Common knowledge, common priors and speculative trade

A model

- Ω is a finite space of states of nature
- $I = \{1, 2\}$, the set of players
- $P_i : \Omega \to 2^{\Omega} \backslash \{\varnothing\}$, player i's *information structure*
  - $P_i(\omega) \subseteq S$ means that in state ω, player i "knows" the event S
  - Player i has a *partitional* information structure if there exists a partition of Ω such that for every $\omega \in \Omega$, the set $P_i(\omega)$ is the cell in the partition, which includes ω

Definition 1 *An event $A \subseteq \Omega$ is **self-evident** between 1 and 2 if for all $\omega \in A$ we have $P_i(\omega) \subseteq A$ for $i = 1, 2$.*

Definition 2 *An event $B \subseteq \Omega$ is **common knowledge** between 1 and 2 in the state $\omega \in \Omega$ if there is a self-evident event A for which $\omega \in A \subseteq B$.*

An event E is self-evident between two individuals if whenever this event occurs, both individuals know it did, and it is common-knowledge between the two individuals at a state ω, if whenever this state is realized, both individuals know that some event, which implies E, has occured.

Example 1

$$\begin{aligned}
\Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} \\
P_1 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \{\omega_6\}\} \\
P_2 &= \{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_5\}, \{\omega_6\}\}
\end{aligned}$$

- The event $B = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is *not* common knowledge between 1 and 2 at any $\omega \in B$. To see why, note that for any $A \subseteq B$ such that $\omega \in A$, there is some player i and some state ω' such that $\omega' \in \cup_{\omega \in A} P_i(\omega)$ but $\omega' \notin \cup_{\omega \in A} P_j(\omega)$ (e.g., if $A = \{\omega_1, \omega_2\}$, then $i = 2$ and $\omega' = \omega_3, \omega_4$). This means that there is no state in B, in which both players agree that an event implying B has occured.

- The event $B' = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ *is* common knowledge between 1 and 2 at any $\omega \in B'$. To see why, note that B' itself is self-evident.

Remark 1 *If P_1 and P_2 are partitional information structures and the event A is self-evident between* 1 *and* 2*, then $A = \cup_{\omega \in A} P_1(\omega) = \cup_{\omega \in A} P_2(\omega)$.*

Suppose two individuals have different posterior beliefs about some event A, where the difference in beliefs is solely due to asymmetric information: each individual observed a different realization of a random variable, whose distribution is commonly known (and agreed upon) by the two individuals. Will these individuals agree to bet on whether or not A will realize? The above framework allows us to address this question, by investigating whether or not it can be common knowledge that two individuals with the same prior belief assigns different posterior beliefs to the same event.

To do this, let $f : 2^\Omega \to \{T, F\}$ and denote by $\mathbf{F}$ the set of all such functions. We interpret the statement, "$f(P_i(\omega)) = T$" to mean that i "knows" that f gets the value "$True$" in state ω.

- f is *preserved under union* if for all sets A and B such that $f(A) = T$ and $f(B) = T$, we have $f(A \cup B) = T$

- f is *preserved under disjoint union* if for all disjoint sets A and B such that $f(A) = T$ and $f(B) = T$, we have $f(A \cup B) = T$

- f is *preserved under difference* if for all sets A and B such that $A \subset B$, $f(A) = T$ and $f(B) = T$, we have $f(B - A) = T$

Example 2 *Let x be a random variable defined on Ω and let α be a number. Define*

$$
\begin{aligned}
f(A) &= T \Longleftrightarrow E(x|A) = \alpha \\
g(A) &= T \Longleftrightarrow E(x|A) \geq \alpha
\end{aligned}
$$

	preserved under union	*preserved under disjoint union*	*preserved under difference*
f		✓	✓
g		✓	

To get a sense of why this is true, consider the following x :

ω	ω_1	ω_2	ω_3
$\Pr(\omega)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$x(\omega)$	2	4	2

To see why both f *and* g *are not preserved under union, let* $A = \{\omega_1, \omega_2\}$, $B = \{\omega_2, \omega_3\}$ *and* $\alpha = 3$. *Then*

$$\begin{aligned} E(x|A) &= 3 \\ E(x|B) &= 3 \\ E(x|A \cup B) &= \frac{8}{3} \end{aligned}$$

To see why g *is not preserved under difference, let* $A = \{\omega_2\}$, $B = \{\omega_1, \omega_2, \omega_3\}$ *and* $\alpha = \frac{8}{3}$. *Then*

$$\begin{aligned} E(x|A) &= 4 \\ E(x|B) &= \frac{8}{3} \\ E(x|B - A) &= 2 \end{aligned}$$

Proposition 1 *Assume that the information structures of the two individuals are partitional. If* $f, g \in \mathbf{F}$ *such that*

(i) for no $A \subseteq \Omega$ *both* $f(A) = T$ *and* $g(A) = T$,

(ii) f *and* g *are preserved under disjoint union*

then there is no $\omega^* \in \Omega$ *at which the set* $\{\omega \in \Omega : f(P_1(\omega)) = T, g(P_2(\omega)) = T\}$ *is common knowledge.*

Proof. Suppose that such a state ω^* exists. Then by the definition of common knowledge, there exists a self evident event A satisfying

$$\begin{aligned} \omega^* &\in A \\ A &\subseteq \{\omega \in \Omega : f(P_1(\omega)) = T\} \\ A &\subseteq \{\omega \in \Omega : g(P_2(\omega)) = T\} \end{aligned}$$

Our assumption that P_1 and P_2 are partitional implies that if $\omega, \omega' \in A$ but $P_i(\omega) \neq P_i(\omega')$, then $P_i(\omega) \cap P_i(\omega') = \varnothing$. Since A is self evident, $A = \cup_{\omega \in A} P_1(\omega)$ and $A = \cup_{\omega \in A} P_2(\omega)$. Because $f(P_1(\omega)) = T$ and $g(P_2(\omega)) = T$ for all $\omega \in A$, and because f and g satisfy (ii), it follows that both $f(A) = T$ and $g(A) = T$, in contradiction to (i). ■

Corollary 1 *Let x be a random variable on Ω and let α and β be two distinct numbers. Then there is no $\omega^* \in \Omega$ at which it is common knowledge that, conditional on his information, player 1 believes that the expectation of x is α, and conditional on his information, player 2 believes that the expectation of x is β.*

Proof. For any $A \subseteq \Omega$, let $f_\alpha(A) = T$ iff $E(x|A) = \alpha$, and let $f_\beta(A) = T$ iff $E(x|A) = \beta$. Note, first, that it cannot be the case that both $f_\alpha(A) = T$ and $f_\beta(A) = T$. Note also that $f_\alpha(A)$ and $f_\beta(A)$ are preserved under disjoint union (recall Example 2). Hence, the required conclusion follows from Proposition 1. ■

Corollary 2 *Let x be a random variable on Ω and let α be a number. Then there is no $\omega^* \in \Omega$ at which it is common knowledge that, conditional on his information, player 1 believes that the expectation of x is strictly above α, and conditional on his information, player 2 believes that the expectation of x is at most α.*

Proof. For any $A \subseteq \Omega$, let $f_\alpha(A) = T$ iff $E(x|A) > \alpha$, and let $g_\alpha(A) = T$ iff $E(x|A) \leq \alpha$. Note, first, that it cannot be the case that both $f_\alpha(A) = T$ and $g_\alpha(A) = T$. Note also that $f_\alpha(A)$ and $g_\alpha(A)$ are preserved under disjoint union (recall Example 2). Hence, the required conclusion follows from Proposition 1. ■

Remark 2 *Let x be a random variable on Ω and let α be a number. Then there can be a state $\omega^* \in \Omega$ at which it is common knowledge that, conditional on his information, player 1 believes that the expectation of x is α, and conditional on his information, player 2 believes that the expectation of x is different from α.*

To see why this is true, consider the following example:

$$\begin{aligned}
\Omega &= \{\omega_1, \omega_2\} \\
\Pr(\omega_1) &= \Pr(\omega_2) \\
P_1(\omega_i) &= \Omega \\
P_2(\omega_i) &= \{\omega_i\} \\
x(\omega_1) &= 1 \\
x(\omega_2) &= 2
\end{aligned}$$

Corollaries 1 and 2 imply two well known results on the impossibility of speculative trade under common priors.

Proposition 2 *(Aumann's "agreeing to disagree" result) If two players have partitional information structures and a common prior, then it cannot be common knowledge among them that player* 1 *assigns probability* α *to some event and player* 2 *assigns prpbability* $\beta \neq \alpha$ *to the same event.*

Proof. Let $B \subseteq \Omega$ and denote by $\Pr(B|A)$ the probability of B, conditional on A. Consider the random variable $x(\omega) = 1_B$, i.e., $x(\omega) = 1$ if $\omega \in B$, otherwise, $x(\omega) = 0$. Then for any $A \subseteq \Omega$, $E(x|A) = \Pr(B|A)$. Consider then the pair of functions defined in the proof of Corollary 1 : $f_\alpha(A) = T$ iff $E(x|A) = \alpha$ and $f_\beta(A) = T$ iff $E(x|A) = \beta$. We, therefore, have that $f_\alpha(A) = T$ iff $\Pr(x|A) = \alpha$ and $f_\beta(A) = T$ iff $\Pr(x|A) = \beta$, and the required result follows from Corollary 1. ■

Milgrom and Stokey established that if two traders with partitional information structures agree on an ex-ante efficient allocation of goods, then after the two get new information, there is no transaction with the property that it is common knowledge that both traders are willing to carry it out. To show that this result follows from Corollary 2, consider the followig set-up.

- Let Z be a set of outcomes
- A *contingent contract* is a function $c : \Omega \to Z$.
- C is the set of all contingent contracts

- Each player i has a vNM utility function u_i defined on $Z \times \Omega$, i.e., $u_i(z, \omega)$ *is* player i's utility of the outcome z in state ω

- For any $c \in C$, define $U_i(c)$ be the random variable on Ω that gets the value $u_i(c(\omega), \omega)$ at each $\omega \in \Omega$

- For each $c \in C$ and $A \subseteq \Omega$, we have that $E[U_i(c)|A]$ is player i's expected utility from a contract c, conditional on A.

Definition 3 *A contingent contract c is ex-ante efficient if there is no contract c' satisfying that for both i, $E[U_i(c')] > E[U_i(c)]$.*

Proposition 3 *(Milgrom and Stokey's No-Trade Theorem) If a contingent contract c is ex-ante efficient, then there is no $\omega^* \in \Omega$ at which the set*

$$D^* = \{\omega \in \Omega : E[U_i(c')|P_i(\omega)] > E[U_i(c)|P_i(\omega)],\ i = 1, 2\}$$

is common knowledge.

Proof. Assume, by contradiction, that there exists a state $\omega^* \in \Omega$ in which D^* is common knowledge. Then there exists a self-evident event $D \subseteq D^*$, which contains ω^*. By the definition of a self-evident event, $D = \cup_{\omega \in D} P_1(\omega) = \cup_{\omega \in D} P_2(\omega)$. This implies that for all $\omega \in D$,

$$E[U_i(c')|P_i(\omega)] > E[U_i(c)|P_i(\omega)] \quad (1)$$

for $i = 1, 2$.

Suppose there is a state $\omega \in D$ for which $E[U_1(c')|P_2(\omega)] > E[U_1(c)|P_2(\omega)]$. Consider then the contract c^* defined as follows:

$$c^*(\omega) = \begin{cases} c'(\omega) & if \quad \omega \in P_2(\omega) \\ c(\omega) & if \quad \omega \in \Omega \backslash P_2(\omega) \end{cases}$$

Note that

$$E[U_2(c^*)] = \sum_{\omega \in P_2(\omega)} \Pr(\omega) \cdot E[U_2(c')|P_2(\omega)] + \sum_{\omega \in \Omega \backslash P_2(\omega)} \Pr(\omega) \cdot E[U_2(c)|P_2(\omega)]$$

Hence, by (1),

$$E[U_2(c^*)] > E[U_2(c)]$$

Because c is ex-ante efficient,

$$E[U_1(c^*)] \leq E[U_1(c)]$$

This inequality may be written as,

$$\sum_{\omega \in \Omega \setminus P_2(\omega)} \Pr(\omega) \cdot E[U_1(c')|P_2(\omega)] + \sum_{\omega \in \Omega \setminus P_2(\omega)} \Pr(\omega) \cdot E[U_1(c)|P_2(\omega)]$$
$$\leq \sum_{\omega \in P_2(\omega)} \Pr(\omega) \cdot E[U_1(c)|P_2(\omega)] + \sum_{\omega \in \Omega \setminus P_2(\omega)} \Pr(\omega) \cdot E[U_1(c)|P_2(\omega)]$$

which implies that $E[U_1(c')|P_2(\omega)] \leq E[U_1(c)|P_2(\omega)]$, a contradiction. Therefore,

$$E[U_1(c')|P_2(\omega)] \leq E[U_1(c)|P_2(\omega)]$$

for all $\omega \in D$.

Define x to be be the random variable on Ω that gets the value $u_1(c'(\omega), \omega) - u_1(c(\omega), \omega)$ at each $\omega \in \Omega$. We, therefore, obtain that for all $\omega \in D$,

$$\begin{aligned} E[x|P_2(\omega)] &\leq 0 \\ E[x|P_1(\omega)] &> 0 \end{aligned}$$

Since D is self-evident, the above inequalities must be common knowledge at $\omega^* \in \Omega$, in contradiction to Corollary 2. ■

The previous results established that it cannot be common knowledge among two parties with partitional information structures and a common prior, that there is a bet, which is strictly profitable for both. This raises the question of whether there always exists a bet, which is strictly profitable for two parties with partitional information structures but *non-common* priors?

What does it mean for individuals to have different *prior* beliefs about a future event? It means that their disagreement is not purely due to asymmetric information, but rather coming from some inherent bias they may have in forming their beliefs such as optimism, pessimism, idealology or overconfidence in their ability to process information. Hence, such individuals may still hold different beliefs even if they were to share all their information. The following are some examples of such situations:

- a die-hard Mets fan, who watches or reads the same sports commentary as a die-hard Yankees fan, would still disagree with the latter on the likelihood that his team would win the World Series.

- entrepreneurs often do not share the same views as venture capitalists with regards to the profitability of their start-ups

- a fashion retailer may believe that his daily contact with consumers makes him better capable of predicting future demand than his supplier, who holds the opposite view owing to his experience with a large number of retailers.

- traders in a market often arrive at different conclusions about what the current economic data implies with regards to future prices.

- voters, who watch or read the same news commentary, often have different opinions about the likelihood that a particular candidate would win

Consider two agents with partitional information structures, P_1 and P_2. For each state $\omega \in \Omega$ define $\theta_i^\omega \in \Delta^\Omega$ to be agent i's *type* at ω, where Δ^Ω is the simplex in R^Ω, which we consider as the set of probability distributions over Ω. An agent's type satisfies the following two conditions:

1. For every $\omega \in \Omega$, $\sum_{\omega' \in P_i(\omega)} \theta_i^\omega(\omega') = 1$ for $i = 1, 2$

2. For every i, $\theta_i^\omega = \theta_i^{\omega'}$ if both ω and ω' belong to the same cell in i's information structure

We interpret θ_i^ω to be i's posterior beliefs, conditional on knowing that the event $P_i(\omega)$ has occured.

Define agent i's prior belief π_i to be a probability measure on Ω that satisfies the following condition: for every $\omega \in \Omega$ with $\sum_{\omega' \in P_i(\omega)} \pi_i(\omega') > 0$,

$$\theta_i^\omega(\hat{\omega}) = \pi_i[\hat{\omega}|P_i(\omega)] = \begin{cases} \frac{\pi_i(\hat{\omega})}{\sum_{\omega' \in P_i(\omega)} \pi_i(\omega')} & if \quad \hat{\omega} \in P_i(\omega) \\ 0 & if \quad \hat{\omega} \notin P_i(\omega) \end{cases}$$

Let Π_i denote agent i's set of prior beliefs. Note that this set includes the set of i's types, $\{\theta_i^\omega\}^{\omega \in \Omega}$. Hence, Π_i is the convex hull of all of i's types. It is, therefore, *convex* and *closed*.

Definition 4 *A probability distribution* $\pi^* \in \Delta^\Omega$ *is a* ***common prior*** *on the type space if* $\pi^* \in \Pi_1 \cap \Pi_2$.

Define a *bet* to be a function $t : \Omega \to R$. A bet is *agreeable* in state $\hat{\omega}$ if

$$\sum_{\omega \in \Omega} \theta_1^{\hat{\omega}}(\omega) t(\omega) > 0 \text{ and } \sum_{\omega \in \Omega} \theta_2^{\hat{\omega}}(\omega)[-t(\omega)] > 0$$

Proposition 4 *If two agents do not have a common prior, then there exists an agreeable bet at every* $\omega \in \Omega$.

Proof. Suppose Π_1 and Π_2 are disjoint. Then by the *Separating Hyperplane Theorem* there exists $h \in R^\Omega$ with $h \neq 0$, and a value $c \in R$, such that $h \cdot \pi_1 > c > h \cdot \pi_2$ for every $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$. In particular, $h \cdot \theta_1^\omega > c > h \cdot \theta_2^\omega$ for every $\omega \in \Omega$. Define t to be the bet that assigns the value $h(\omega) - c$ for every state ω. It follows that for each state $\hat{\omega}$, $\sum_{\omega \in \Omega} \theta_1^{\hat{\omega}}(\omega) t(\omega) > 0$ and $\sum_{\omega \in \Omega} \theta_2^{\hat{\omega}}(\omega)[-t(\omega)] > 0$. ■

The above result implicitly assumes that a bet made directly on the states of nature can be enforced. This may be true if the two agents can reliably verify whether or not a state of nature was realized. What if this was not true? Can two agents with non-common prior beliefs always find a bet, which they would agree to sign, even if the bet was made on states of nature, which are not observed by both? We address this question with the following example.

	ω_1	ω_2	ω_3
P_1	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
P_2	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$

Let $p = (p_1, p_2, p_3)$ be a prior belief of agent 1, and let $q = (q_1, q_2, q_3)$ be a prior belief of agent 2. Note that if $p \neq q$, then there exists a bet t, which is ex-ante agreeable to both agents:

$$\begin{aligned} p_1 t_1 + p_2 t_2 + p_3 t_3 &> 0 \\ -q_1 t_1 - q_2 t_2 - q_3 t_3 &> 0 \end{aligned}$$

Suppose the states of nature are unverifiable and so in order to bet on a state each agent i annouces a cell $\hat{s}_i$ in his information structure (assume 1's action set is

$\{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and 2's action set is $\{\{\omega_1, \omega_2\}, \{\omega_3\}\}$) and for each pair of announcements,

$$t(\hat{s}_1, \hat{s}_2) = \begin{cases} t(\omega_k) & if \quad \omega_k = \hat{s}_1 \cap \hat{s}_2 \\ 0 & if \quad \hat{s}_1 \cap \hat{s}_2 = \emptyset \end{cases}$$

A bet is incentive-compatible (IC) if at every state, each agent weakly prefers to send a true report, assuming the other agent is also honest. Is there an *ex-ante* agreeable bet, which is also incentive-compatible?

Assume, by contradiction, that there exists such a bet t. If t is IC, then assuming the other agent is honest,

- when agent 1 observes $\{\omega_1\}$, he has no incentive to report $\{\omega_2, \omega_3\}$:

$$t_1 \geq t_2 \tag{2}$$

- when agent 1 observes $\{\omega_2, \omega_3\}$, he has no incentive to report $\{\omega_1\}$:

$$p_2 t_2 + p_3 t_3 \geq p_2 t_1 \tag{3}$$

- when agent 2 observes $\{\omega_1\}$, he has no incentive to report $\{\omega_2, \omega_3\}$:

$$-t_3 \geq -t_2 \tag{4}$$

- when agent 2 observes $\{\omega_1, \omega_2\}$, he has no incentive to report $\{\omega_3\}$:

$$-q_1 t_1 - q_2 t_2 \geq -q_2 t_3 \tag{5}$$

By (2) and (3), $p_3 t_3 \geq 0$, which from the fact that $p_3 \geq 0$ implies that $t_3 \geq 0$. By (4) and (5), $-q_1 t_1 \geq 0$, which from the fact that $q_1 \geq 0$ implies that $t_1 \leq 0$. But by (2) and (4),

$$t_1 \geq t_2 \geq t_3$$

which can only be true if

$$t_1 = t_2 = t_3 = 0$$

This implies that the only IC bet is the null bet, hence there is no IC bet, which is ex-ante agreeable.