Problem set 3 (subgame-perfect implementation)

1. *Subgame-perfect implementation of the Nash bargaining solution.* A bargaining problem is a tuple $\langle X, D, \succsim_1, \succsim_2 \rangle$ where

- X (the set of "agreements") is a compact set
- D (the "disagreement" outcome) is a member of X
- $\succsim_1$ and $\succsim_2$ are continuous preference relations on the set of lotteries over X that satisfy the vNM assumptions of expected utility.
- $x \succsim_i D$ for all $x \in X$ for $i = 1, 2$, and there exists $x \in X$ such that $x \succ_1 D$ and $x \succ_2 D$.
- (convexity) for any $x \in X$, $y \in X$ and $p \in [0, 1]$ there exists $z \in X$ such that both agents are indifferent between z and a lottery that draws x with probability p and y with probability $1 - p$.
- (non-redundancy) if $x \in X$ then there is no $x' \in X$ with $x' \neq x$ such that $x \sim_i x'$ for $i = 1, 2$.
- (unique best agreement) for each player i there is a unique agreement $B_i \in X$ with $B_i \succsim_i x$ for all $x \in X$.
- for each player i we have $B_i \sim_j D$ for $j \neq i$.

Denote by $p \cdot x$ the lottery that draws x with probability p and D with probability $1 - p$. The *Nash bargaining solution* is a function that assigns to every bargaining problem an agreement $x^* \in X$ with the following property: if $p \cdot x \succ_i x^*$ for some $p \in [0, 1]$ and $x \in X$, then $p \cdot x^* \succsim_j x$ for $j \neq i$.

Fix a set X and an outcome D, and assume a planner wants to implement the Nash solution for all pairs $(\succsim_1, \succsim_2)$ for which $\langle X, D, \succsim_1, \succsim_2 \rangle$ is a bargaining problem. Consider the extensive game form consisting of the following stages.

- Player 1 chooses $y \in X$.
- Player 2 chooses $x \in X$ and $p \in [0, 1]$.
- With probability $1 - p$ the game ends, with the outcome D, and with probability p it continues.
- Player 1 chooses either x or the lottery $p \cdot y$; this choice is the outcome.

Show that this game form implements the Nash solution in subgame-perfect equiibrium (SPE).

2. *Renegotiation-proof implementation.* Consider a buyer and a seller who are interested in signing a contract for the sale of a certain unit. The reservation values, denoted by s and b, respectively, are taken from the finite sets S and

B. Let $s_{\max}$, $b_{\max}$, $s_{\min}$ and $b_{\min}$ denote the maxima and minima of S and B, respectively. Assume $b_{\min} > s_{\max}$ (so there are gains from trade). When the contract is signed the parties know only S and B, but before it is carried out, the true values of s and b are realized and are common knowledge among the parties. The possible outcomes with which this interaction may end are exchanges for some price p, and the no-sale outcome D. The preferences of the parties over these outcomes are given by the utility functions $p - s$ and $b - p$ for the seller and the buyer, respectively; both parties assign a utility of zero to D.
(a) Let P be a function that assigns a price to each pair of values $(s, b) \in S \times B$. We say that P is strictly individually-rational if for all (s, b), $s < P(s, b) < b$. Show that the following extensive game form SPE-implements any function SIR P, that is non-decreasing in s and b.

- *Stage* 1 : *The Announcement Stage.*
  - The seller announces a number $v_s \in S$
  - The buyer challenges the seller or announces a pair (v_B, v'_S), where $v_B \in B$, $v_S \in S$ and $v'_S \geq v_S$.
    - If the buyer challenges the seller, the game continues to stage 2.
    - If the buyer chooses (v_B, v'_S), the seller may challenge the buyer. If he does, the game continues to stage 3, if he does not, the unit is exchanged for a price $P(v_B, v'_S)$.
- *Stage* 2 : *The buyer can make a "take-it-or-leave-it" price offer below* v_S.
  - The buyer can choose a price offer $p < v_S$.
  - The seller either accepts, in which case the good is exchanged for p, or rejects, in which case the outcome is D.
- *Stage* 3 : *The seller can make a "take-it-or-leave-it" price offer above* v_B.
  - The seller can choose a price offer $p > v_B$.
  - The buyer either accepts, in which case the good is exchanged for p, or rejects, in which case the outcome is D.

(b) A price function $P(s, b)$ is called "renegotiation-proof implementable" if it is SPE-implementable, and the game form that implements it is such that, for all s and b, all nodes (including terminal ones), and every SPE, the SPE outcome in the subgame starting at that node is efficient. Show that the only renegotiation-proof implementable, strictly individually-rational, price functions are the constant functions $P(s, b) = p$, where $p \in (s_{\max}, b_{\min})$.