Second-Order Ambiguous Beliefs

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Abstract

This paper axiomatizes models of second-order ambiguous beliefs in the original domain of preferences of Anscombe and Aumann (1963) by weakening the (first-stage) independence postulate. The models we propose include the Second-Order Subjective Expected Utility (SOSEU) of Seo (2009) as a particular case. We characterize the intersection of our models of second-order ambiguity with the canonical models of (first-order) ambiguity aversion, and provide a further generalization of SOSEU by relaxing the completeness axiom.

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1 Introduction

Anscombe and Aumann (1963) gave a simple derivation of subjective probabilities. Differently from Savage (1954), they considered acts that map states into a set of roulette lotteries and relied on a second randomization device by taking as domain for preferences the set of roulette lotteries of acts.[1] Using the same setup of Anscombe and Aumann (1963), Seo (2009) provided an alternative axiomatization for the smooth ambiguity decision model of Klibanoff, Marinacci, and Mukerji (2005) by relaxing the Reversal of Order axiom and imposing a monotonicity condition that is implied by the Anscombe and Aumann's (1963) axiomatization. According to his Second-Order Subjective Expected Utility (SOSEU) model, the decision maker forms a (not necessarily unique) belief m over the set $\Delta(S)$ of priors on a set S of states, and evaluates the lottery of acts P according to

$$\int_{\Delta(\Delta(S))} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dP(f) \right] dm(\mu), \tag{1}$$

where u is a von Neumann-Morgenstern (vN-M) utility function, and ϕ is a real-valued mapping.[2,3]

One of the main motivations for the SOSEU model is the classic Ellsberg Paradox, due to Ellsberg (1961). When the function ϕ is concave, preferences represented by (1) can account for the so-called "Ellsbergian choices". Such choices reflect the decision maker's aversion to bets on events for which probabilities are not specified, and they have been supported by experimental evidence.[4] Other than SOSEU there are alternative models that explain the Ellsbergian choices. The most popular have as domain the set of Anscombe-Aumann acts and were axiomatized by Schmeidler (1989), Gilboa and Schmeidler (1989), and Maccheroni,

[1] We use the terms "roulette lottery", "lottery", and "objective lottery" interchangeably.

[2] For alternative foundations of models of second-order beliefs that achieve a representation similar to (1) in the domain of acts, see Ergin and Gul (2009) and Nau (2006).

[3] In our framework (and Seo's) the order of integration is immaterial, and (1) can also be written as the SOSEU was originally stated.

[4] See Camerer (1995) for a survey of the experimental work testing Ellsberg's (1961) predictions.

Marinacci, and Rustichini (2006).[5] We refer to them as models of "first-order ambiguity", or "first-order beliefs". As opposed to the models of first-order ambiguity, the explanation of SOSEU to the same sort of behavior relies on the presence of a second-order belief and the non-reduction of compound lotteries as captured by the concavity of ϕ.

There are at least two major shortcomings of the current models of second-order beliefs. The first one was suggested by Epstein (2009).[6] He claims that the SOSEU model accounts for the typical Ellsbergian choices for Anscombe-Aumann acts, but has the non-intuitive implication of not accounting for counterfactual "second-order Ellsbergian choices". That is, if bets were available on the probabilities μ's of the states in S, then the SOSEU would fail to account for the "second-order" intuitive counterpart of the Ellsberg Paradox. As opposed to the derivation of a single second-order subjective belief like in the SOSEU model, one would expect second-order beliefs to be ambiguous.[7] The second problem in the literature is that it says little on how models of second-order beliefs relate to the canonical models of first-order beliefs. To the best of our knowledge, the only results along these lines are the intersection of SOSEU and SEU under reduction of compound lotteries (Seo (2009)), and the intersection of the smooth ambiguity model with maxmin preferences when ambiguity aversion is taken to infinite (Klibanoff et al. (2005)).

Motivated by those two problems, we develop a more general model of second-order beliefs that we name "Second-Order Variational Preference" (SOVP). Our model is a generalization of Seo (2009), and as a consequence a further generalization of the original Anscombe and Aumann (1963) model. The SOVP representation is completely described by a triple (u, ϕ, c), where u is a vN-M utility, ϕ is a real-valued function, and $c : \Delta(\Delta(S)) \to \mathbb{R}_+ \cup \{+\infty\}$ is a cost function on the set of all second-order beliefs. The decision maker ranks the lottery of

[5]Chateauneuf (1991) independently gave an axiomatization of maxmin preferences.

[6]Although his critique is focused on the smooth ambiguity model of preferences, Epstein's (2009) remarks also apply to the SOSEU model. See also the reply of Klibanoff, Marinacci, and Mukerji (2009).

[7]In fact, models of ambiguous second-order beliefs have already been used in applications: e.g., Hansen and Sargent (2007) and Ju and Miao (2009).

acts P according to

$$\min_{m\in\Delta(\Delta(S))}\left\{\int_{\Delta(\Delta(S))}\left[\int_{\mathcal{F}}\phi\left(\int_{S}u(f)\,d\mu\right)dP(f)\right]dm(\mu)+c(m)\right\}. \tag{2}$$

One possible interpretation of (2) is similar to the one Maccheroni et al. (2006) gave to their variational preferences.[8] The decision maker believes there is a true probability measure μ describing the odds of the states, and he will eventually reduce each possible act in the support of the lottery P according to that measure using expected utility. Given μ, P is viewed as a two-stage lottery: each act in the support of P is collapsed to the second-stage lottery $\sum_{s\in S}\mu(s)\,f(s)$. Nevertheless, the true law μ is unknown. The decision maker forms second-order beliefs and is pessimistic about the true model μ describing the environment. He behaves as if he is playing a zero-sum game against a "malevolent nature". While trying to minimize the decision maker's payoff by choosing m, the malevolent nature pays a penalty $c(m)$ for doing so.

In our representation, "second-order bets" are induced by objective lotteries of acts. Any such bet is identified with the vector of payoffs $\mu\mapsto\int_{\mathcal{F}}\phi\left(\int_S u(f)\,d\mu\right)dP(f)$, for some $P\in\Delta(\mathcal{F})$. Using the same argument as to why standard maxmin and variational preferences can explain the Ellsberg paradox, one can argue that the second-order variational preferences also explains a counterfactual second-order version of that paradox. In order to obtain the representation in (2) we modify Seo's axioms in two ways. We not only replace the independence axiom he imposes on the lotteries of acts by a weaker condition, but we also impose a stronger version of his dominance postulate.[9] Our stronger version of dominance is implied by his axioms when independence is satisfied, but need not hold if we drop independence.

[8]The interpretation of variational preferences given by Maccheroni et al. (2006) was originally suggested by Hansen and Sargent (2000) in the context of multiplier preferences.

[9]The motivation for relaxing the independence axiom is clear: there are plenty of evidence showing it is often violated (see, e.g., Camerer (1995) and Davis and Holt (1993, chapter 8)).

One important contribution of our representation is to connect the models of second-order beliefs with the canonical models of first-order ambiguity. In particular, we show that those two general classes of models intersect exactly when reduction of compound lotteries holds (see Figure 1). Such an intersection provides evidence for the generality of our representation, which encompasses SOSEU, and the canonical models of first-order ambiguity (in the subdomain of Anscombe-Aumann acts) as particular cases.

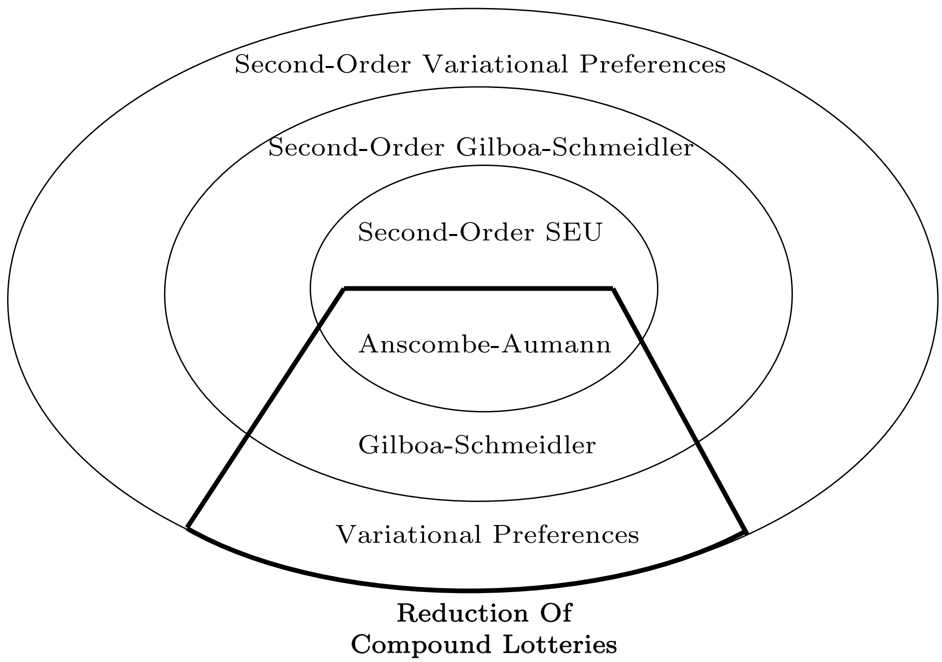

Figure 1: Intersection of first- and second-order models of ambiguity

A further generalization of the SOSEU model is achieved by relaxing the Order axiom of Seo (2009). After imposing a weaker version of completeness, we obtain a second-order version of Bewley's (1986) representation. In this case there exists a set M of second-order beliefs and, when restricted to the subdomain of Anscombe-Aumann acts, the decision maker weakly prefers f to g iff

$$\int_{\Delta(\Delta(S))} \phi\left(\int_S u(f)\, d\mu\right) dm(\mu) \geq \int_{\Delta(\Delta(S))} \phi\left(\int_S u(g)\, d\mu\right) dm(\mu) \text{ for all } m \in M.$$

The intersection of this class of Bewley preferences with the original (first-order) Bewley representation occurs under reduction of compound lotteries.

The paper is organized as follows. In section 2 we introduce the basic setup. Section 3 describes the main axioms. In section 4 we characterize SOVP, and provide a uniqueness result. Section 5 characterizes the second-order versions of maxmin and SEU preferences under our axiomatization, and shows the intersection between the classes of first- and second-order ambiguity models. In section 6 we introduce a second-order version of Bewley's (1986) representation. While section 7 concludes the paper with a discussion, the appendix contains the proofs of our main results.

2 Setup

The set X is a separable metric space, and $\Delta(X)$ stands for the set of all (Borel) probability measures on X. We endow $\Delta(X)$ with any metric that induces the topology of weak convergence. The state space S is assumed to be finite. An act is a mapping $f : S \to \Delta(X)$. Define $\mathcal{F} := \Delta(X)^S$, the set of all acts, and endow it with the product topology. Also define $\mathcal{F}_c := \{f \in \mathcal{F} : \exists\, p \in \Delta(X) \text{ s.t. } f(s) = p \text{ for all } s \in S\}$, the set of all constant acts. The domain of preferences is the set of all objective lotteries of acts $\Delta(\mathcal{F})$, which is endowed with any metric that induces the topology of weak convergence. Note that $\Delta(X)$, $\mathcal{F}$ and $\Delta(\mathcal{F})$ are separable metric spaces, and that $\Delta(\mathcal{F}_c)$ (the set of lotteries of constant acts) is a separable metric space as well. As a matter of notation, when Y is a metric space we denote by $\mathcal{B}(Y)$ the Borel σ-algebra on Y.

We identify an act $f \in \mathcal{F}$ with the degenerate lottery $\delta_f \in \Delta(\mathcal{F})$, and a constant act $f \in \mathcal{F}_c$ with the lottery $p \in \Delta(X)$ satisfying $f(s) = p$ for all $s \in S$. Therefore, we view $\Delta(X)$ as a subset of $\mathcal{F}$, and $\mathcal{F}$ as a subset of $\Delta(\mathcal{F})$. We write $\lambda f + (1-\lambda) g$ in place of $\lambda\delta_f + (1-\lambda)\delta_g$, and $\lambda f \oplus (1-\lambda) g$ in place of $\delta_{\lambda f+(1-\lambda)g}$. Given $\mu \in \Delta(S)$ and $f \in \mathcal{F}$, we denote by $\Psi(\mu, f)$ the (one-stage) lottery $\delta_{\int f d\mu}$.[10] In general, for any $P \in \Delta(\mathcal{F})$, we define $\Psi(\mu, P) \in \Delta(\mathcal{F}_c)$ so that $\Psi(\mu, P)(B) = P(\{f \in \mathcal{F} : \Psi(\mu, f) \in B\})$ for all $B \in \mathcal{B}(\mathcal{F}_c)$. For example, if $P = \lambda\delta_f + (1-\lambda)\delta_g$, then $\Psi(\mu, P)$ is the (two-stage) lottery that yields $\int f d\mu$ with probability λ, and $\int g d\mu$ with probability $1-\lambda$.

Our setup is the same as in Seo (2009). The original state space is S. Our axioms will induce a larger state space, which is $\Delta(S)$, the set of all probability measures on S. Note that S is embedded in $\Delta(S)$ by $s \mapsto \delta_{\{s\}}$. Having $\Delta(S)$ as an induced state space gives rise to second-order beliefs $m \in \Delta(\Delta(S))$ in a natural way. We endow the set $\Delta(\Delta(S))$ with the weak* topology, that is, a net (m_α) in $\Delta(\Delta(S))$ converges to m iff the net $\left(\int \eta dm_\alpha\right)$ converges to $\int \eta dm$ for all $\eta \in C(\Delta(S))$. One can also define a second-order act as a mapping $F : \Delta(S) \to \Delta(\Delta(X))$, which assigns to each possible belief a two-stage lottery on X.[11]

3 Axioms

We use the following set of axioms to characterize preferences.

Axiom A1 (Weak Order). *$\succcurlyeq$ is complete and transitive.*

Axiom A2 (Continuity). *If $(P_n), (Q_n) \in \Delta(\mathcal{F})^\infty$ are such that $P_n \succcurlyeq Q_n$ for all n, $P_n \to P \in \Delta(\mathcal{F})$, and $Q_n \to Q \in \Delta(\mathcal{F})$, then $P \succcurlyeq Q$.*

Axiom A3 (C-Dominance). *For all $P \in \Delta(\mathcal{F})$, and $P_c, Q_c \in \Delta(\mathcal{F}_c)$: (a) if $P_c \succcurlyeq \Psi(\mu, P)$ for all $\mu \in \Delta(S)$, then $P_c \succcurlyeq P$, and (b) if $\Psi(\mu, P) \succcurlyeq Q_c$ for all $\mu \in \Delta(S)$, then $P \succcurlyeq Q_c$.*

Axiom A4 (S-Dominance). *For all $P, Q \in \Delta(\mathcal{F})$, $P_c, Q_c \in \Delta(\mathcal{F}_c)$, and $\lambda \in [0,1]$: if $\lambda\Psi(\mu, P) + (1-\lambda) P_c \succcurlyeq \lambda\Psi(\mu, Q) + (1-\lambda) Q_c$ for all $\mu \in \Delta(S)$, then $\lambda\widetilde{P}_c + (1-\lambda) P_c \succcurlyeq$*

[10] By $\int f d\mu$ we mean $\sum_{s\in S} \mu(s) f(s)$.

[11] Two-stage lotteries on X have the interpretation of constant second-order acts.

$\lambda\widetilde{Q}_c + (1-\lambda) Q_c$ *for any* $\widetilde{P}_c, \widetilde{Q}_c \in \Delta(\mathcal{F}_c)$ *such that* $\widetilde{P}_c \succcurlyeq P$ *and* $Q \succcurlyeq \widetilde{Q}_c$.

Axiom A5 (Convexity). *For all* $P, Q \in \Delta(\mathcal{F})$, *and* $\lambda \in (0,1)$: *if* $P \sim Q$, *then* $\lambda P + (1-\lambda) Q \succcurlyeq Q$.

Axiom A6 (Partial First-Stage Independence). *For all* $P_c, Q_c, R_c \in \Delta(\mathcal{F}_c)$, *and* $\lambda \in (0,1)$: *if* $P_c \succcurlyeq Q_c$, *then* $\lambda P_c + (1-\lambda) R_c \succcurlyeq \lambda Q_c + (1-\lambda) R_c$.

Axiom A7 (Second-Stage Independence). *For all* $p, q, r \in \Delta(X)$, *and* $\lambda \in (0,1)$: *if* $p \succcurlyeq q$, *then* $\lambda p \oplus (1-\lambda) r \succcurlyeq \lambda q \oplus (1-\lambda) r$.

Axiom A8 (Nondegeneracy). $\succ \neq \emptyset$.[12]

Axioms A1, A2, and A8 are standard.[13] Second-Stage Independence requires the independence axiom to be satisfied in the subdomain of constant acts. This condition holds in all models of (first-order) ambiguity aversion and SOSEU. In order to explain the remaining postulates, we first need to introduce the following axioms that are used, in different combinations, to characterize the representations of Anscombe and Aumann (1963) and Seo (2009).

Axiom B1 (Dominance). *For all* $P, Q \in \Delta(\mathcal{F})$: *if* $\Psi(\mu, P) \succcurlyeq \Psi(\mu, Q)$ *for all* $\mu \in \Delta(S)$, *then* $P \succcurlyeq Q$.

Axiom B2 (AA-Dominance). *For all* $f, g \in \mathcal{F}$: *if* $f(s) \succcurlyeq g(s)$ for all $s \in S$, then $f \succcurlyeq g$.

Axiom B3 (Reversal of Order). *For all* $f, g \in \mathcal{F}$, *and* $\lambda \in [0,1]$: $\lambda f \oplus (1-\lambda) g \sim \lambda f + (1-\lambda) g$.

Axiom B4 (First Stage Independence). *For all* $P, Q, R \in \Delta(\mathcal{F})$, *and* $\lambda \in (0,1)$: *if*

[12] $\succ$ stands for the asymmetric part of $\succcurlyeq$.

[13] For the axiomatization of the SOVP, one could have replaced the Continuity axiom by the following weaker version: for all $P \in \Delta(\mathcal{F})$, the sets $\{P_c \in \Delta(\mathcal{F}_c) : P \succcurlyeq P_c\}$ and $\{P_c \in \Delta(\mathcal{F}_c) : P_c \succcurlyeq P\}$ are closed.

$P \succcurlyeq Q$, then $\lambda P + (1-\lambda) R \succcurlyeq \lambda Q + (1-\lambda) R$.

Lemma 4.1 of Seo (2009) shows that the Dominance axiom is implied by A1, A2, B2 and B3. It is easy to verify that First Stage Independence and Dominance imply C-Dominance and S-Dominance. The original axioms that characterize (nontrivial) SOSEU preferences are A1, A2, B1, B4, A7, and A8. The main contribution of our axioms is to relax B4. Seo (2009) drops the Reversal of Order postulate from the Anscombe-Aumann framework, but retains Dominance. Similarly, we weaken the First-Stage Independence axiom but retain C-Dominance and S-Dominance. We replace the First-Stage Independence axiom by Convexity and Partial First-Stage Independence.

The Convexity axiom has already been discussed in the literature, and was recently used to derive a dual version of maxmin preferences in the domain of lotteries (see Maccheroni (2002)). We interpret Convexity as a form of "second-order uncertainty aversion". The decision maker evaluates each lottery P knowing that he is going to associate each act f in its support to the second-stage lottery $\int f d\mu$, for some prior $\mu \in \Delta(S)$.[14] He is uncertain about the "correct" prior over the states, though. Because of that association, he transforms each lottery into a second-order act F_P, so that $F_P(\mu) = \Psi(\mu, P)$ for all $\mu \in \Delta(S)$. Note that the elements of $\Delta(\Delta(X))$ correspond to the second-order constant acts, since no second-order uncertainty w.r.t. μ matters to evaluate a lottery of constant acts. The indifference $P \sim Q$ is reflected in the indifference between F_P and F_Q. Now, given $\lambda \in (0,1)$, the same intuition for the uncertainty aversion postulate of Schmeidler (1989) applies to motivate the preference of $\lambda F_P + (1-\lambda) F_Q$ over F_Q, and hence the ranking $\lambda P + (1-\lambda) Q \succcurlyeq Q$. Partial First-Stage Independence restricts the subset of lotteries in which independence holds. That is, axiom A6 imposes independence in the subdomain of lotteries of constant acts. Because we can associate a constant second-order act to each element of $\Delta(\mathcal{F}_c)$, the motivation for the Partial First-Stage Independence axiom becomes clear: it is the same why in models of

[14]Note that the motivation for the collapsing of the acts in the support of P to $\int f d\mu$ is due to Seo (2009).

first-order ambiguity aversion the independence axiom is applied to constant acts only.

4 Representation

We now state our main result. It generalizes the representations of Anscombe and Aumann (1963) and Seo (2009).

Theorem 1. *The following are equivalent.*

(i) $\succcurlyeq$ *satisfies axioms A1-A8.*

(ii) There exists $W \in C_b(\Delta(\mathcal{F}))$ *such that, for all* $P, Q \in \Delta(\mathcal{F})$, $P \succcurlyeq Q$ *iff* $W(P) \geq W(Q)$, *where*

$$W(R) = \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dR(f) \right] dm(\mu) + c(m) \right\}, \tag{3}$$

$u \in C_b(\Delta(X))$ *is affine and nonconstant,* $\phi \in C_b(u(\Delta(X)))$ *is strictly increasing, and* $c : \Delta(\Delta(S)) \to \mathbb{R}_+ \cup \{+\infty\}$ *is a lower semicontinuous (l.s.c.), convex and grounded cost function.*[15] *In particular, the function* $U : \mathcal{F} \to \mathbb{R}$, *as defined by*

$$U(f) = \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \phi \left(\int_S u(f)\, d\mu \right) dm(\mu) + c(m) \right\},$$

represents $\succcurlyeq |_{\mathcal{F}}$. *Moreover, given* ϕ *and* u, *there exists a unique minimal cost function* $c^* : \Delta(\Delta(S)) \to \mathbb{R}_+ \cup \{+\infty\}$, *which is defined by*

$$c^*(m) = \sup_{R \in \Delta(\mathcal{F})} \left\{ W(R) - \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dR(f) \right] dm(\mu) \right\}.$$

It follows from Theorem 1 that a SOVP representation for $\succcurlyeq$ is completely characterized

[15] c is grounded if $\inf_{m \in \Delta(\Delta(S))} c(m) = 0$.

by a triple (u, ϕ, c). This representation is unique in the sense we establish next.[16]

Proposition 2. *Let* (u_0, ϕ_0, c_0) *and* (u_1, ϕ_1, c_1) *be two alternative SOVP representations of* $\succcurlyeq$. *Then there exist* $\left(\widetilde{\alpha}, \widetilde{\beta}\right), \left(\widehat{\alpha}, \widehat{\beta}\right) \in \mathbb{R}_{++} \times \mathbb{R}$ *such that* $u_0 = \widetilde{\alpha} u_1 + \widetilde{\beta}$, $\phi_0 \circ u_0 = \widehat{\alpha} \phi_1 \circ u_1 + \widehat{\beta}$, *and*

$$\min_{m \in \Delta(\Delta(S))} \left\{ \int \xi dm + c_0(m) \right\} = \min_{m \in \Delta(\Delta(S))} \left\{ \int \xi dm + \widehat{\alpha} c_1(m) \right\},$$

for all $\xi \in C(\Delta(S))$ *such that there exists* $P \in \Delta(\mathcal{F})$ *satisfying* $\xi(\mu) = \int \phi_0 \left(\int u_0(f) \, d\mu \right) dP(f)$ *for all* $\mu \in \Delta(S)$.

The representation in (3) is similar to the (first-order) variational preferences of Maccheroni et al. (2006) in the set of second-order util acts $\left\{ \mu \mapsto \int_{\mathcal{F}} \phi \left(\int_S u(f) \, d\mu \right) dR(f) : R \in \Delta(\mathcal{F}) \right\}$. The intuition stems from the discussion of the Convexity axiom in the previous section, so that the SOVP (and even SOSEU) comes at little surprise once the proper analogy with the canonical models of first-order beliefs is made. One should also note that the induced domain of second-order util acts is a "small" subset of $C(\Delta(S))$. In fact, the main difficulty with the uniqueness of SOSEU and SOVP comes from this observation. This is in contrast with the canonical variational preferences, where under plausible assumptions one can show a sharp uniqueness result for the cost function, and the SEU, where the uniqueness of the subjective probability is readily obtained.[17]

We note, in passing, that our axiomatization of SOVP is new. Our derivation of the SOVP model here follows a slightly different set of axioms than the original variational preferences in Maccheroni et al. (2006). It is based on the idea that one can replace the weakening of the independence axiom Maccheroni et al. (2006) proposed by independence on constant acts plus a stronger monotonicity condition.[18] In our framework, the analogue of the inde-

[16] Observe that Proposition 2 is equivalent to lemma C.1 of Seo (2009) in the subclass of SOSEU preferences.

[17] It is also in contrast with the literature on preferences over menus of Dekel, Lipman, and Rustichini (2001), Epstein, Marinacci, and Seo (2007), and Ergin and Sarver (2009). The set of menus is identified with a set of support functions. Such set is also relatively "small", but the set of second-order util acts $\Phi \ni 0$ in our setup (and Seo's) is even "smaller", in the sense that the set $\cup_{\alpha>0} \alpha\Phi - \cup_{\alpha>0} \alpha\Phi$ is not dense in $C(\Delta(S))$.

[18] This idea is developed further in Nascimento and Riella (2009).

pendence axiom for second-order acts is First-Stage Independence, while First-Stage Partial Stage Independence is equivalent to independence in the subdomain of constant second-order acts. S-Dominance and C-Dominance together can be regarded as a strengthening of the monotonicity axiom for second-order acts.

5 Relation to Other Classes of Preferences

5.1 Second-Order Beliefs

We say that $\succcurlyeq$ admits a second-order multiple priors representation (SOMEU) if there exists a nonempty, closed and convex set $M \subseteq \Delta(\Delta(S))$, an affine function $u \in C_b(\Delta(X))$, and a strictly increasing mapping $\phi \in C_b(u(\Delta(X)))$ such that the function

$$P \mapsto \min_{m \in M} \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dP(f) \right] dm(\mu)$$

represents $\succcurlyeq$. In particular, the restriction of $\succcurlyeq$ to $\mathcal{F}$ is represented by

$$f \mapsto \min_{m \in M} \int_{\Delta(S)} \phi \left(\int_S u(f)\, d\mu \right) dm(\mu),$$

which is similar to the maxmin preferences of Gilboa and Schmeidler (1989). Note that SOMEU preferences are completely characterized by the triple (u, ϕ, M). The next axiom is a strengthening of S-Dominance and characterizes SOMEU.

Axiom A9 (S-Dominance*). *For all $P, Q \in \Delta(\mathcal{F})$, $P_c, Q_c \in \Delta(\mathcal{F}_c)$, and $\lambda_1, \lambda_2 \in [0,1]$: if $\lambda_1 \Psi(\mu, P) + (1-\lambda_1) P_c \succcurlyeq \lambda_2 \Psi(\mu, Q) + (1-\lambda_2) Q_c$ for all $\mu \in \Delta(S)$, then $\lambda_1 \widetilde{P}_c + (1-\lambda_1) P_c \succcurlyeq \lambda_2 \widetilde{Q}_c + (1-\lambda_2) Q_c$ for any $\widetilde{P}_c, \widetilde{Q}_c \in \Delta(\mathcal{F}_c)$ such that $\widetilde{P}_c \succcurlyeq P$ and $Q \succcurlyeq \widetilde{Q}_c$.*

It is easy to verify that S-Dominance* is implied by Dominance and First-Stage Independence, and therefore holds in the framework of Seo (2009) (but need not hold after we relax First-Stage Independence). Theorem 3 is a particular case of our SOVP representation, but is still more general than the representations of Anscombe and Aumann (1963) and Seo (2009).

Theorem 3. *The following are equivalent.*

(i) $\succcurlyeq$ *satisfies A1-A3,A5-A8, and A9.*

(ii) $\succcurlyeq$ *admits a SOMEU representation.*

As a particular case of SOMEU, we obtain the SOSEU model of Seo (2009), which is characterized by the following further strengthening of the S-Dominance axiom.

Axiom A10 (S-Dominance).** *For all* $P, Q, R_1, R_2 \in \Delta(\mathcal{F})$, *and* $\lambda_1, \lambda_2 \in [0,1]$*: if* $\lambda_1 \Psi(\mu, P) + (1-\lambda_1)\Psi(\mu, R_1) \succcurlyeq \lambda_2 \Psi(\mu, Q) + (1-\lambda_2)\Psi(\mu, R_2)$ *for all* $\mu \in \Delta(S)$*, then* $\lambda_1 \widetilde{P}_c + (1-\lambda_1)\widetilde{R}_{1c} \succcurlyeq \lambda_2 \widetilde{Q}_c + (1-\lambda_2)\widetilde{R}_{2c}$ *for any* $\widetilde{P}_c, \widetilde{Q}_c, \widetilde{R}_{1c}, \widetilde{R}_{2c} \in \Delta(\mathcal{F}_c)$ *such that* $\widetilde{P}_c \succcurlyeq P$, $\widetilde{R}_{1c} \succcurlyeq R_1$, $Q \succcurlyeq \widetilde{Q}_c$, *and* $R_2 \succcurlyeq \widetilde{R}_{2c}$.

Theorem 4 derives the SOSEU representation of Seo (2009) using our general representation, and also shows that his axioms, which are listed in part (ii), are equivalent to ours.

Theorem 4. *The following are equivalent.*

(i) $\succcurlyeq$ *satisfies A1-A3,A5-A8, and A10.*

(ii) $\succcurlyeq$ *satisfies A1, A2, B1, B4, A7, and A8.*

(iii) $\succcurlyeq$ *admits a SOSEU representation.*

5.2 First-Order Beliefs

We say that $\succcurlyeq$ admits a (first-order) variational preference representation if the restriction of $\succcurlyeq$ to $\mathcal{F}$ can be represented by

$$f \mapsto \min_{\mu \in \Delta(S)} \left\{ \int \widetilde{u}(f)\, d\mu + \widetilde{c}(\mu) \right\}, \tag{4}$$

where $\widetilde{u} \in C_b(\Delta(X))$ is affine and nonconstant, and $\widetilde{c} : \Delta(S) \to \mathbb{R}_+ \cup \{+\infty\}$ is l.s.c., grounded and convex. Preferences that admit a representation as in (4) were axiomatized by Maccheroni et al. (2006) in the domain of Anscombe-Aumann acts.

The binary relation $\succcurlyeq$ admits a (first-order) maxmin representation if the restriction of $\succcurlyeq$ to $\mathcal{F}$ can be represented by

$$f \mapsto \min_{\mu \in \widetilde{M}} \left\{ \int \widetilde{u}(f)\, d\mu \right\}, \tag{5}$$

where $\widetilde{M}$ is a nonempty, closed, and convex subset of $\Delta(S)$. Gilboa and Schmeidler (1989) provided axiomatic foundations for preferences that admit a maxmin representation as in (5).

Finally, we say that $\succcurlyeq$ admits a (first-order) SEU representation if the restriction of $\succcurlyeq$ to $\mathcal{F}$ can be represented by

$$f \mapsto \int \widetilde{u}(f)\, d\mu,$$

where $\mu \in \Delta(S)$ (Anscombe and Aumann (1963)).

The following axiom characterizes the intersection of models of first- and second-order beliefs.

Axiom B5 (Reduction of Compound Lotteries – ROCL). *For all* $p, q \in \Delta(X)$, $\lambda \in [0,1] : \lambda p \oplus (1-\lambda) q \sim \lambda p + (1-\lambda) q$.

Seo (2009) argued that, if a SOSEU preference relation satisfies ROCL, then it is also SEU. As the next proposition shows, his observation can be generalized for the classes SOVP and SOMEU.[19]

Proposition 5. *If $\succcurlyeq$ is SOVP [SOMEU] and satisfies ROCL, then it is also variational [maxmin].*

6 Incomplete Preferences

Many have argued that the completeness assumption present in A1 is restrictive and unrealistic.[20] The completeness axiom that is used to characterize SOSEU can in fact be relaxed. A model of incomplete preferences that derives a set of second-order subjective probabilities in the same way as Bewley (1986) derived a set of first-order probabilities can be obtained. We drop A1 and use the following two axioms instead. They replace the completeness requirement in A1 by a weaker condition: completeness holds only in the subdomain of lotteries of constant acts. For notation, define $\succcurlyeq^{\bullet} := \succcurlyeq|_{\Delta(\mathcal{F}_c)}$.

Axiom I1 (Preference Relation). *The binary relation $\succcurlyeq$ is a preorder (i.e., reflexive and transitive).*

Axiom I2 (Partial Completeness). *The binary relation $\succcurlyeq^{\bullet}$ is complete.*

The next theorem establishes a second-order Bewley representation. We basically replace Axiom 1 of Seo (2009) by I1 and I2, and assume (as a useful technical condition) that the metric space X is compact.

Theorem 6. *Assume that X is a compact metric space. The following are equivalent.*

[19] Note that the converse of Proposition 5 also holds: that is, maxmin expected utility and variational preferences can be mapped into a model of second-order beliefs in an obvious way.

[20] See, for example, discussions in Aumann (1962), Bewley (1986), Schmeidler (1989), and Ok (2002).

(i) $\succcurlyeq$ *satisfies I1, I2, A2, B1, B4, A7, and A8.*

(ii) There exists a nonempty, closed and convex set $M \subseteq \Delta(\Delta(S))$*, such that, for all* $P, Q \in \Delta(\mathcal{F})$*,* $P \succcurlyeq Q$ *iff*

$$\int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dP(f) \right] dm(\mu) \geq \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi \left(\int_S u(f)\, d\mu \right) dQ(f) \right] dm(\mu)$$

$$\text{for all } m \in M, \quad (6)$$

$u \in C(\Delta(X))$ *is affine and nonconstant, and* $\phi \in C(u(\Delta(X)))$ *is strictly increasing.*

Note that each Second-Order Bewley representation is completely characterized by the triple (u, ϕ, M), which is unique in the sense we establish next.

Proposition 7. *If* (u_0, ϕ_0, M_0) *and* (u_1, ϕ_1, M_1) *are two Second-Order Bewley representations, then there exist* $\left(\widetilde{\alpha}, \widetilde{\beta}\right), \left(\widehat{\alpha}, \widehat{\beta}\right) \in \mathbb{R}_{++} \times \mathbb{R}$ *such that* $u_0 = \widetilde{\alpha} u_1 + \widetilde{\beta}$*,* $\phi_0 \circ u_0 = \widehat{\alpha} \phi_1 \circ u_1 + \widehat{\beta}$*, and for all* $m_i \in M_i$ *there exists* $m_j \in M_j$*,* $j \neq i$*, such that*

$$\int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_i \left(\int_S u_i(f)\, d\mu \right) dP(f) \right] dm_i(\mu) = \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_i \left(\int_S u_i(f)\, d\mu \right) dP(f) \right] dm_j(\mu), \quad (7)$$

for all $P \in \Delta(\mathcal{F})$*, and* $i = 0, 1$*.*

The original Bewley representation is characterized by an affine $\widetilde{u} \in C(\Delta(X))$ and a set of first-order priors $\widetilde{M}_{Bewley} \subseteq \Delta(S)$ such that the restriction of $\succcurlyeq$ to $\mathcal{F}$ admits the following representation: for all $f, g \in \mathcal{F}$,

$$f \succcurlyeq g \text{ iff } \int_S \widetilde{u}(f)\, d\mu \geq \int_S \widetilde{u}(g)\, d\mu \text{ for all } \mu \in \widetilde{M}_{Bewley}.$$

We refer to the pair $\left(\widetilde{u}, \widetilde{M}_{Bewley}\right)$ as a Bewley representation. Like in the other models of second-order ambiguity, we characterize the intersection of the class of second-order Bewley with the standard Bewley model.

Proposition 8. *If $\succcurlyeq$ is Second-Order Bewley and satisfies ROCL, then it has a Bewley representation.*

7 Discussion

Seo (2009) provided an alternative axiomatization of the Klibanoff et al. (2005) smooth ambiguity model of preferences in the original domain of Anscombe and Aumann (1963). The key idea there was the observation that in such domain it was possible to write axioms that used the set of all first-order beliefs as an implicit state space. Seo's Dominance axiom is simply the version of the standard monotonicity axiom w.r.t. this larger state space. Building on that idea, we axiomatized second-order beliefs versions of some of the canonical models used in the study of choice under uncertainty, and characterized their intersections with their respective first-order counterparts.[21]

Given our defense of our models of second-order ambiguous beliefs, a natural question is, "How seriously can we take such beliefs?" Our uniqueness results are far from supporting any positive answer to this kind of question, and the same observation applies to the SOSEU model of Seo (2009). Surely one could have derived similar versions of our models with better uniqueness properties by taking as primitive an enlarged state space like in Klibanoff et al. (2005) and imposing a suitable set of axioms. Nevertheless, our models still contribute to connect the original Anscombe and Aumann (1963) model to the more popular models of first-order ambiguity. Although maxmin and variational preferences work in an "Anscombe-Aumann framework", the question of axiomatizing those models in the original setup of Anscombe and Aumann (1963) has never been addressed.

There are two alternative axiomatizations of variational preferences in the setup of

[21] The Dominance axiom, or the stronger versions we study here, induce a domain of second-order acts. There exists a very close analogy between such a domain and the current models of first-order beliefs (and the techniques involved in their derivation). The set of second-order util acts is "small", though, and the techniques that we use to characterize our more general model are a mixture of the ones employed by Maccheroni et al. (2006) and Ergin and Sarver (2009).

Anscombe and Aumann (1963). One assumes that the decision maker is an expected utility maximizer w.r.t. the lotteries of acts. Once the objective uncertainty is resolved and an act is realized, he evaluates that act pessimistically according to the functional in (4). This is accomplished by imposing Weak Order, First-Stage Independence, Continuity, and ROCL, plus the standard axioms of Maccheroni et al. (2006) in the subdomain $\mathcal{F}$.[22] In the other, as pursued in this paper, we add ROCL on top of axioms A1-A8. The difference between the two axiomatizations is that in the second one the decision maker already reduces the subjective uncertainty of each possible act in the first stage by using a probability measure over states, even though he is unsure about the correct measure. In the first axiomatization he lets the uncertainty unfold "naturally": initially the objective uncertainty of the lottery of acts is resolved, and then the subjective uncertainty regarding the state takes place.

Finally, we note that one can extend the notion of comparative ambiguity aversion of Ghirardato and Marinacci (2002) to our models of second-order beliefs. Given any two preference relations $\succcurlyeq_0$ and $\succcurlyeq_1$ on $\Delta(\mathcal{F})$, we say that $\succcurlyeq_1$ is more second-order ambiguity averse than $\succcurlyeq_0$ if they induce the same preferences on $\Delta(\mathcal{F}_c)$ and, for all $(P_c, P) \in \Delta(\mathcal{F}_c) \times \Delta(\mathcal{F})$, $P_c \succcurlyeq_0 P$ implies $P_c \succcurlyeq_1 P$. We say that a preference relation $\succcurlyeq$ on $\Delta(\mathcal{F})$ is second-order ambiguity averse if it is more second-order ambiguity averse than some SOSEU preference relation. It is routine to show that SOMEU and SOVP are second-order ambiguity averse in the sense just defined.

A Proofs

A.1 Proof of Theorem 1

Necessity. The Weak Order and Nondegeneracy axioms are clearly satisfied. The function $f \mapsto \phi\left(\int_S u(f)\, d\mu\right)$ is continuous and bounded. Therefore, if $\Delta(\mathcal{F})^\infty \ni (P_n) \to P$, then

[22]In this representation, Dominance does not necessarily hold. In fact, Dominance would be satisfied only if preferences are SOSEU.

$\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP_n(f) \rightarrow \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$. An application of the Dominated Convergence Theorem (Aliprantis and Border (1999, p.407)) implies that $\mu \mapsto \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$ is continuous. Hence the function $(P, \mu) \mapsto \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f) =: \Lambda(\mu, P)$ is jointly continuous (and bounded), and by a second application of the Dominated Convergence Theorem one can show that $P \mapsto \int_{\Delta(S)} \Lambda(\mu, P)\, dm(\mu)$ is continuous. For all $P \in \Delta(\mathcal{F})$, there exists $m_P \in \Delta(\Delta(S))$ such that $W(P) = \int_{\Delta(S)} \Lambda(\mu, P)\, dm_P(\mu) + c(m_P)$. Hence, after rearranging two inequalities obtained from the definition of W, for all $P, Q \in \Delta(\mathcal{F})$ one has

$$\begin{aligned} |W(P) - W(Q)| &\leq \max_{R \in \{P,Q\}} \left| \int_{\Delta(S)} [\Lambda(\mu, P) - \Lambda(\mu, Q)]\, dm_R(\mu) \right| \\ &\leq \sup_{\mu \in \Delta(S)} |\Lambda(\mu, P) - \Lambda(\mu, Q)|, \end{aligned}$$

which yields continuity of W. Hence $\succcurlyeq$ satisfies Continuity.

Now we show that axioms A3, A4, and A5 are satisfied. For any $Q_c \in \Delta(\mathcal{F}_c)$, $W(Q_c) = \int_{\Delta(X)} \phi(u(p))\, dQ_c(p)$. For any $(P_c, P) \in \Delta(\mathcal{F}_c) \times \Delta(\mathcal{F})$, and $\mu \in \Delta(S)$, $P_c \succcurlyeq \Psi(\mu, P)$ iff $\int_{\Delta(X)} \phi(u(p))\, dP_c(p) \geq \int_{\Delta(X)} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$. Take any $m_0 \in \Delta(\Delta(S))$. If the last inequality holds for all $\mu \in \Delta(S)$, then

$$W(P_c) + c(m_0) \geq \int_{\Delta(S)} \left[\int_{\Delta(X)} \phi\left(\int_S u(f)\, d\mu \right) dP(f) \right] dm_0(\mu) + c(m_0) \geq W(P).$$

Because m_0 was arbitrary and c is grounded, we conclude that $W(P_c) \geq W(P)$. This establishes part (a) of C-Dominance. One can employ a similar argument to show that S-Dominance and part (b) of C-Dominance are satisfied. As for Convexity, if $\lambda \in [0,1]$ and $P \sim Q$, then

$$\begin{aligned} W(\lambda P + (1-\lambda) Q) &= \min_{m \in \Delta(\Delta(S))} \left\{ \lambda \int_{\Delta(S)} \Lambda(\mu, P)\, dm(\mu) + (1-\lambda) \int_{\Delta(S)} \Lambda(\mu, Q)\, dm(\mu) + c(m) \right\} \\ &\geq \lambda W(P) + (1-\lambda) W(Q) = W(Q). \end{aligned}$$

The argument for necessity of the Second-Stage Independence axiom is the same as in Seo (2009). Finally, the Partial First-Stage Independence axiom follows from $W|_{\Delta(\mathcal{F}_c)}$ being expected utility.

Sufficiency. We first show the existence of nontrivial cardinal representations for the restrictions of $\succcurlyeq$ to $\Delta(X)$ and $\Delta(\mathcal{F}_c)$. Using axioms A1, A2, and A7, it follows from the Expected Utility Theorem (Grandmont (1972, Theorem 2)) that there exists an affine function $u \in C_b(\Delta(X))$ that represents $\succcurlyeq|_{\Delta(X)}$. Another consequence of that same theorem (Grandmont (1972, Theorem 3)), given axioms A1, A2, and A6, is that there exists a function $w \in C_b(\Delta(X))$ such that the functional $\widetilde{W} : \Delta(\mathcal{F}_c) \to \mathbb{R}$, as defined by $\widetilde{W}(P_c) = \int w dP_c$, represents $\succcurlyeq|_{\Delta(\mathcal{F}_c)}$. Since u and w are both continuous representations of $\succcurlyeq|_{\Delta(X)}$, there exists a strictly increasing function $\phi \in C_b(u(\Delta(X)))$ such that $w = \phi \circ u$ (see lemma B.9 of Seo (2009)). Note that if $P_c \sim Q_c$ for all $P_c, Q_c \in \Delta(\mathcal{F}_c)$, then A3 would imply that $P \sim Q$ for all $P, Q \in \Delta(\mathcal{F})$. So, there must exist $P_c, Q_c \in \Delta(\mathcal{F}_c)$ such that $P_c \succ Q_c$. This implies that w and consequently u cannot be constant. Finally, because the function w is unique up to a positive affine transformation, it is w.l.o.g. to assume that $\sup_{p\in\Delta(X)} \phi(u(p)) = 1 > -1 = \inf_{p\in\Delta(X)} \phi(u(p))$. This implies that $0 \in int(\phi(u(\Delta(X))))$.

Claim A.1.1. *If $P \in \Delta(\mathcal{F})$, then there exists $P_c \in \Delta(\mathcal{F}_c)$ such that $P \sim P_c$.*

Proof of Claim A.1.1. Let $P \in \Delta(\mathcal{F})$. Define $\overline{\varphi} : \mathcal{F} \to \mathcal{F}_c$ so that

$$\overline{\varphi}(f) = \left\{ f(\overline{s}) \in f(S) : u(f(\overline{s})) = \max_{s\in S} u(f(s)) \text{ and } u(f(\overline{s})) > u(f(s)) \text{ for all } s < \overline{s} \right\}.$$

Note that $\overline{\varphi}$ is a well defined function. Since u is affine and represents $\succcurlyeq|_{\Delta(X)}$, $\overline{\varphi}(f) \succcurlyeq \int_S f d\mu$ for all $\mu \in \Delta(S)$. Fix any $s^* \in S$. If $B \subseteq \Delta(X)$ is open, then the set $B_o(s^*) := \{f \in \mathcal{F} : f(s^*) \in B\}$ is open. Moreover $B_{oo}(s^*) := \{f \in \mathcal{F} : u(f(s^*)) > u(f(s)) \text{ for all } s < s^* \}$ is always open, and $B_{ooo}(s^*) := \{f \in \mathcal{F} : u(f(s^*)) \geq u(f(s)) \text{ for all } s \in S \}$ is always closed. Therefore $\overline{\varphi}^{-1}(B) = \bigcup_{s^*=1}^{|S|} B_o(s^*) \cap B_{oo}(s^*) \cap B_{ooo}(s^*) \in \mathcal{B}(\mathcal{F})$, thus implying that $\overline{\varphi}$ is measurable. Define $\overline{P}_c \in \Delta(\mathcal{F}_c)$ such that $\overline{P}_c(B) = P(\{f \in \mathcal{F} : \overline{\varphi}(f) \in B\})$, and fix any

$\mu \in \Delta(S)$. The functions $f \mapsto \phi(u(\overline{\varphi}(f)))$ and $f \mapsto \phi\left(\int_S u(f)\, d\mu\right)$ are P-integrable, and $\phi(u(\overline{\varphi}(f))) - \phi\left(\int_S u(f)\, d\mu\right) \geq 0$ for all $f \in \mathcal{F}$. Therefore $\int_{\mathcal{F}} \phi(u(\overline{\varphi}(f)))\, dP(f) \geq \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$. Measurability of $\overline{\varphi}$ implies that the transformation $T : (\mathcal{F}_c, \mathcal{B}(\mathcal{F}_c)) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$ as defined by $T(B) = \{f \in \mathcal{F} : \overline{\varphi}(f) \in B\}$ is measurable. Because $P = \overline{P}_c T^{-1}$, it follows from the Change of Variables Theorem (Aliprantis and Border (1999, p.452)) that $\int_{\mathcal{F}} \phi(u(\overline{\varphi}(f)))\, dP(f) = \int_{\Delta(X)} \phi(u(\overline{\varphi}(T(p))))\, d\overline{P}_c(p) = \int_{\Delta(X)} \phi(u(p))\, d\overline{P}_c(p)$. Another application of the Change of Variables Theorem shows that $\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f) = \int_{\Delta(X)} \phi(u(p))\, d\Psi(\mu, P)$. Therefore $\overline{P}_c \succcurlyeq \Psi(\mu, P)$ for all $\mu \in \Delta(S)$. By the C-Dominance axiom, $\overline{P}_c \succcurlyeq P$. One can use a similar argument to construct $\underline{P}_c$ such that $P \succcurlyeq \underline{P}_c$. Hence the sets $\left\{\lambda \in [0,1] : \lambda \overline{P}_c + (1-\lambda)\underline{P}_c \succcurlyeq P\right\}$ and $\left\{\lambda \in [0,1] : P \succcurlyeq \lambda \overline{P}_c + (1-\lambda)\underline{P}_c\right\}$ are nonempty. Using axiom A2, one can show that they are also closed. Therefore, by A1 and standard arguments, they must have a nonempty intersection. We conclude that there exists $\lambda^* \in [0,1]$ such that $\Delta(\mathcal{F}_c) \ni \lambda^* \overline{P}_c + (1-\lambda^*)\underline{P}_c \sim P$. □

Define $W : \Delta(\mathcal{F}) \to \mathbb{R}$ so that $W(P) = \widetilde{W}(P_c)$ for any $P_c \in \Delta(\mathcal{F}_c)$ such that $P \sim P_c$. The existence of at least one such P_c is guaranteed by claim A.1.1. Since $\widetilde{W}$ represents $\succcurlyeq|_{\Delta(\mathcal{F}_c)}$, the function W is well-defined and agrees with $\widetilde{W}$ on $\Delta(\mathcal{F}_c)$. Define

$$\Phi := \left\{\xi \in C(\Delta(S)) : \exists\, P \in \Delta(\mathcal{F}) \text{ s.t. } \xi(\mu) = \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f) \text{ for all } \mu \in \Delta(S)\right\}. \tag{8}$$

Φ is convex and represents the set of all second-order util acts. We identify any $a \in \mathbb{R}$ with $a\mathbf{1}_{\Delta(S)} \in C(\Delta(S))$. Note that $\phi(u(\Delta(X))) \subseteq \Phi$.

Let $I : \Phi \to \mathbb{R}$ be such that $I(\xi) = W(P)$, for any $P \in \Delta(\mathcal{F})$ satisfying $\xi(\mu) = \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$ for all $\mu \in \Delta(S)$. The function I is well-defined. In fact, if ξ and ζ are associated, respectively, with the lotteries of acts P and Q, then $\xi = \zeta$ iff $\xi(\mu) = \zeta(\mu)$ for all $\mu \in \Delta(S)$ iff $\Psi(\mu, P) \sim \Psi(\mu, Q)$ for all $\mu \in \Delta(S)$, so that by the S-Dominance axiom (take $\lambda = 1$) and claim A.1.1 we obtain $P \sim Q$, and hence $I(\xi) = I(\zeta)$.

Claim A.1.2. *The function I satisfies the following properties.*

(i) I is normalized, that is, $I(a) = a$ for all $a \in \phi(u(\Delta(X)))$.

(ii) For all $\xi, \zeta \in \Phi$, $I(\xi) - I(\zeta) \leq \sup_{\mu \in \Delta(S)} [\xi(\mu) - \zeta(\mu)]$.

(iii) I is monotone ($\xi \geq \zeta$ implies $I(\xi) \geq I(\zeta)$), and satisfies $I(\xi + a) = I(\xi) + I(a) = I(\xi) + a$ whenever $\{\xi, a, \xi + a\} \subseteq \Phi$.

Proof of Claim A.1.2. If $\xi = a$ for some $a \in \phi(u(\Delta(X)))$, then $I(\xi) = W(\delta_p)$, where $p \in \Delta(X)$ satisfies $\phi(u(p)) = a$. By definition $W(\delta_p) = \phi(u(p)) = a$, and this shows part (i). Now let $\mu^* \in \Delta(S)$ be such that $\xi(\mu^*) - \zeta(\mu^*) = \max_{\mu \in \Delta(S)} [\xi(\mu) - \zeta(\mu)]$. Note that continuity of ξ and ζ, and compactness of $\Delta(S)$ ensure that the maximum is attained. Therefore $\frac{1}{2}\xi(\mu^*) + \frac{1}{2}\zeta(\mu) \geq \frac{1}{2}\xi(\mu) + \frac{1}{2}\zeta(\mu^*)$ for all $\mu \in \Delta(S)$. Let P and Q be the lotteries of acts associated with ξ and ζ respectively. That is, $\xi(\mu) = \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)$ and $\zeta(\mu) = \int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dQ(f)$ for all $\mu \in \Delta(S)$. Let $p, q \in \Delta(X)$ be such that $\xi(\mu^*) = \phi(u(p))$ and $\zeta(\mu^*) = \phi(u(q))$. Hence, for all $\mu \in \Delta(S)$,

$$\frac{1}{2}\phi(u(p)) + \frac{1}{2}\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dQ(f) \geq \frac{1}{2}\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f) + \frac{1}{2}\phi(u(q)),$$

which translates into

$$\frac{1}{2}\delta_p + \frac{1}{2}\Psi(\mu, Q) \succcurlyeq \frac{1}{2}\Psi(\mu, P) + \frac{1}{2}\delta_q,$$

for all $\mu \in \Delta(S)$. It follows from axiom A4 that $\frac{1}{2}\delta_p + \frac{1}{2}\widetilde{Q}_c \succcurlyeq \frac{1}{2}\widetilde{P}_c + \frac{1}{2}\delta_q$ for any $\widetilde{P}_c, \widetilde{Q}_c \in \Delta(\mathcal{F}_c)$ such that $P \sim \widetilde{P}_c$ and $Q \sim \widetilde{Q}_c$. Therefore we obtain $\frac{1}{2}I(\xi(\mu^*)) + \frac{1}{2}I(\zeta) \geq \frac{1}{2}I(\xi) + \frac{1}{2}I(\zeta(\mu^*))$. Because I is normalized, $I(\xi) - I(\zeta) \leq \xi(\mu^*) - \zeta(\mu^*) = \sup_{\mu \in \Delta(S)} [\xi(\mu) - \zeta(\mu)]$. This shows (ii). Finally, note that (iii) is an immediate consequence of (i) and (ii). □

We now prove that I is concave in Φ. Define the set

$$\Phi_0 := \{\xi \in \Phi : \exists\, a > 0 \text{ s.t. } \xi + a \in \Phi\}.$$

It follows that $a \in \Phi$ for any $a \in (-1, 1)$, and Φ_0 is nonempty.

Claim A.1.3. *Φ_0 is convex and dense in Φ.*

Proof of Claim A.1.3. If $\xi, \zeta \in \Phi_0$, $\lambda \in (0,1)$, then let $a_\xi, a_\zeta > 0$ be such that $\{\xi + a_\xi, \zeta + a_\zeta\} \subseteq \Phi$. Because Φ is convex, $\lambda\xi + (1-\lambda)\zeta + (\lambda a_\xi + (1-\lambda) a_\zeta) \in \Phi$, which implies that $\lambda\xi + (1-\lambda)\zeta \in \Phi_0$. Therefore Φ_0 is convex. Now let $\xi \in \Phi$. Note that $\left\{\lambda\xi + (1-\lambda)\frac{1}{2}, \lambda\xi\right\} \subseteq \Phi$ for all $\lambda \in (0,1)$. Hence $\lambda\xi \in \Phi_0$ for all $\lambda \in (0,1)$, and we conclude that Φ_0 is dense in Φ. □

Claim A.1.4. *$I|_{\Phi_0}$ is concave.*

Proof of Claim A.1.4. For any pair $\xi, \zeta \in \Phi_0$, $\xi \neq \zeta$, define $\widetilde{I}_{\xi,\zeta} : [0,1] \to \mathbb{R}$ so that $\widetilde{I}_{\xi,\zeta}(\alpha) = I(\alpha\xi + (1-\alpha)\zeta)$. Continuity of I implies that $\widetilde{I}_{\xi,\zeta}$ is (uniformly) continuous. There exists $\delta > 0$ such that, for any $a \in (0,\delta)$ and $\alpha \in [0,1]$, $\alpha\xi + (1-\alpha)\zeta + a \in \Phi_0$ (e.g. $\delta = \min\{b_\xi, b_\zeta\}$ for any $b_\xi, b_\zeta > 0$ such that $\{\xi + b_\xi, \zeta + b_\zeta\} \subseteq \Phi$). Fix any $\alpha^* \in (0,1)$ and let $\varepsilon > 0$ be such that $\alpha^* - \varepsilon > 0$, $\alpha^* + \varepsilon < 1$ and $\left|\widetilde{I}_{\xi,\zeta}(\alpha_1) - \widetilde{I}_{\xi,\zeta}(\alpha_2)\right| < \delta$ for all $\alpha_1, \alpha_2 \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$. Suppose w.l.o.g. that $0 \leq \widetilde{I}_{\xi,\zeta}(\alpha_1) - \widetilde{I}_{\xi,\zeta}(\alpha_2) =: d$. For any $\alpha \in [0,1]$ let $\sigma_\alpha := \alpha\xi + (1-\alpha)\zeta$. By the properties of I we know that $I(\sigma_{\alpha_1}) = I(\sigma_{\alpha_2} + d)$. By axiom A5, this implies that, for any $\alpha \in (0,1)$,

$$I(\alpha\sigma_{\alpha_1} + (1-\alpha)(\sigma_{\alpha_2} + d)) \geq \alpha I(\sigma_{\alpha_1}) + (1-\alpha) I(\sigma_{\alpha_2} + d).$$

If follows from part (iii) of claim A.1.2 that $I(\alpha\sigma_{\alpha_1} + (1-\alpha)\sigma_{\alpha_2}) \geq \alpha I(\sigma_{\alpha_1}) + (1-\alpha) I(\sigma_{\alpha_2})$ or, equivalently, $\widetilde{I}_{\xi,\zeta}(\alpha\alpha_1 + (1-\alpha)\alpha_2) \geq \alpha\widetilde{I}_{\xi,\zeta}(\alpha_1) + (1-\alpha)\widetilde{I}_{\xi,\zeta}(\alpha_2)$. That is, $\widetilde{I}_{\xi,\zeta}$ is locally concave in $(0,1)$. As a consequence of a well-known result in convex analysis (see Hormander (2007, p.58)) and continuity of $\widetilde{I}_{\xi,\zeta}$, the function $\widetilde{I}_{\xi,\zeta}|_{(0,1)}$ is concave. Using continuity of $\widetilde{I}_{\xi,\zeta}$ once again one can easily show that $\widetilde{I}_{\xi,\zeta}$ is concave. Finally, note that concavity of $\widetilde{I}_{\xi,\zeta}$ for any pair $(\xi,\zeta) \in (\Phi_0)^2$ implies that $I|_{\Phi_0}$ is concave (e.g., Papadopoulos (2005, p.169)). □

If follows from claims A.1.3 and A.1.4, and continuity of I, that I is also concave. We say that a functional $\widehat{I}: C(\Delta(S)) \to \mathbb{R}$ is a niveloid if $\widehat{I}(\xi) - \widehat{I}(\zeta) \leq \sup_{\mu\in\Delta(S)} [\xi(\mu) - \zeta(\mu)]$ for all $\xi, \zeta \in C(\Delta(S))$. Note that any niveloid is monotone and vertically invariant ($\widehat{I}(\xi + a) = \widehat{I}(\xi) + a$ for all $a \in \mathbb{R}$). Dolecki and Greco (1995) call a niveloid a function $J: \overline{\mathbb{R}}^{\Delta(S)} \to \overline{\mathbb{R}}$ that is monotone and vertically invariant. Our abuse of notation of this term is explained in the working paper version of Maccheroni et al. (2006). The remaining of the proof is a direct extension of their results to our framework.

Define $\widehat{I}: C(\Delta(S)) \to \mathbb{R}$ such that $\widehat{I}(\xi) = \sup_{\xi'\in\Phi} \left\{ I(\xi') + \inf_{\mu\in\Delta(S)} [\xi(\mu) - \xi'(\mu)] \right\}$ for all $\xi \in C(\Delta(S))$.

Claim A.1.5. *$\widehat{I}$ is the least niveloid that extends I to $C(\Delta(S))$. Moreover, $\widehat{I}$ is concave and normalized.*

Proof of Claim A.1.5. The function $\widehat{I}$ is defined as in Dolecki and Greco (1995), who show that $\widehat{I}$ is in fact the least niveloidal extension to all of $\overline{\mathbb{R}}^{\Delta(S)}$. It should also be the least niveloid that extends I to $C(\Delta(S)) \subseteq \overline{\mathbb{R}}^{\Delta(S)}$. At the same time $\widehat{I}$ is also normalized. In fact, for all $a \in \mathbb{R}$, $\widehat{I}(a) = \sup_{\xi'\in\Phi} \left\{ I(\xi') + \inf_{\mu\in\Delta(S)} [a - \xi'(\mu)] \right\} = a + \sup_{\xi'\in\Phi} \left\{ I(\xi') - \sup_{\mu\in\Delta(S)} \xi'(\mu) \right\} = a$, where the last equality follows from the fact $0 \in \Phi$, as this implies $I(\xi') - \sup_{\mu\in\Delta(S)} \xi'(\mu) \leq 0$ for all $\xi' \in \Phi$ and $\sup_{\xi'\in\Phi} \left\{ I(\xi') - \sup_{\mu\in\Delta(S)} \xi'(\mu) \right\} \geq I(0) = 0$. To show that $\widehat{I}$ is concave, let $\xi, \zeta \in C(\Delta(S))$, and $\lambda \in (0,1)$. By definition

$$\widehat{I}(\lambda\xi + (1-\lambda)\zeta) = \sup_{\sigma'\in\Phi} \left\{ I(\sigma') + \inf_{\mu\in\Delta(S)} [\lambda\xi(\mu) + (1-\lambda)\zeta(\mu) - \sigma'(\mu)] \right\}.$$

Let $(\xi'_n), (\zeta'_n) \in \Phi^\infty$ be such that $I(\xi'_n) + \inf_{\mu\in\Delta(S)} [\xi(\mu) - \xi'_n(\mu)] \to \widehat{I}(\xi_n)$ and $I(\zeta'_n) + \inf_{\mu\in\Delta(S)} [\zeta(\mu) - \zeta'_n(\mu)] \to \widehat{I}(\zeta_n)$. Therefore

$$\begin{aligned}\widehat{I}(\lambda\xi + (1-\lambda)\zeta) &\geq I(\lambda\xi'_n + (1-\lambda)\zeta'_n) + \inf_{\mu\in\Delta(S)} [\lambda(\xi(\mu) - \xi'_n(\mu)) + (1-\lambda)(\zeta(\mu) - \zeta'_n(\mu))] \\ &\geq \lambda \left\{ I(\xi'_n) + \inf_{\mu\in\Delta(S)} [\xi(\mu) - \xi'_n(\mu)] \right\} + (1-\lambda) \left\{ I(\zeta'_n) + \inf_{\mu\in\Delta(S)} [\zeta(\mu) - \zeta'_n(\mu)] \right\},\end{aligned}$$

and $\widehat{I}(\lambda\xi + (1-\lambda)\zeta) \geq \lambda\widehat{I}(\xi) + (1-\lambda)\widehat{I}(\zeta)$ after taking limit as $n \to \infty$. $\square$

The norm dual of $C(\Delta(S))$ is isometrically isomorphic to the space of Borel measures on $\Delta(S)$, $ca(\Delta(S))$. This fact plus the Fenchel–Moreau theorem imply the next claim.

Claim A.1.6. *For all $\xi \in C(\Delta(S))$, $\widehat{I}(\xi) = \inf_{m\in ca(\Delta(S))}\left\{\int \xi dm - \widehat{I}^*(m)\right\}$, where $\widehat{I}^*(m) = \inf_{\xi\in C(\Delta(S))}\left\{\int \xi dm - \widehat{I}(\xi)\right\}$.*

Let $\widetilde{c} := -\widehat{I}^*$. Note that $\widetilde{c}$ is l.s.c. and convex. Moreover $0 = \widehat{I}(0) = \inf_{m\in ca(\Delta(S))}\{\widetilde{c}(m)\}$. Hence $\widetilde{c}$ takes values on $\mathbb{R}_+ \cup \{+\infty\}$ and is grounded. The proof of Theorem 1 is complete once we show that

$$\inf_{m\in ca(\Delta(S))}\left\{\int \xi dm + \widetilde{c}(m)\right\} = \min_{m\in\Delta(\Delta(S))}\left\{\int \xi dm + c(m)\right\}, \tag{9}$$

where $c = \widetilde{c}|_{\Delta(\Delta(S))}$. If $m_0 \in ca(\Delta(S))$ does not induce a positive linear functional in the norm dual of $C(\Delta(S))$, then there exists $\zeta \in C(\Delta(S))$ satisfying $\zeta \geq 0$ and $\int \zeta dm_0 < 0$. Therefore $\widetilde{c}(m_0) = \sup_{\xi\in C(\Delta(S))}\left\{\widehat{I}(\xi) - \int \xi dm_0\right\} \geq \widehat{I}(n\zeta) - n\int \zeta dm_0 \geq n\left|\int \zeta dm\right|$ for all $n \in \mathbb{N}$. As a consequence, $\widetilde{c}(m_0) = +\infty$ if m_0 is not a positive measure. In fact, if $\widetilde{c}(m_0) < \infty$, then for all $n \in \mathbb{N}$

$$\begin{aligned} n = \widehat{I}(n) &= \inf_{m\in ca(\Delta(S))}\left[n\int \mathbf{1}_{\Delta(S)}dm + \widetilde{c}(m)\right] \leq n \times m_0(\Delta(S)) + \widetilde{c}(m_0), \\ n = -\widehat{I}(-n) &= \sup_{m\in ca(\Delta(S))}\left[n\int \mathbf{1}_{\Delta(S)}dm - \widetilde{c}(m)\right] \geq n \times m_0(\Delta(S)) - \widetilde{c}(m_0), \end{aligned}$$

so that $m_0(\Delta(S)) = 1$ after taking the limit as $n \to \infty$.

It follows from a standard result in convex analysis (e.g., Ekeland and Turnbull (1983, proposition 2,p.112)) plus claim A.1.6 and (9) that the superdifferential of $\widehat{I}$ satisfies $\partial\widehat{I}(\xi) = \partial I(\xi) = \arg\min_{m\in\Delta(\Delta(S))}\left\{\int \xi dm + c(m)\right\} \neq \emptyset$ for all $\xi \in \Phi$. Now define $c^* : \Delta(\Delta(S)) \to \mathbb{R}_+ \cup \{+\infty\}$ so that $c^*(m) = \sup_{\xi\in\Phi}\left\{I(\xi) - \int \xi dm\right\}$ for all $m \in \Delta(\Delta(S))$. Using the definition of c^*, if c^{**} is any other cost function satisfying (9), then $c^* \leq c^{**}$. Finally, we observe that, given $\xi \in \Phi$, for any fixed $m \in \partial I(\xi)$ we obtain $I(\xi) \geq \sup_{\zeta \in \Phi} \left[I(\zeta) + \int (\xi - \zeta)\, dm \right] \geq \min_{m \in \Delta(S)} \left[\int \xi dm + c^*(m) \right] \geq I(\xi)$, so that (9) is satisfied for any $\xi \in \Phi$ when we replace c by c^*.

A.2 Proof of Proposition 2

Because u_0 and u_1 are two affine representation of $\succcurlyeq|_{\Delta(X)}$, it follows from cardinal uniqueness that there exists $\left(\widetilde{\alpha}, \widetilde{\beta}\right) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $u_0 = \widetilde{\alpha} u_1 + \widetilde{\beta}$. Similarly, because $\phi_0(u_0)$ and $\phi_1(u_1)$ induce two affine representations of $\succcurlyeq|_{\Delta(\mathcal{F}_c)}$ by integration, it follows from the same cardinal uniqueness argument that there exists $\left(\widehat{\alpha}, \widehat{\beta}\right) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $\phi_0 \circ u_0 = \widehat{\alpha} \phi_1 \circ u_1 + \widehat{\beta}$. Now fix any $P \in \Delta(\mathcal{F})$. Using the same notation of the proof of Theorem 1, the triples (u_0, ϕ_0, c_0) and (u_1, ϕ_1, c_1) induce two representations W_0 and W_1 of $\succcurlyeq$, and two cardinal representations of $\succcurlyeq|_{\Delta(\mathcal{F}_c)}$, $\widetilde{W}_0$ and $\widetilde{W}_1$, satisfying $\widetilde{W}_i(P_c) = \int_{\Delta(X)} \phi_i(u_i(p))\, dP_c(p)$ for all $P_c \in \Delta(\mathcal{F}_c)$, and $W_i|_{\Delta(\mathcal{F}_c)} = \widetilde{W}_i$, $i = 0, 1$. Using the existence of $P_c \in \Delta(\mathcal{F}_c)$ such that $P \sim P_c$ as established by claim A.1.1, we obtain

$$\begin{aligned}
W_0(P) &= \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_0 \left(\int_S u_0(f)\, d\mu \right) dP(f) \right] dm(\mu) + c_0(m) \right\} \\
&= \widetilde{W}_0(P_c) \\
&= \widehat{\alpha} \widetilde{W}_1(P_c) + \widehat{\beta} \\
&= \widehat{\alpha} W_1(P) + \widehat{\beta} \\
&= \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left[\int_{\mathcal{F}} \left[\widehat{\alpha} \phi_1 \left(\int_S u_1(f)\, d\mu \right) + \widehat{\beta} \right] dP(f) \right] dm(\mu) + \widehat{\alpha} c_1(m) \right\} \\
&= \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_0 \left(\int_S u_0(f)\, d\mu \right) dP(f) \right] dm(\mu) + \widehat{\alpha} c_1(m) \right\}.
\end{aligned}$$

A.3 Proof of Theorem 3

The proof that the axioms are necessary is similar to the necessity of the axioms in Theorem 1, and thus omitted. We now prove (i)⇒(ii). Clearly the axioms for SOMEU imply the axioms for SOVP. Therefore assume that $\succcurlyeq$ has a SOVP representation W induced by (u, ϕ, c). Define $M := \{m \in \Delta(\Delta(S)) : c(m) = 0\}$, which is nonempty, closed and convex. We show that $\succcurlyeq$ has a SOMEU representation with set of priors M. Fix any $P \in \Delta(\mathcal{F})$, and let $P_c \in \Delta(\mathcal{F}_c)$ and $\lambda \in (0,1)$. Define $\Delta(\mathcal{F}) \ni P_\lambda := \lambda P + (1-\lambda) P_c$. Therefore $\Psi(\mu, P_\lambda) = \lambda \Psi(\mu, P) + (1-\lambda)\Psi(\mu, P_c)$. By S-Dominance*, for any $\widetilde{P}_c \in \Delta(\mathcal{F}_c)$ that is indifferent to P we obtain

$$W(P_\lambda) = \lambda W\left(\widetilde{P}_c\right) + (1-\lambda) W(P_c) = \lambda W(P) + (1-\lambda) W(P_c). \tag{10}$$

Now let $m^* \in \Delta(\Delta(S))$ be such that

$$W(P_\lambda) = \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP_\lambda(f)\right] dm^* + c(m^*).$$

Note that

$$\begin{aligned} W(P_\lambda) &= \lambda \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)\right] dm^* + (1-\lambda) W(P_c) + c(m^*) \\ &\geq \lambda \left\{\int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)\right] dm^* + c(m^*)\right\} + (1-\lambda) W(P_c) \qquad (11) \\ &\geq \lambda W(P) + (1-\lambda) W(P_c). \qquad (12) \end{aligned}$$

Because of (10), the last two inequalities are satisfied with equality. This implies that $c(m^*) = 0$. Now using (11) and (12) we obtain

$$W(P) = \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)\right] dm^*.$$

Since P was arbitrary, we conclude that $\succcurlyeq$ admits a SOMEU representation with set of priors M.

A.4 Proof of Theorem 4

We show (i)⇒(iii)⇒(ii)⇒(i). The proof of (iii)⇒(ii) is contained in Seo (2009). It is clear that Seo's axioms imply ours, so that (ii)⇒(i) holds. Now we show (i)⇒(iii). The axioms for SOSEU imply the axioms for SOMEU. Hence assume that $\succcurlyeq$ has a SOMEU representation W induced by (u, ϕ, M). Because $\Delta(\mathcal{F})$ is separable, take $(P_n) \in \Delta(\mathcal{F})^\infty$ such that $cl(\{P_1, P_2, ...\}) = \Delta(\mathcal{F})$. Define $\Delta(\mathcal{F}) \ni P_\infty := \sum_{n=1}^\infty \frac{1}{2^n} P_n$, and for each $n \in \mathbb{N}$

$$P_{-n} := \left(1 - \frac{1}{2^n}\right)^{-1} \sum_{l \in \mathbb{N} \setminus \{n\}} \frac{1}{2^l} P_l \in \Delta(\mathcal{F}).$$

Note that, for each $n \in \mathbb{N}$, $P_\infty = \frac{1}{2^n} P_n + \left(1 - \frac{1}{2^n}\right) P_{-n}$. For all $\mu \in \Delta(S)$,

$$\Psi(\mu, P_\infty) = \frac{1}{2^n} \Psi(\mu, P_n) + \left(1 - \frac{1}{2^n}\right) \Psi(\mu, P_{-n}).$$

The S-Dominance** axiom now implies that

$$W(P_\infty) = \frac{1}{2^n} W(P_n) + \left(1 - \frac{1}{2^n}\right) W(P_{-n}). \tag{13}$$

Now let $m^* \in M$ be such that

$$W(P_\infty) = \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP_\infty(f)\right] dm^*(\mu),$$

and define $W_{m^*} : \Delta(\mathcal{F}) \to \mathbb{R}$ so that

$$W_{m^*}(P) = \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi\left(\int_S u(f)\, d\mu\right) dP(f)\right] dm^*(\mu).$$

Note that

$$\begin{aligned} W(P_\infty) &= W_{m^*}(P_\infty) \\ &= \frac{1}{2^n} W_{m^*}(P_n) + \left(1 - \frac{1}{2^n}\right) W_{m^*}(P_{-n}) \\ &\geq \frac{1}{2^n} W(P_n) + \left(1 - \frac{1}{2^n}\right) W(P_{-n}). \end{aligned} \tag{14}$$

If follows from (13) that (14) is satisfied with equality. By the SOMEU representation of $\succcurlyeq$, we have $W_{m^*}(P_n) \geq W(P_n)$ and $W_{m^*}(P_{-n}) \geq W(P_{-n})$. This plus the previous observation imply that $W_{m^*}(P_n) = W(P_n)$. Because W_{m^*} and W agree in a dense subset of $\Delta(\mathcal{F})$ and are both continuous, $\succcurlyeq$ admits a SOSEU representation with second-order prior m^*.

A.5 Proof of Proposition 5

If $\succcurlyeq$ has a SOVP representation and satisfies ROCL, then ϕ is affine. Define $\widetilde{u} := \phi \circ u$, and note that $\widetilde{u}$ is nonconstant, affine, and continuous. The function U, as given by

$$\begin{aligned} U(f) &= \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left(\int_S \phi(u(f))\, d\mu \right) dm(\mu) + c(m) \right\} \\ &= \min_{m \in \Delta(\Delta(S))} \left\{ \int_{\Delta(S)} \left(\int_S \widetilde{u}(f)\, d\mu \right) dm(\mu) + c(m) \right\}, \end{aligned}$$

represents $\succcurlyeq|_{\mathcal{F}}$. For all $\mu \in \Delta(S)$, let $A_\mu := \{m \in \Delta(\Delta(S)) : \mu = \mathbb{E}_m(\widetilde{\mu})\}$, where $\mathbb{E}_m(\widetilde{\mu})$ is a $|S|$-vector whose i-th component is $\int_{\Delta(S)} proj_i(\widetilde{\mu})\, dm(\widetilde{\mu})$. Note that $A_\mu \neq \emptyset$. Let $\widetilde{c} : \Delta(S) \to \mathbb{R}_+ \cup \{+\infty\}$ be such that $\widetilde{c}(\mu) = \min_{m \in A_\mu} c(m)$ for all $m \in \Delta(\Delta(S))$. Because c is l.s.c. and the set A_μ is closed (hence compact), the function $\widetilde{c}$ is well defined.

Claim A.5.1. *$\widetilde{c}$ is convex, l.s.c. and grounded.*

Proof of Claim A.5.1. Let $\lambda \in [0,1]$, and $\mu_1, \mu_2 \in \Delta(S)$. Also let $m_{\mu_1}, m_{\mu_2} \in \Delta(\Delta(S))$ be such that $\widetilde{c}(\mu_1) = c(m_{\mu_1})$ and $\widetilde{c}(\mu_2) = c(m_{\mu_2})$. Then, because $\mathbb{E}_{\lambda m_{\mu_1} + (1-\lambda) m_{\mu_2}}(\widetilde{\mu}) =$

$\lambda\mu_1 + (1-\lambda)\mu_2$ and c is convex, we obtain

$$\begin{aligned}\widetilde{c}(\lambda\mu_1 + (1-\lambda)\mu_2) &= \min\{c(m) : m \in \Delta(\Delta(S)) \text{ and } \lambda\mu_1 + (1-\lambda)\mu_2 = E_m(\widetilde{\mu})\} \\ &\leq c(\lambda m_{\mu_1} + (1-\lambda) m_{\mu_2}) \\ &\leq \lambda c(m_{\mu_1}) + (1-\lambda) c(m_{\mu_2}) = \lambda\widetilde{c}(\mu_1) + (1-\lambda)\widetilde{c}(\mu_2).\end{aligned}$$

Therefore $\widetilde{c}$ is convex. Note that $H : \Delta(S) \rightrightarrows \Delta(\Delta(S))$ as defined as $H(\mu) = A_\mu$ is an upper hemicontinuous and compact-valued correspondence. Lower semicontinuity of $\widetilde{c}$ now follows from a version of the Maximum Theorem (see Aliprantis and Border (1999, lemma 16.30)). Finally, because c is l.s.c. and grounded, and $\Delta(\Delta(S))$ is compact, there exists $m^* \in \Delta(\Delta(S))$ such that $c(m^*) = \inf_{m\in\Delta(\Delta(S))} c(m)$. Let $\mu^* := \mathbb{E}_{m^*}(\widetilde{\mu})$, and note that $\widetilde{c}(\mu^*) = c(m^*) = 0$. □

Define $\mu_m := \mathbb{E}_m(\widetilde{\mu})$. Note that any $m_f \in \arg\min_{m\in\Delta(\Delta(S))} \left\{\int_S \widetilde{u}(f)\, d\mu_m + c(m)\right\}$ also belongs to $\arg\min_{m\in A_{\mu_{m_f}}} c(m)$. Hence $U(f) = \int_S \widetilde{u}(f)\, d\mu_{m_f} + c(m_f) = \int_S \widetilde{u}(f)\, d\mu_{m_f} + \widetilde{c}(\mu_{m_f}) \geq \min_{\mu\in\Delta(S)} \left\{\int_S \widetilde{u}(f)\, d\mu + \widetilde{c}(\mu)\right\}$. If the inequality is strict, then there exists $\mu^* \in \Delta(S)$ such that $U(f) > \int_S \widetilde{u}(f)\, d\mu^* + \widetilde{c}(\mu^*) = \int_S \widetilde{u}(f)\, d\mu_{m^*} + c(m^*)$ for some $m^* \in A_{\mu^*}$, a contradiction. Therefore $U(f) = \min_{\mu\in\Delta(S)} \left\{\int_S \widetilde{u}(f)\, d\mu + \widetilde{c}(\mu)\right\}$.

Now assume that $\succcurlyeq$ has a SOMEU representation with an affine ϕ. Then, there exists a nonvoid closed and convex set $M \subseteq \Delta(\Delta(S))$ such that

$$\begin{aligned}U(f) &= \min_{m\in M} \int_{\Delta(S)} \left(\int_S \widetilde{u}(f)\, d\mu\right) dm(\mu) \\ &= \min_{m\in M} \int_{\Delta(S)} \int_S \widetilde{u}(f)\, d\mu_m.\end{aligned}$$

Define $\widetilde{M} := \{\mu \in \Delta(S) : A_\mu \cap M \neq \emptyset\}$.

Claim A.5.2. *$\widetilde{M}$ is nonempty, closed and convex.*

Proof of Claim A.5.2. The fact $\widetilde{M}$ is nonempty is a straightforward consequence of M being nonempty. Now let $\widetilde{M}^{\infty} \ni (\mu_n) \to \mu$, and $(m_n) \in M^{\infty}$ be such that $\mu_n = \mathbb{E}_{m_n}(\widetilde{\mu})$ for all n. Because M is compact, passing to a convergent subsequence if necessary, we note that $\mu = \lim_{n_k} \mu_{n_k} = \lim \mathbb{E}_{m_{n_k}}(\widetilde{\mu}) = \mathbb{E}_m(\widetilde{\mu})$ for some $m \in M$. Therefore $\mu \in \widetilde{M}$. Finally, let $\mu_1, \mu_2 \in \widetilde{M}$, and $\lambda \in [0,1]$. If $m_1, m_2 \in M$ are such that $\mu_{m_i} = \mu_i$, $i = 1,2$, then $\lambda\mu_1 + (1-\lambda)\mu_2 = \lambda\mathbb{E}_{m_1}(\widetilde{\mu}) + (1-\lambda)\mathbb{E}_{m_2}(\widetilde{\mu}) = \mathbb{E}_{\lambda m_1+(1-\lambda)m_2}(\widetilde{\mu})$. Because M is convex, $\lambda m_1 + (1-\lambda)m_2 \in M \cap A_{\lambda\mu_1+(1-\lambda)\mu_2}$, so that $\lambda\mu_1 + (1-\lambda)\mu_2 \in \widetilde{M}$. □

Now one can use claim A.5.2 and easily adapt the arguments used in the case of SOVP to show that $U(f) = \min_{\mu\in\widetilde{M}} \int_S \widetilde{u}(f)\, d\mu$.

A.6 Proof of Theorem 6

The proof of the direction (ii)⇒(i) is standard, and thus omitted. We only show (i)⇒(ii). In view of axioms I1 and I2, the same arguments used in the proof of Theorem 1 can be employed to obtain nontrivial cardinal representations for the restrictions of $\succcurlyeq$ to $\Delta(X)$ and $\Delta(\mathcal{F}_c)$. That is, there exist an affine function $u \in C(\Delta(X))$ that represents $\succcurlyeq|_{\Delta(X)}$, and a function $w \in C(\Delta(X))$ such that the functional $\widetilde{W} : \Delta(\mathcal{F}_c) \to \mathbb{R}$, as defined by $\widetilde{W}(P_c) = \int w dP_c$, represents $\succcurlyeq|_{\Delta(\mathcal{F}_c)}$. The set $\mathcal{F}$ is a compact metric space, and the preorder $\succcurlyeq$ is continuous and satisfies independence. If follows from the Expected Multi-Utility Theorem of Dubra, Maccheroni, and Ok (2004) that there exists a set $\mathcal{U} \subseteq C(\mathcal{F})$ such that for all $P, Q \in \Delta(\mathcal{F})$, $P \succcurlyeq Q$ iff $\int U dP \geq \int U dQ$ for all $U \in \mathcal{U}$. Assume w.l.o.g. that $\mathcal{U}$ only contains nonconstant functions. If follows from axiom A8 that $\mathcal{U} \neq \emptyset$. For each $U \in \mathcal{U}$, define $W_U : \Delta(\mathcal{F}) \to \mathbb{R}$ so that $W_U(P) = \int U dP$ for all $P \in \Delta(\mathcal{F})$.

Claim A.6.1. *If $U \in \mathcal{U}$, then $W_U|_{\Delta(\mathcal{F}_c)}$ represents $\succcurlyeq^{\bullet}$.*

Proof of Claim A.6.1. It suffices to show that if $P_c, Q_c \in \Delta(\mathcal{F}_c)$ satisfy $P_c \succ^{\bullet} Q_c$, then $W_U(P_c) > W_U(Q_c)$. Because X is a compact metric space and $\succcurlyeq^{\bullet}$ is a continuous and complete preorder, there exist $\overline{P}_c, \underline{P}_c \in \Delta(\mathcal{F}_c)$ such that $\overline{P}_c \succcurlyeq^{\bullet} P_c \succcurlyeq^{\bullet} \underline{P}_c$ for all $P_c \in \Delta(\mathcal{F}_c)$.

Clearly $\overline{P}_c \succ^\bullet \underline{P}_c$ for otherwise axiom A3 would imply that $\succ = \emptyset$, thus contradicting A8. Moreover, we have $W_U\left(\overline{P}_c\right) > W_U\left(\underline{P}_c\right)$, for otherwise W_U would be constant. It follows from standard arguments that for any $P_c \in \Delta(\mathcal{F}_c)$ there exists a unique $\lambda_{P_c} \in [0,1]$ such that $P_c \sim \lambda_{P_c}\overline{P}_c + (1-\lambda_{P_c})\underline{P}_c$. Note that, whenever $P_c, Q_c \in \Delta(\mathcal{F}_c)$ satisfy $P_c \succ^\bullet Q_c$, we must have $\lambda_{P_c} > \lambda_{Q_c}$. If, in addition, we have $W_U(P_c) = W_U(Q_c)$, then

$$\begin{aligned} W_U(P_c) &= \lambda_{P_c} W_U\left(\overline{P}_c\right) + (1-\lambda_{P_c}) W_U\left(\underline{P}_c\right) \\ &= \lambda_{Q_c} W_U\left(\overline{P}_c\right) + (1-\lambda_{Q_c}) W_U\left(\underline{P}_c\right) = W_U(Q_c), \end{aligned}$$

so that $(\lambda_{P_c} - \lambda_{Q_c})\left(W_U\left(\overline{P}_c\right) - W_U\left(\underline{P}_c\right)\right) = 0$. In view of $W_U\left(\overline{P}_c\right) > W_U\left(\underline{P}_c\right)$ this requires $\lambda_{P_c} = \lambda_{Q_c}$, a contradiction. □

For any $U \in \mathcal{U}$, $W_U|_{\Delta(\mathcal{F}_c)}$ and $\widetilde{W}$ are two affine and continuous representations of $\succsim^\bullet$. It follows from cardinal uniqueness that there exists $(\alpha_U, \beta_U) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $W_U|_{\Delta(\mathcal{F}_c)} = \alpha_U \widetilde{W} + \beta_U$. Each W_U induces a complete preorder on $\Delta(\mathcal{F})$ that satisfies all the assumptions of part (ii) of Theorem 4. Therefore, given any $U \in \mathcal{U}$, there exists a triple (u_U, ϕ_U, m_U) such that, for all $P, Q \in \Delta(\mathcal{F})$, $W_U(P) \geq W_U(Q)$ iff

$$\int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_U\left(\int_S u_U(f)\, d\mu\right) dP(f)\right] dm_U(\mu) \geq \int_{\Delta(S)} \left[\int_{\mathcal{F}} \phi_U\left(\int_S u_U(f)\, d\mu\right) dQ(f)\right] dm_U(\mu).$$

Note that we can normalize $u_U = u$, and $\phi_U \circ u_U = \phi \circ u$, so that the set $M := \overline{co}\{m_U : U \in \mathcal{U}\}$ yields the representation in (6).

A.7 Proof of Proposition 7

Because of axioms I1 and I2, the part that relates u_0 and u_1, and $\phi_0 \circ u_0$ and $\phi_1 \circ u_1$ follows from standard arguments for cardinal uniqueness. Now note that, for $i = 0, 1$, the set of utility functions

$$\mathcal{U}_i := \left\{ U \in C(\mathcal{F}) : \exists\, m \in M_i \text{ s.t. } U(f) = \int_{\Delta(S)} \phi \left(\int_S u(f)\, d\mu \right) dm(\mu) \text{ for all } f \in \mathcal{F} \right\}$$

induces an Expected Multi-Utility representation in $\Delta(\mathcal{F})$. That is, for a fixed i: for all $P, Q \in \Delta(\mathcal{F})$, $P \succcurlyeq Q$ iff $\int U dP \geq \int U dQ$ for all $U \in \mathcal{U}_i$. Therefore, alternatively a Second-Order Bewley Representation is characterized by a set $\mathcal{U} \subseteq C(\mathcal{F})$. Since $\mathcal{F}$ is a compact metric space, it follows from the uniqueness result of Dubra et al. (2004) that

$$cl_{\|\cdot\|_\infty} \left(cone\left(\mathcal{U}_0\right) + \{\theta \mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\} \right) = cl_{\|\cdot\|_\infty} \left(cone\left(\mathcal{U}_1\right) + \{\theta \mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\} \right), \tag{15}$$

where $cone\left(\mathcal{U}_i\right)$ stands for the smallest convex cone that contains $\mathcal{U}_i$, $i = 0, 1$.[23] The following claim, which characterizes the nonconstant functions in the LHS of (15), also holds when we replace $\mathcal{U}_0$ by $\mathcal{U}_1$.

Claim A.7.1. *If $U \in cl_{\|\cdot\|_\infty} \left(cone\left(\mathcal{U}_0\right) + \{\theta \mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\} \right)$ is nonconstant, then there exists $m^* \in M_0$, and $(\alpha^*, \beta^*) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $U(f) = \int_{\Delta(S)} \alpha^* \phi_0 \left(\int_S u_0(f)\, d\mu \right) dm^*(\mu) + \beta^*$.*

Proof of Claim A.7.1. If $U \in cone\left(\mathcal{U}_0\right) + \{\theta \mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\}$ is nonconstant, then by definition there exists $(m_l)_{l=1}^L$ in M_0, $(\alpha_l)_{l=1}^L \in \mathbb{R}_+^L \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that

$$U(f) = \sum_{l=1}^{L} \int_{\Delta(S)} \alpha_l \phi_0 \left(\int_S u_0(f)\, d\mu \right) dm_l(\mu) + \beta.$$

Define $\alpha := \sum_{l=1}^L \alpha_l$, and $\rho_l := \frac{\alpha_l}{\alpha}$, and note that $U(f) = \alpha \int_{\Delta(S)} \phi_0 \left(\int_S u_0(f)\, d\mu \right) dm(\mu) + \beta$, where $m = \sum_{l=1}^L \rho_l m_l \in M_0$. Now note that if $(U_n) \in cone\left(\mathcal{U}_0\right) + \{\theta \mathbf{1}_{\mathcal{F}} : \theta \in \mathbb{R}\}$, $U_n \to U$, and U is nonconstant, then there exists a subsequence of nonconstant functions (U_{n_k}) converging to U. Note that $U_{n_k}(f) = \alpha_{n_k} \int_{\Delta(S)} \phi_0 \left(\int_S u_0(f)\, d\mu \right) dm_{n_k}(\mu) + \beta_{n_k}$. It follows from axioms I1, I2, A2 and A6 that there exists $p, q \in \Delta(X)$ such that $p \succ q$, so that

[23] The symbol $\|\cdot\|_\infty$ stands for the sup norm.

$\phi_0(u_0(p)) > \phi_0(u_0(q))$. Because (U_{n_k}) also converges pointwise, we obtain $U_{n_k}(p) - U_{n_k}(q) = \alpha_{n_k}(\phi_0(u_0(p)) - \phi_0(u_0(q))) \to \alpha^*(\phi_0(u_0(p)) - \phi_0(u_0(q)))$, for some $\alpha^* \geq 0$, which is in fact strictly positive because U is nonconstant. One can now easily show that $\beta_{n_k} \to \beta^*$, for some $\beta^* \in \mathbb{R}$. Because M_0 is compact, passing to a subsequence if necessary, we obtain $m_{n_k} \to m^*$, for some $m^* \in M_0$. Using uniqueness of the limit, we conclude that $U(f) = \alpha^* \int_{\Delta(S)} \phi_0\left(\int_S u_0(f)\, d\mu\right) dm^*(\mu) + \beta^*$. □

It follows from claim A.7.1 and (15) that for any $m_0 \in M_0$, there exists $m_1 \in M_1$ such that

$$\alpha_0 \int_{\Delta(S)} \phi_0\left(\int_S u_0(f)\, d\mu\right) dm_0(\mu) + \beta_0 = \alpha_1 \int_{\Delta(S)} \phi_1\left(\int_S u_1(f)\, d\mu\right) dm_1(\mu) + \beta_1, \quad (16)$$

for all $f \in \mathcal{F}$. By cardinal uniqueness of $\phi_1 \circ u_1$, one can show that it is w.l.o.g. to replace $\phi_1 \circ u_1$ by $\phi_0 \circ u_0$ in (16). Using nontriviality of preferences and completeness on the subdomain $\Delta(\mathcal{F}_c)$, clearly $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. This shows (7).

A.8 Proof of Proposition 8

If $\succcurlyeq$ admits a Second-Order Bewley representation and satisfies ROCL, then ϕ is affine. Fix any $m \in M$. When ϕ is affine,

$$\int_{\Delta(S)} \phi\left(\int_S u(f)\, d\mu\right) dm(\mu) = \int_{\Delta(S)} \left(\int_S \widetilde{u}(f)\, d\mu\right) dm(\mu),$$

where $\widetilde{u} = \phi \circ u \in C(\Delta(X))$ is also affine. Using the notation of the proof of Proposition 5, define $\Delta(S) \ni \mu_m := \mathbb{E}_m(\widetilde{\mu})$ and note that the set $\widetilde{M} := \{\mu_m : m \in M\}$ induces a First-Order Bewley representation.

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