Problem Set (Nash implementation)

1. *Nash implementation with two agents.* This question demonstrates the difficulty of Nash implementation when there are only two agents.

Let $N = \{1, 2\}$ and $C = \{a, b, c\}$, and suppose that there are two possible preference profiles:

$\succ_1$	$\succ_2$		$\succ'_1$	$\succ'_2$
a	c		c	b
c	b		a	c
b	a		b	a

Let $F(\succ) = \{a\}$ and $F(\succ') = \{b\}$. Show that F is monotonic and has limited veto-power, yet it is not Nash-implementable.

2. *Nash implementation with stochastic mechanisms.* This question illustrates that we can weaken the sufficient condition of limited veto-power by allowing for randomizations off the equilibrium.

Let $\langle N, C, \mathcal{P}, \mathcal{G}\rangle$ be an environment in which $n \geq 3$ and $\mathcal{G}$ is the set of all strategic game forms. For any $b, c \in C$ define $l(b, c)$ as the lottery that draws b or c with equal probability. Let C^* be the extended set of consequences $C^* = C \cup \{l(b, c) : b, c \in C\}$. Let $\mathcal{P}^*$ denote the set of all preference profiles over C^*. For any $\succeq \in \mathcal{P}$, a profile $\succeq^* \in \mathcal{P}^*$ is a monotonic extension of $\succeq$ if for every i and for every $b, c \in C$,

$$b \succeq_i c \Longleftrightarrow b \succeq^*_i c$$

and

$$b \succ_i c \Longrightarrow b \succ^*_i l(b, c) \succ^*_i c$$

Consider the extended environment $\langle N, C^*, \mathcal{P}^*, \mathcal{G}^*\rangle$, where $\mathcal{G}^*$ is the set of strategic game forms with consequences in C^*. Let F be a SCR that assigns a subset of C (i.e., the *original* set of consequence) to each profile in P (the *original* set of preference profiles). A game form $G^* \in \mathcal{G}^*$ with outcome function g^* is said to Nash-implement F if for every $\succeq \in \mathcal{P}$, and for every $\succeq^* \in \mathcal{P}^*$, which is a monotonic extension of $\succeq$,

$$g[NE(G^*, \succeq^*) = F(\succeq)$$

Define $\max_{\succeq_i} = \{c \in C : c \succeq_i b \text{ for all } b \in C\}$. The SCR F is weakly unanimous if, for all $\succeq \in \mathcal{P}$ we have $c \in F(\succeq)$ whenever $c \in \max_{\succeq_i} C$ for all $i \in N$ and $\{c\} = \max_{\succeq_i} C$ for some $i \in N$.

Suppose that that P satisfies the "top coincidence" condition: for any $\succeq \in P$ there are at least two individuals with a unique best consequence (i.e., $|\max_{\succeq_i} C| = 1$). Prove that if F is monotonic and weakly unanimous, then it is Nash implementable by some $G^* \in \mathcal{G}^*$.

3. *Unions of Nash implementable SCRs.* When a SCR is not Nash implementable it may be because for some preference profile, not all 'desirable' outcomes (i.e., those outcomes picked by the SCR) can be sustained in a Nash equilibrium. However, there may still exist an implementable *subcorrespondence* of

F. If there is such a correspondence, the one of most interest would naturally be the largest, as this would be the closet to the original correspondence. But does there exist such a largest subcorrespondence? Since this subcorrespondence is the union of all Nash implementable subcorrespondences, the asnwer depends on whether the union of Nash implementable SCRs is itself Nash implementable.

Let $\langle N, C, \mathcal{P}, \mathcal{G} \rangle$ be an environment in which $n \geq 3$ and $\mathcal{G}$ is the set of all strategic game forms. Show that if F and G are a pair of Nash implementable SCRs, then the SCR defined by $F(\succeq) \cup G(\succeq)$ is also Nash implementable.

4. *Implementing the efficient allocation of pollution.* While most of our discussion of Nash implementation has been abstract, this question provides a simple "real-life" application of this concept.

There are n firms indexed by i. Let Q_i denote firm i's level of pollution and let $B_i(Q_i)$ denote firm i's net private benefit from producing Q_i (strictly concave and differentiable). The social cost from firms' pollution in monetary terms is $C(Q_1, \ldots, Q_n)$ (convex and differentiable). The benefit and cost functions are common knowledge among firms, and the regulator only knows the functional form of $C(\cdot)$.

The regulator's problem is to implement the socially optimal allocation of pollution, that is the solution to

$$\max_{Q_1 \geq 0, \ldots, Q_n \geq 0} \left[\sum_{i=1}^{n} B_i(Q_i) \right] - C(Q_1, \ldots, Q_n)$$

subject to

$$Q_1 \geq 0, \ldots, Q_n \geq 0$$

Assume this problem has a solution, $Q^* = (Q_1^*, \ldots, Q_n^*)$, which by the strict concavity must be unique. Impose the appropriate Inada-type conditions on $C(\cdot)$ and each $B()$ to ensure an interior solution, so that the social optimum is characterized by the condition that each firm's marginal benefit equals the marginal social cost of pollution, i.e., for every i,

$$\frac{dB_i}{dQ_i}(Q_i^*) = \frac{\partial C}{\partial Q_i}(Q_1^*, \ldots, Q_n^*)$$

Finally, assume that the planner knows some bound K such that $Q_i^* < K$ for all i.

Consider the following normal-form mechanism. Each firm i reports a pair $\left(\hat{Q}_i, \bar{Q}_{-i}^i\right) \in [0, K]^2$, where $\hat{Q}_i$ is firm i's demand and $\bar{Q}_{-i}$ is the demand of its "neighbor", firm $i-1$ (we consider firm n to be firm 1's neighbor). Based on these reports, firm i pays the following amount:

$$\hat{Q}_i \frac{\partial C}{\partial Q_i} \left(\hat{Q}_1, \ldots, \hat{Q}_{i-1}, \bar{Q}_i, \hat{Q}_{i+1}, \ldots \hat{Q}_n \right) + \left| \bar{Q}_{-i} - \hat{Q}_{-i} \right|$$

(a) Show that this mechanism Nash-implements the socially optimal level of pollution.

(b) Show that (a) is true even if we allow firms to use mixed-strategies.
(c) Suggest a way to modify the mechanism in order to achieve budget balance in equilibrium.
(d) Suggest a way to modify the mechanism in order to achieve budge balance also out-of-equilibrium.