V31.0018: Statistics (Lab)
Matthias Kredler
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Solution for Homework 6

Suppose there are three meteorologists who have their different theories about the weather in June:

- Meteorologist 1: The probability that it rains on a given day in June is .25 (for all days, independent of what happened the days before).
- Meteorologist 2: The probability that it rains on a given day in June is 1/3 (for all days, independent of what happened the days before).
- Meteorologist 3: The probability that it rains on a given day in June is .4 (for all days, independent of what happened the days before).

Now, suppose we have actually observed that it has rained 11 out of the 30 days in June. Let us infer by the Maximum-Likelihood Principle which researcher has the best theory!

(a) What is the distribution of the random variable of interest here? Which parameters does it have for the respective theories?

(b) Set up the likelihood for each of the three theories (i.e. the probability that the actually observed outcome occurred given the parameters in the respective theory).

(c) Draw a graph of the likelihood versus the parameters. Which parameter yields the maximal likelihood? Which parameter would yield the maximal likelihood if we didn't constrain ourselves to the three parameters proposed by the meteorologists but, allowed for *any* parameter value between 0 and 1? (You can make a guess for this last part.)

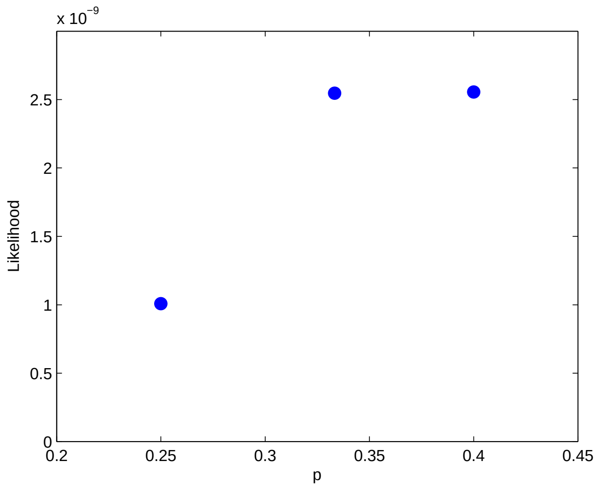
Solution:

(a) The random variable of interest here is the number of days that it rained in June, which we will denote by x. x follows a **binomial distribution** with parameters p and N, where p is the probability of success (i.e. rain in this case) and N ist the number of times the experiment is carried out (i.e. the number of days in June in this case: $N = 30$). Denote the different values of p suggested by the meteorologists as follows:

$$p_1 = .25$$
$$p_2 = 1/3$$
$$p_3 = .4$$

(b)

$$\begin{aligned}
\mathcal{L}(\text{data}, p_1) &= p_1^{11} \times (1 - p_1)^{30-11} = .25^{11} \times .75^{19} &&= 1.008 \times 10^{-9} \\
\mathcal{L}(\text{data}, p_2) &= p_2^{11} \times (1 - p_2)^{30-11} = (1/3)^{11} \times (2/3)^{19} &&= 2.546 \times 10^{-9} \\
\mathcal{L}(\text{data}, p_3) &= p_3^{11} \times (1 - p_3)^{30-11} = .4^{11} \times .6^{19} &&= 2.556 \times 10^{-9}
\end{aligned}$$

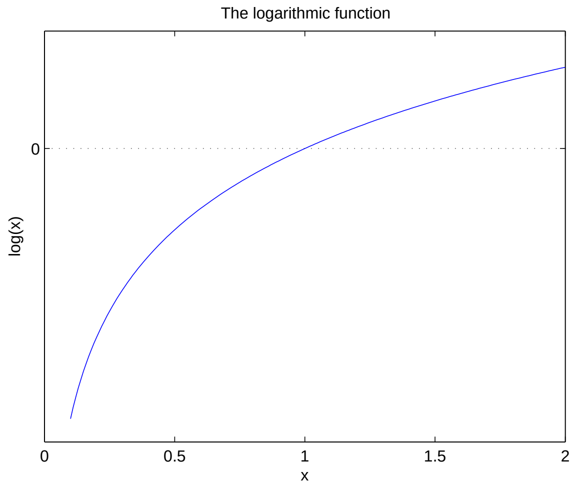


(c)

- $p_3 = .4$ yields the maximal likelihood.
- If we did not constrain ourselves to the given parameters, the Maximum-Likelihood Estimate of p would be $11/30 = .3667$, which coincides with the estimator that we studied in the lecture for p in the binomial distribution.

Remark: (for interested people — this is not relevant for the exam!)

- Notice that finding the maximal likelihood would have been a lot easier had we used the following trick: Instead of comparing the likelihood yielded by the three parameters, we could have compared the *logarithm of the likelihood*. Since the logarithmic function is increasing, the logarithm of the likelihood for p_3 is larger then the logarithm of the likelihood (we also say: the *log-likelihood*) for parameter p_2 – for example – if and only if the likelihood itself is larger for p_3 than for p_2. Hence, comparing the pure likelihood and the log-likelihood are



equivalent.

- We can see that this simplifies the calculations a lot:

$$log\mathcal{L}(\text{data}, p_1) = \log\left(p_1^{11} \times (1-p_1)^{30-11}\right) =$$

$$\log(.25^{11}) + \log(.75^{19}) = 11 \times \log(.25) + 19 \times \log(.75) = -20.7152$$

For the other parameters we get:

$$log\mathcal{L}(\text{data}, p_2) = -19.7886$$

$$log\mathcal{L}(\text{data}, p_3) = -19.7849$$

Of course we get the same result as before: p_3 yields the maximal likelihood among the three parameters.

- If you are familiar with derivatives, then you will be able to follow the derivation of the Maximum-Likelihood Estimator for the case where we don't constrain ourselves to the three given values for p, but consider any value of p between 0 and 1:

- First, write the log-likelihood for any parameter p given our data:

  $$log\mathcal{L}(\text{data}, p) = \log\left(p^{11} \times (1-p)^{30-11}\right) = 11\times\log(p)+19\times\log(1-p)$$

- You may have studied in a maths course that at the maximum of a function (here our log-likelihood), the derivative with respect to the parameter that we can choose (here p) is zero (If it was *not* zero, we could get a higher function value by going a tiny bit to the left or to the right!). So we should have a look at the derivative of the log-likelihood with respect to p.
- Recall that the derivative of the logarithmic function is as follows:

  $$[\log(x)]' = \frac{1}{x}$$

  By the chain rule for derivatives, we also have:

  $$[\log(1-x)]' = \frac{1}{1-x} \times (-1) = \frac{-1}{1-x}$$

  (Notice that in our problem, p plays the role of x in these formulas.)
- Now, do the calculations:

  $$\begin{aligned} [log\mathcal{L}(\text{data}, p)]' &= 11 \times \frac{1}{p} - 19 \times \frac{1}{1-p} = 0 \\ \frac{11}{p} &= \frac{19}{1-p} \\ 11 - 11p &= 19p \\ 11 &= 30p \\ p^* &= \frac{11}{30} = .3667 \end{aligned}$$

  Since p^* is the only value of p where the derivative is zero, this is the only possible maximizer and thus our Maximum-Likelihood Estimator!

8.11 (a) $H_0: \quad p = .45$
$H_a: \quad p \neq .45$

(b) $H_0: \quad \mu = 2.5$
$H_a: \quad \mu \neq 2.5$

8.23 (a) $H_0: \quad \mu = 100$
$H_a: \quad \mu > 100$
$\alpha = .05$
$n = 100$
Since $n \geq 30$, $\bar{x}$ is normally distributed and we can work with the corresponding z-values. We are dealing with a one-sided test, hence $z_\alpha^{one-s.}$ is chosen such that $P(z > z_\alpha^{one-s.}) = \alpha$, which yields $z_\alpha^{one-s.} = 1.64$.

$$\sigma_{\bar{x}} = 60/\sqrt{100} = 60/10 = 6$$

$$z = \frac{110 - 100}{6} = \frac{10}{6} = 5/3 = 1.67 > 1.64 = z_\alpha^{one-s.}$$

We *reject* H_0 since the z-value exceeds the required level.

(b) $H_0: \quad \mu = 100$
$H_a: \quad \mu \neq 100$
$\alpha = .05$
Note that nothing has changed apart from the fact that we are dealing with a *two-sided* test now. This means that we choose z_α such that $P(z \geq z_\alpha^{two-s.}) + P(z \leq z_\alpha^{two-s.}) = \alpha$. This yields $z_\alpha^{two-s.} = 1.96$.

$$z = 1.67 < 1.96 = z_\alpha^{two-s.}$$

Hence we *cannot reject* H_0.

(c) The two-sided test makes it harder to reject H_0 since the probability mass α has to be evenly split among the two tails of the distribution when calculating $z_\alpha^{two-s.}$. On the contrary, in the one-sided case the probability mass α is concentrated on the right side of the distribution beyond $z_\alpha^{one-s.}$. Hence $z_\alpha^{one-s.}$ is smaller than $z_\alpha^{two-s.}$, and z surpasses the former but not the latter.

8.33 (a) No. The confidence interval was constructed with the following formula:

$$\bar{x}_M \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

A two-sided hypothesis test of $H_0: \ \mu_M = 60,000$ would be rejected if

$$\frac{\bar{x}_M - 60,000}{\sigma_{\bar{x}_M}} > z_{\alpha/2}$$

Work on this expression to get a connection to the above confidence interval:

$$\bar{x}_M - 60,000 > z_{\alpha/2}\sigma_{\bar{x}_M}$$
$$\bar{x}_M > 60,000 + z_{\alpha/2}\sigma_{\bar{x}_M} = 60,000 + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

(If we didn't know that $\bar{x}_M$ is smaller than 60,000, we should also include the following statement for the rejection region of H_0:

$$\bar{x}_M < 60,000 + z_{\alpha/2}\frac{s}{\sqrt{n}})$$

We see that the 'non-rejection region' around 60,000 has the same width as the confidence interval around $\bar{x}_M$. Hence, since 60,000 is in the confidence interval around $\bar{x}_M$, $\bar{x}_M$ will also lie in the 'non-rejection region' for H_0.

(b) Answered by (a).

(c) Answered by (a).

(d) By the same reasoning as in (a), there is no evidence to say that μ_F differs from 33,000.

(e) By (a), it is clear that we will fail to reject H_0.

(f) Answered by (a).

8.45 $H_0: \quad \mu = 75$
$p = .1032$ (two-tailed)

(a) The two-tailed p-value gives us the following information:

$$P\left(z < \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}\right) + P\left(z > \frac{-(\bar{x} - \mu)}{\sigma_{\bar{x}}}\right) = p$$

[Notice that this notation assumes that $\bar{x} < \mu$, which is implied by the negative z-value that we are given in the problem.] From this, we can infer that

$$P\left(z < \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}\right) = \frac{p}{2} = .0516$$

This is consistent with the z-value -1.63 that we are given in the problem. Compare this to the criterion for selecting z_α for a one-sided test:

$$P(z < z_\alpha) = \alpha = .05$$

The last two equations together tell us that $z_\alpha < \frac{\bar{x}-\mu}{\sigma_{\bar{x}}}$, so we *fail to reject* H_0. [Have a look at the graph of the normal distribution and look at the probability areas as we did in the lab if this isn't clear to you!]

(b) The positive z-value given in the problem indicates that $\bar{x}$ is in fact *larger* than μ. Hence it is impossible that we reject H_0 in favor of H_a, which states that μ is *smaller* than 75.

(c) In this case, $\bar{x} > \mu$ by the positive z. Also, as in (a):

$$P(z > z_\alpha) = \alpha = .1$$

$$P\left(z > \frac{\bar{x}-\mu}{\sigma_{\bar{x}}}\right) = \frac{p}{2} = .0516,$$

using the p-value given in the beginning. These two equations tell us that $\frac{\bar{x}-\mu}{\sigma_{\bar{x}}} > z_\alpha$, so we *reject* H_0.

(d) In this case, we are dealing with a two-sided test.

$$P(z < z_\alpha) = \alpha/2 = .005$$

This z_α is even smaller than the one in (a), so definitely we have $z_\alpha < \frac{\bar{x}-\mu}{\sigma_{\bar{x}}}$ and we *fail to reject* H_0.