\theta_i} y(n-1)y^{n-2} dy$$