Homework Assignment 1 (Strategy-proof Implementation)

1. Let A denote a finite set of at least three alternatives. Consider a set N of $n \geq 3$ individuals. Let $\succ = (\succ_1, \ldots, \succ_n)$, a profile of n strict linear orders on A, and let P denote the set of all such profiles. Let R be the following set of binary relations on A with the property that a relates to every other element, but all other alternatives are not related: $R = \{(a, b) : b \in A\}$. The interpretation is, that at R, a is considered to be the unique socially desirable alternative, but all the other alternatives cannot be ranked by society. Given a pair of alternatives, $a, b \in A$ and a preference profile $\succ \in P$, let $aF(\succ)b$ stand for "a relates to b when the preference profile is $\succ$".

A function $F : P \to R$ is called a social aggregator. Consider the following properties of social aggregators.

- *Monotonicity*: If a pair of profiles $\succ, \succ' \in P$ satisfies $aF(\succ)x$ for all $x \in A \backslash \{a\}$, and $a \succ'_i b$ if $a \succ_i b$ for every individual i and for every alternative $b \in A$, then $aF(\succ')x$ for all $x \in A \backslash \{a\}$.
- *Preference Reversal*: if for every pair of alternatives, $a, b \in A$, if $aF(\succ)b$, b does not relate to a according to $F(\succ)$ but $bF(\succ')a$, then there must be an individual i that satisfies $a \succ_i b$ and $b \succ'_i a$.

Show that if $F : P \to R$ is monotonic, then it must also satisfy Preference Reversal.

The next two questions address the following problem.

A House allocation problem with existing tenants consists of:

- a finite set of existing tenants I_E,
- a finite set of new applicants I_N,
- a finite set of occupied houses $H_O = \{h_i\}_{i \in I_E}$,
- a finite set of vacant houses H_V, and
- a list of strict preference relations $P = (P_i)_{i \in I_E \cup I_N}$, where the null house (no house) h_0 is assumed to be at the bottom of each agent's preference relation.

A house matching μ is an assignment of houses to agents such that:

1. every agent is assigned one house, and
2. only the null house h_0 can be assigned to more than one agent

A matching is individually rational if no existing tenant strictly prefers his current house to his assigned house. A housing lottery is a probability distribution over all matchings. It is ex-post individually rational if it gives positive weight to only individually rational matchings and it is ex-post Pareto efficient if it it gives positive weight to only Pareto efficient matchings.

2. Suppose $I_E = \{i_1\}$, $I_N = \{i_2, i_3\}$, $H_O = \{h_1\}$, $H_V = \{h_2, h_3\}$ and P is represented by the vNM utility numbers in the following matrix:

	h_1	h_2	h_3
i_1	3	4	1
i_2	4	3	1
i_3	3	4	1

2.1. Suppose the agents are randomly ordered and the first agent is assigned his or her top choice, the next agent his or her top choice among the remaining houses and so on. Agent 1 who already occupies h_1 is given two options: keep the house or enter the lottery. Show that this procedure is not ex-post Pareto efficient.

2.2. Propose a procedure that is ex-post Pareto efficient.

3. Consider the following mechanism for allocating houses.

- An ordering f of agents is chosen from a given distribution of agents.
- The first agent is tentatively assigned his or her top choice among all houses, the next agent is tentaively assigned his top choice among the remaining houses, and so on, until a *squatting conflict* occurs.
- A *squatting conflict* occurs if it is the turn of an existing tenant but every remaining house is worse than his or her current house. This means someone else, the conflicting agent, is tentatively assigned the existing tenant's house. when this happens:
  - the existing tenant is assigned his or her current house and removed from the process, and
  - all tentative assignments starting with the conflicting agent and up to the existing tenant are erased.
- The procedure starts over again with the conflicting agent. Every squatting conflict that occurs afterwards is resolved in a similar manner.
- The process terminates when there are no houses or agents left. At this point all tentative assignments are finalized.

Show that this procedure is not Pareto efficient.