Homework on connection between SED and PDE

1. **(The generalized Ito Rule)** Consider a diffusion X_t and another process Y_t which is calculated as a smooth function $f(\cdot)$ on X_t and time:

$$dX_t = a(X_t)dt + s(X_t)dW_t, \qquad X_0 \text{ given}$$
$$Y_t = f(t, X_t),$$

where $a(\cdot)$, $s(\cdot)$ and $f(\cdot)$ are well-behaved functions, i.e. we can differentiate them a couple of times.

(a) Take a Taylor expansion of second order (as in the derivation for the Ito rule) for $f(t + \Delta t, X_{t+\Delta t})$ around the point (t, X_t). Note that here we are fixing a particular realization of the Brownian Motion W_t driving the process X_t and hence a fixed realization of the process X_t itself.

(b) Now, let's wave our hands: For very small Δt, the SDE for X_t above has its drift roughly fixed at $a(X_t)$ and its volatility roughly fixed at $s(X_t)$ over the short interval $[t, t+\Delta t]$, since we are dealing with continuous functions. So we can replace

$$\Delta X_t = a(X_t)\Delta t + s(X_t)\Delta W_t$$

in the Taylor expansion (If you're not happy with this sloppy argumentation, take Taylor expansions of $a(X_{t+h})$ around $a(X_t)$ for $0 < h \leq \Delta t$ to see that what we are missing is of second order.)

(c) Now, use the usual rules we have derived for stochastic calculus a la Ito:

$$(\Delta W_t)^2 \to \Delta t$$
$$\Delta W_t \Delta t \to 0$$

What is left of the Taylor expansion after you apply these rules?

You should get the following generalized version of the Ito Rule:

$$
\begin{aligned}
dY_t = df(X_t) &= \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 g(t, X_t)}{\partial x^2}(dX_t)^2 = \\
&= \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} a(X_t)dt + \\
&+ \frac{1}{2}\frac{\partial^2 g(t, X_t)}{\partial x^2}[s(X_t)]^2 dt + \frac{\partial g(t, X_t)}{\partial x} s(X_t)dW_t
\end{aligned}
$$

[Note that the Ito Rule of course also holds for Ito-processes that are *not* diffusions – however, we will very rarely deal with these processes, so I left this out here; the rule is just analogous.]

2. **(European Option)** Let us price a European call option. The underlying asset price X_t moves according to a geometric Brownian Motion:

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

where we set $\alpha = 0.3$ and $\sigma = 0.1$; a unit of time is equal to one year. The option price at $t = 0$ is $X_0 = 100$, and it expires in one year ($T = 1$) with a strike $K = 120$. A risk-neutral agent who discounts at a constant rate r values the option as follows:

$$V(x, t) = e^{-r(T-t)} E\left[(X_T - K)^+ | X_t\right]$$

(a) Write down the density of the random variable X_1.

(b) How could we compute an approximation to $(X_0, 0)$ using random draws from the random varible X_T? You do not have to compute these values, describing the scheme is enough.

(c) What is the probability that the option ends up "in the money", i.e. $Prob(X_1 \geq K|X_0)$?

(d) Now, take a grid over time (say $t = 0, 0.01, \ldots, 1$) and over prices (say $x = 10, 11, \ldots, 300$). Calculate the price $V(x, t)$ of the option using the closed form solution for the Black-Scholes formula (look it up on `wikipedia.org`) for each point on this grid.

(e) What would a recursive scheme to calculate $V(x, t)$ for fixed t – given we knew $V(x, t + \Delta t)$ for all values of x a step Δt into the future– look like? You don't have to code this, just describe the scheme in words to build some intuition.

(f) Approximate the t- and the x-derivative of this function using the following common-sensical scheme:

$$V_x(x_i,t) = \frac{V(x_i,t)-V(x_{i-1},t)}{\Delta x}$$
$$V_{xx}(x_i) = \frac{V_x(x_{i+1},t)-V_x(x_i,t)}{\Delta x} = \frac{V(x+\Delta x)-2V(x)+V(x-\Delta x)}{(\Delta x)^2}$$

where $\Delta x = 1$ is the grid size. Calculate terms in the PDE that we derived in class from these approximations and see if they really sum up approximately to zero as the PDE claims.

(g) Use the approximation scheme for second-order PDEs that is described in the file `TrinomTreePDE.pdf` to approximate the value function. How close is this to the true solution? Experiment with different bounding values for the lattice and with different grid sizes!

(h) Look at the value function in the region where $t \to T$ and $x \sim K$. Describe what happens here. Which effect is at work? What is $\partial V(x,t)/\partial t$ at the point $(x = K, t = T)$?

3. **(Connection to first-order PDE/deterministic motion)** Suppose an individual's human capital X_t follows the following deterministic process:

$$X_t = X_0 + \alpha t, \qquad X_0 \text{ given}$$

At time T, for some weird reason the model ends and the agent is rewarded with some payoff that is related to his human-capital level at this point. The value function is just the discounted value of this payoff:

$$V(x,t) = e^{-r(T-t)}g(X_T)$$

where $g(\cdot)$ is a given function.

(a) Using a pencil and a ruler, make a graph and find a closed-form solution for the value function for all $0 \le t \le T$ and for all x by finding out where the agent will end up if he is at a certain point x at a given t.

(b) Now, look at the problem from a PDE-perspective: Take the steps analogous to the ones we took in class to get a PDE for the value function. Check if the function you found in 3a indeed fulfills this PDE.

(c) Recall that the path that X_t follows was called a "characteristic curve" for first-order PDEs. Look again at the derivation of our second-order PDE for the European Option and convince yourself that the realizations of the path for the price of the underlying ($dX_t = \alpha X_t dt + \sigma X_t dW_t$) play the role of these characteristic curves. Why do we average over a bunch of "curves" in this case?

4. Consider a general random walk with drift

$$dX_t = \alpha dt + \sigma dW_t, \qquad X_0 = x$$

Let the value function be an expectation of where the process ends up at $T > 0$:

$$V(x,t) = e^{-r(T-t)} E\left[g(X_T) | X_t = x\right]$$

where $g(\cdot)$ is a given function.

(a) Write down the solution for $V(x,t)$ as an integral using the normal density of the hitting distribution and the function $g(\cdot)$.

(b) Derive the PDE that has to hold for $V(x,t)$; note that it has to be closely connected to the PDE we derived in class for the European Option.

(c) Check if your solution for V from 4a fulfills the PDE in 4b.