**Solution to problem set 2**

**1.** This question is taken from Martin Osborne and Ariel Rubinstein's *A Course in Game Theory*.

The choice function is monotonic since $a \succeq_1 c$ and $c \succ_1' a$, and $b \succeq_2' c$ and $c \succ_2 b$. Suppose that a game form $G$ with outcome function $g$ Nash-implements $f$. Then $(G, \succeq)$ has a Nash equiibrium, say $(s_1, s_2)$, for which $g(s_1, s_2) = a$. Since $(s_1, s_2)$ is a Nash equilibrium, $g(s_1, s_2') \preceq_2 a$ for all actions $s_2'$ of player 2, so that $g(s_1, s_2') = a$ for all actions $s_2'$ of player 2. That is, by choosing $s_1$, player 1 guarantees that the outcome is $a$. Since $a \succeq_1' b$, it follows that $(G, \succeq')$ has no Nash equilibrium $(t_1, t_2)$ for which $g(t_1, t_2) = b$. Hence, $f$ is not Nash-implementable.

**2.** This question is based on Jean Pierre Benoit and Efe A. Ok. "Nash Implementation Without No-Veto Power." forthcoming in *Games and Economic Behavior*.

Let $F$ be a monotonic and weakly unanimous SCR. Let $G$ be a normal game-form with outcome function $g$ in which every agent $i$ announces a triple $a_i = (\trianglerighteq_i, c_i, k_i)$, where $\trianglerighteq_i$ is a preference profile in $P$, $c_i$ is a consequence in $C$, and $k_i$ is an integer. For any profile of actions $a$, define $g$ as follows:

(i) if $(p_i, c_i) = (\trianglerighteq, c)$ for all $i \in N$ and $c \in F(\trianglerighteq)$, then $g(a) = c$,

(ii) if there exists $j \in N$ such that $(p_i, c_i) = (\trianglerighteq, c)$ for all $i \in N \backslash \{j\}$ with $c \in F(\trianglerighteq)$, and $(p_j, c_j) \neq (\trianglerighteq, c)$, then

$$g(a) = \begin{cases} l(c, c_j) & if \quad c \trianglerighteq c_j \\ c & if \quad c \ntrianglerighteq c_j \end{cases}$$

(iii) if neither the antecedent of (i) nor (ii) applies, then $g(a) = c_{i^*}$ where $i^*$ is the smallest indexed agent who announces the highest integer $k_i$.

Fix any $\succeq \in P$ and any monotonic extension of it, $\succeq^*$. Note first that for any $c \in F(\succeq)$, we have $c = g(a) \in NE(G, \succeq^*)$, where $a_i = (\succeq, c, 1)$ for all $i \in N$. Conversely, take any $a \in NE(G, \succeq^*)$, we need to show that $g(a) \in F(\succeq)$.

Suppose first that case (i) applies. Assume, by contradiction, that $g(a) = c \notin F(\succeq)$. Because $a \in NE(G, \succeq^*)$, no $j \in N$ has any incentive to unilaterally alter the outcome from $c$ to $l(c, c_j)$, for any $c_j \trianglelefteq_j c$. This means that for any such $c_j$, $c \succeq_j^* l(c, c_j)$, which by the definition of a monotonic extension, implies that $c \succeq_j^* c_j$. Hence, by Maskin monotonicity, $c \in F(\succeq)$.

Suppose next that case (ii) applies. Assume $g(a) = l(c, c_j)$. If $c \succ_i c_j$ for some $i \in N \backslash \{j\}$, then $c \succ_i^* l(c, c_j)$ and $i$ can unilaterally deviate to some $(\cdot, c, \hat{k}_i)$, where $\hat{k}_i > \max_{j \neq i} k_j$, such that the outcome is determined according to case (iii) and $i$ wins the integer game implementing $c$. Similarly, it cannot be the case that $c_j \succ_i c$ for any $i \in N \backslash \{j\}$. If, on the other hand, $c \sim_i c_j$ for all $i \in N \backslash \{j\}$, then by the top coincidence condition, neither $c$ nor $c_j$ belong to $\max_{\succeq_l} C$ for some $l \in N \backslash \{j\}$. But then there exists some $b \in C$ such that $b \succ_l c \sim_l^* l(c, c_j)$ implying that player $l$ has a profitable deviation, a contradiction. It follows that $g(a) \neq l(c, c_j)$. Therefore, $g(a) = c$. But then $a$ cannot be an equilibrium of $\langle G, \succeq^* \rangle$ unless $c$ is a top choice for every $i \in N \backslash \{j\}$ since each $i \neq j$ can force the mechanism to choose any alternative in $C$ by a unilateral deviation. Hence, $c \in \max_{\succeq_i} C$ for all $i \in N \backslash \{j\}$. But then given that $g(a) = c$ and $a \in NE(G, \succeq^*)$, it is also an equilibrium for all agents to send the message $(\trianglerighteq, c, 1)$. Therefore, by the argument of the previous paragraph, $c \in F(\succeq)$.

Finally, assume that case (iii) applies, Then $g(a) \in \max_{\succeq_i} C$ for all $i$. By the top coincidence condition, $g(a) = \cap \{\max_{\succeq_i} C : i \in N\}$, and by weak unanimity of $F$, we have that $g(a) \in F(\succeq)$.

**3.** This question is based on Jean Pierre Benoit, Efe A. Ok and M.Remzi Sanver. "On Combining Implementable Social Choice Rules." forthcoming in *Games and Economic Behavior.*
Let $G^H$ be a game form with outcome function $g^H$ that Nash-implements $H \in \{F, G\}$. For each $i \in N$, define $A_i \equiv A_i^F \times A_i^G \times \{F, G\}$. Moreover, let $A \equiv A_i \times \cdots \times A_n$ and define the outcome function $h : A \to C$ as follows:

$$h[(a_1^F, a_1^G, H_1), \ldots, (a_n^F, a_n^G, H_n)] = \begin{cases} g^H(a_1^H, \ldots, a_n^H) & if \quad |i \in N : H_i = H \text{ for some } H \in \{F, G\}| \geq n-1 \\ g^F(a_1^F, \ldots, a_n^F) & if \quad |i \in N : H_i = H \text{ for some } H \in \{F, G\}| < n-1 \end{cases}$$

Let $G \equiv \langle A, h \rangle$. Take any $\succeq \in P$, and let $[(a_1^F, a_1^G, H_1), \ldots, (a_1^F, a_1^G, H_1)] \in NE(G, \succeq)$. By the definition of $h$, there must exist $H \in \{F, G\}$ such that

$$h[(a_1^F, a_1^G, H_1), \ldots, (a_1^F, a_1^G, H_1)] = g^H(a_1^H, \ldots, a_n^H)$$

It follows that $a_i^H$ is a best response in the game $G^H$ to $a_{-i}^H$ for each $i$. Then $(a_1^H, \ldots, a_n^H) \in NE(G^H, \succeq)$. Hence,

$$h[(a_1^F, a_1^G, H_1), \ldots, (a_1^F, a_1^G, H_1)] \in NE(G^H, \succeq) = H(\succeq) \subseteq F(\succeq) \cup G(\succeq)$$

Conversely, let $c \in F(\succeq) \cup G(\succeq)$. Then $c \in H(\succeq)$ for some $H \in \{F, G\}$. Take any $(a_i^F, a_i^G) \in A_i^F \times A_i^G$, $i \in N$, with $(a_1^H, \ldots, a_n^H) \in NE(G^H, \succeq)$, and define

$$\hat{a} \equiv [(a_1^F, a_1^G, H), \ldots, (a_n^F, a_n^G, H)] \in A$$

Clearly, $\hat{a} \in NE(G, h)$ and $h(\hat{a}) = g^H(a_1^H, \ldots, a_n^H) = c$.

**4.** This question is based on John Duggan and Joanne Roberts. "Implementing the Efficient Allocation of Pollution." *American Economic Review*, Sep. 2002, 92(4), 1070-1078.
**(a)** See the proof of Proposition 1.
**(b)** See Section 3(a), p.1073
**(c)** See Section 3(b), pp.1073-1074
**(d)** See Section 3(b), p.1074