Virtual Implementation in SPE

1. Model

$N = \{1, \ldots, n\}$
$C^* \equiv$ finite set of deterministic consequences
$C = \{(L, m) : L$ is a lottery over C^* and $m \in \Re\}$
$u_i : C^* \to \Re$ player i's *VNM* utility function so that his preferences over C can be represented by

$$E_L u(c^*) - m \equiv u_i(L) - m$$

$P = U^n$ where U is a finite set that excludes the constant function and has the property that if $u \in U,$ then any affine transformation of u is *not* in U.

2. Preliminaries

Since U does not include the constant function, for every pair of utility functions v, v' there is a pair of lotteries $L(v,v'), L'(v,v')$ such that

$$v(L(v,v')) > v(L'(v,v'))$$
$$v'(L'(v,v')) > v(L(v,v'))$$

so that a choice between $L(v,v')$ and $L'(v,v')$ reveals whether player is of type v or v'.

For any three utility functions u, v, v' let $L^*(u,v,v')$ denote the lottery a player of type u would choose from $\{L(v,v'), L'(v,v')\}$. Consequently,

$$u[L^*(u,v,v')] \geq u[L^*(u',v,v')] \text{ for all } u'$$

Furthermore,

$$u[L^*(u,u,u')] > u[L^*(u',u,u')] \text{ for all } u' \neq u$$

Consider some pair of utility functions v, v' and suppose a player is asked to announce his utility function. Based on his announced function u the player receives the lottery $L^*(u,v,v')$. Suppose u is the player's actual utility function. If he announces u' instead, his expected loss (by defintion, $L^*(u,v,v')$ is best for player of type u) is:

$$L^*(u,v,v') - L^*(u',v,v')$$

Suppose we do the following calculations:

1. For a given u we calculate the average loss from announcing some u' over all possible pairs $v, v' \in U$.
2. We do the same calculation for announcing some other $u'' \neq u'$.
3. We repeat steps 1 and 2 for a different utitlity function u^*.
4. Of all the numbers we obtained in steps 1-4 we pick the smallest one. We call this number B so that

$$B = \min_{(u,u')\in W}\left[\frac{1}{M}\sum_{(v,v')\in M}(L^*(u,v,v') - L^*(u',v,v'))\right]$$

where W is the set of all distinct pairs of utitlity functions and M is the number of elements in W so that

$$M = |W| = |U||U-1|$$

To interpret B consider a procedure in which based on a player's announcement of his type (say u) a pair of utitlity functions (v,v') is randomly drawn from W and the player participates in the lottery $L^*(u,v,v')$.

B is the **minimal** expected loss that the player may accrue by lying.

Claim: $B > 0$

Proof: By construction of L^*, for every $u \neq u'$ we have $u[L^*(u,u,u')] > u[L^*(u',u,u')]$ (recall $L^*(u,u,u')$ is the *strictly preferred* lottery in the pair $\{L(u,u'),L'(u,u')\}$ and L and L' are constructed so they will not be indifferent).■

3. Virtual implementation in SPE

Definition: *A choice function $f : P \to C^*$ is virtually SPE-implementable if for any $\varepsilon > 0$ there is an extensive game form Γ such that for any preference profile $u \in P$ the extensive game $\langle\Gamma,u\rangle$ has a unique SPE in which the outcome is $f(u)$ with probability at least $1-\varepsilon$.*

Proposition: *If $n \geq 3$ then every choice function is virtually SPE-implementable*.

Proof: we construct an extensive game form and show that its unique SPE satisfies the following:

1. All players reveal their true utitlity function.
2. A lottery is held which picks $f(u)$ with with probability $1-\varepsilon$.

The game form

Round 1

Stage 1: Player 1 announces $m_1^1 \in U^n$

⋮

Stage i: Player i announces $m_i^1 \in U^n$

⋮

Stage n :Player n announces $m_n^1 \in U^n$

⋮

Round K

Stage 1: Player 1 announces $m_1^K \in U^n$

⋮

Stage n :Player n announces $m_n^K \in U^n$

Round $K+1$

Stage 1: Player 1 announces $m_1^{K+1} \in U$

⋮

Stage n :Player n announces $m_n^{K+1} \in U$

Outcomes

A grand lottery is drawn:

Prob. $1-\varepsilon$: Prob. $\frac{1}{K}$: $f(u)$ if at least $n-1$ players announced u in stage 1, otherwise prize is some fixed c^*.

$\vdots$

Prob. $\frac{1}{K}$: $f(u)$ if at least $n-1$ players announced u in stage K, otherwise prize is some fixed c^*.

Prob. ε: Prob. $\frac{1}{n}$: A lottery $L^*(m_1^{K+1}, v, v')$ is randomly drawn out of the M possible lotteries of this type.

$\vdots$

Prob. $\frac{1}{n}$: A lottery $L^*(m_n^{K+1}, v, v')$ is randomly drawn out of the M possible lotteries of this type.

Fines

For any stage k in which all players but i announced the same u player i pays a fine of $\frac{\delta}{K}$.

If player i is the last player in the first K rounds to announce a utility profile which is different from $(m_1^{K+1}, \dots, m_n^{K+1})$, then he pays a fine of δ.

δ and K are chosen such that:

$$(1-\varepsilon)\frac{D}{K} + \frac{\delta}{K} < \delta < \frac{\varepsilon B}{n}$$

where

$$D = \max_{v,c,c'}\{v(c) - v(c') : v \in U, c \in C^*, c' \in C^*\}$$

That is, D is computed as follows: Start with some v and compute the difference $v(c) - v(c')$ for all the pairs c, c'. Take the next v and carry out the same computation. At the end you'll be left with $|U| \cdot |C^*| \cdot |C^* - 1|$ numbers. Pick the largest one, That number corresponds to some triplet (v, c, c').

Equilibrium

STEP 1: Following any history, it is a dominant strategy to tell the truth in Round $K+1$

Assume player i does not tell the truth in final round.

Suppose he switches to announcing his true utility.

Consider the worst case scenario: i gets the minimal utility from this switch and turns out to be the last player in the first K rounds to contradict $m^{K+!}$.

Minimal expected gain $= \frac{\varepsilon}{n}B$

Maximal expected loss $= \delta$

By construction, deviation is profitable regardless of what others do.

STEP 2: Given that players tell the truth in final round, it is a dominant strategy to tell the truth in every round.

Consider stage i of some round k of the game.

Consider the last player in the first K rounds to disagree with the truthful announcement of everyone in round $K+1$.

Case 1: All other players report the actual u.

Effect on outcome: None (even without i at least $n-1$ players are in agreement).

Effect on fines:

(a). Because i was the only liar in round k before switching to the truth, he no longer needs to pay a fine of $\frac{\delta}{K}$.

(b). i no longer pays δ.

Case 2: At least one other player does not report actual u.

Effect on outcome: Consider the worst case scenario in which player i changes the outcome to one he does not like and accrues the maximal possible loss D (this may occur if before switching to the truth, player i coordinated with all the other players on some non-truthful profile u').

Effect on fines:

(a). i becomes the odd man out and pays $\frac{\delta}{K}$.

(b). i no longer pays δ.

By our choice of δ and K, player i should revert to the truth regardless of others' behavior.

Note

Mechanism has 2 central features:

(1) Dominant strategy to tell truth in round $K+1$

(2) Each player may want to lie in first K rounds, but no one wants to be the last one to do so. Consequently, no one lies in first K rounds.