Known Closed Formulas and their proofs

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Abstract

In this paper, we will try to give every known closed formula for options and also their proof(s). We are going to work under the Black Scholes equation for the diffusion of the process.We will start with a classical call option, try to give a few proofs of its price and then move on to more complex products, such as binary options, quanto options. . .

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Introduction

This article has not only the ambition to contain most of the known closed formulas for options, but also to give their proof and even their proofs when available. It will then be useful for students or just curious mathematicians to remember how to find the price of such options which is often the starting point of the pricing of more complex options.

We will assume a quite general framework since we are just assuming the Black Scholes diffusion equation for the process as well as deterministic drifts and volatilities.

For those who are curious to know more about me, you can visit my website at http://homepages.nyu.edu/~sl1544/index.html. You can also contact me by email at simon.leger@nyu.edu.

1 Framework

As we mentioned, we are going to price some path independent options. We call the underlying S_t which is a function of time, this is the stock price. We assume that it obeys the following equation :

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

As we are going to study first only non path dependent options, we can assume that the drift and the volatility are deterministic functions of time since we just then need to replace the constants by their integrals over the period considered. Indeed, if we consider an option with maturity T, we just need to replace μ and σ with : $\mu = \frac{1}{T}\int_{t=0}^{T}\mu(t)dt$ and $\sigma = \sqrt{\frac{1}{T}\int_{t=0}^{T}\sigma^2(t)dt}$.

This can be easily proved by seeing that if we assume the following equation :

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t$$

we still have, thanks to Itô lemma :

$$S_T = S_0 \times exp\left[\left(\int_{t=0}^{T}\mu(t)dt - \frac{1}{2}\int_{t=0}^{T}\sigma^2(t)dt\right) + \int_{t=0}^{T}\sigma(t)dW_t\right]$$

Since $\sigma(t)$ is deterministic, the integral $\int_{t=0}^{T}\sigma(t)dW_t$ is gaussian with mean zero and variance $\int_{t=0}^{T}\sigma^2(t)dt$ so we can replace our deterministic functions by the constants we gave.

We will not assume in this paper any kind of more complex diffusion equations such as stochastic volatility, local volatility, jump processes since it is beyond the scope of this article. We are now ready to start pricing our first option. We will first see the classic call european type option, but we will try to give a few proofs of the known closed form price.

2 Classic European Call

2.1 Direct calculation

2.1.1 Deriving the price of a call

A *european* call means that the payoff can only be exercised at maturity T, his payoff is : $[S_T - K]^+ = max(S_T - K, 0)$. To price it, we use the solution of the Black-Scholes equation :

$$S_T = S_0 \times e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

We can now price this call using the fact that its price is the discounted expected payoff under the risk-neutral measure :

$$C(K) = \mathbf{E}^{\mathbb{Q}}[e^{-rT}(S_T - K)^+]$$

We can now work out this calculation since r is a constant here (we replaced the deterministic function of time by its integral). Indeed, let us write $S_T = S_0 e^X$ where X is a gaussian variable with mean $\alpha = (r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Then :

$$C(K) = e^{-rT} \int_{-\infty}^{\infty} (S_0 e^X - K)^+ \frac{e^{-\frac{(X-\alpha)^2}{2\sigma^2 T}}}{\sqrt{2\pi}} dX$$

and we can rewrite this with an indicatrice function, since the payoff is strictly positive only when $S_0 e^X > K$ i.e. $X > ln \frac{K}{S_0}$:

$$C(K) = e^{-rT} \int_{ln \frac{K}{S_0}}^{\infty} (S_0 e^X - K) \frac{e^{-\frac{(X-\alpha)^2}{2\sigma^2 T}}}{\sqrt{2\pi}} dX$$

We can now split this computation into two integrals, the first one being :

$$I_1 = \int_{ln \frac{K}{S_0}}^{\infty} S_0 e^X \frac{e^{-\frac{(X-\alpha)^2}{2\sigma^2 T}}}{\sqrt{2\pi}} dX$$

and the second one :

$$I_2 = \int_{ln \frac{K}{S_0}}^{\infty} K \frac{e^{-\frac{(X-\alpha)^2}{2\sigma^2 T}}}{\sqrt{2\pi}} dX$$

The second one is easier, let us set $u = \frac{X-\alpha}{\sigma\sqrt{T}}$:

$$\begin{aligned} I_2 &= K \int_{\frac{ln\frac{K}{S_0}-\alpha}{\sigma\sqrt{T}}}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} dX \\ &= K(1-\mathcal{N}(-d_2)) \\ &= K(\mathcal{N}(d_2) \end{aligned}$$

where :

$$\begin{aligned} -d_2 &= \frac{ln\frac{K}{S_0}-\alpha}{\sigma\sqrt{T}} \\ d_2 &= \frac{ln\frac{S_0}{K}+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{aligned}$$

We can now work out the other integral :

$$\begin{aligned} I_1 &= S_0 \int_{ln\frac{K}{S_0}}^{\infty} e^X \frac{e^{-\frac{(X-\alpha)^2}{2\sigma^2 T}}}{\sqrt{2\pi}} dX \\ I_1 &= S_0 \int_{ln\frac{K}{S_0}}^{\infty} \frac{e^{-\frac{(X-(\alpha+\sigma^2 T))^2}{2\sigma^2 T}}}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2\sigma^2 T}} e^{\frac{(\alpha+\sigma^2 T)^2}{2\sigma^2 T}} dX \end{aligned}$$

by developing the square in the exponential and recomposing it by adding the e^X inside. We can now replace our α to simplify the expression :

$$\begin{aligned} I_1 &= S_0 \int_{ln\frac{K}{S_0}}^{\infty} \frac{e^{-\frac{(X-(r+\frac{\sigma^2}{2}T))^2}{2\sigma^2 T}}}{\sqrt{2\pi}} e^{\frac{2\alpha\sigma^2 T}{2\sigma^2 T}} e^{\frac{\sigma^2 T^2}{2}} dX \\ &= S_0 \int_{ln\frac{K}{S_0}}^{\infty} \frac{e^{-\frac{(X-(r+\frac{\sigma^2}{2}T))^2}{2\sigma^2 T}}}{\sqrt{2\pi}} e^{\alpha+\frac{\sigma^2 T}{2}} dX \end{aligned}$$

We now let u be $u = \frac{X-(r+\frac{\sigma^2}{2}T))}{\sigma\sqrt{T}}$ such that :

$$\begin{aligned} I_1 &= S_0 \int_{\frac{ln\frac{K}{S_0}-(r+\frac{\sigma^2}{2}T)}{\sigma\sqrt{T}}}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} e^{rT} dX \\ &= S_0 e^{rT}(1-\mathcal{N}(-d_1)) \\ &= S_0 e^{rT}\mathcal{N}(d_1) \end{aligned}$$

where :

$$\begin{aligned} -d_1 &= \frac{ln\frac{K}{S_0}-(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_1 &= \frac{ln\frac{S_0}{K}+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{aligned}$$

We can now combine our two results to get our option price :

$$C(K) = e^{-rT}\left[S_0 e^{rT}\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right]$$

such that :

$$C(K) = S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

which is our call price.

2.1.2 Implied results

We can derive the put price the same way or just by using the call-put parity : $C_t - P_t = S_t - Ke^{-rT}$ which leads us to :

$$P(K) = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$

From these equations, we can calculate the greeks by deriving the prices, the results are : (we are not going to do these very boring calculations here, but please be careful when you do them since for exemple the delta can not be computed just by saying that the payoff is linear in S_0 since S_0 occurs also in the $d_1, d_2 \ldots$)

	Calls	**Puts**
delta	$\mathcal{N}(d_1)$	$\mathcal{N}(d_1) - 1$
gamma	$\frac{\phi(d_1)}{S\sigma\sqrt{(T)}}$	
vega	$S\phi(d_1)\sqrt{(T)}$	
theta	$-\frac{S\phi(d_1)\sigma}{2\sqrt{(T)}} - rKe^{-rT}\mathcal{N}(d_2)$	$-\frac{S\phi(d_1)\sigma}{2\sqrt{(T)}} + rKe^{-rT}\mathcal{N}(-d_2)$
rho	$KTe^{-rT}\mathcal{N}(d_2)$	$-KTe^{-rT}\mathcal{N}(-d_2)$

2.2 PDE approach

We derived the price of a call by direct calculation, we will now try to get the same results by solving the Black-Scholes PDE.

2.2.1 Deriving the Black-Scholes PDE

We are going to derive the Black-Scholes PDE by considering a hedging strategy, since this is to my sense the most intuitive way. Let $C(s,t)$ be the price of a call at time t, maturing at T. We have the final condition : $C(s,T) = [s-K]^+$.

We consider now a portfolio Π where we are long a call and short ϕ shares of stocks : $\Pi = C - \phi s$. We consider now the difference in the portfolio value for an arbitrary small time spent : $d\Pi = dC - \phi ds$. One should here wonder why we did not derive ϕ. The argument in the original paper is that the number of shares hold is instantaneously constant, which is not very rigorous but true. The way Peter Carr goes around this problem is to say that we are not actually interested in the differentiation but in the gain of our portfolio which is given by $\Delta\Pi = \Delta C - \phi\Delta s$ since the number of shares hold is constant during this period of time, because we can not anticipate the change in s.

We can now apply Ito's lemma to get :

$$\begin{aligned} d\Pi &= dC - \phi ds = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial s}ds + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 s^2 dt - \phi ds \\ &= \left(\frac{\partial C}{\partial s} - \phi\right)(r + \frac{1}{2}\sigma^2)sdt + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 s^2\right)dt + \left(\frac{\partial C}{\partial s} - \phi\right)\sigma s dW \end{aligned}$$

We know that the choice of ϕ that makes our portfolio deterministic is $\phi = \frac{\partial C}{\partial s}$ but here we have a proof of this. With this ϕ our portfolio now satisfies :

$$d\Pi = dC - \phi ds = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 s^2\right)dt$$

As our portfolio is now deterministic, a non arbitrage relation implies that it grows at the risk free rate : $d\Pi = r\Pi dt$ which gives us :

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 s^2\right)dt = r(C - \phi s)dt$$

which is the Black Scholes PDE that we can rewrite as :

$$\frac{\partial C}{\partial t} + r\frac{\partial C}{\partial s}s + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 s^2 = rC$$

We are now ready to transform our PDE into the heat equation.

2.2.2 Transformation into heat equation

Let us rewrite this equation for any claim $V(S,t)$:

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial s}s + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 s^2 = rV$$

For a call, our PDE has the following boundary conditions :

$$\begin{cases} V(0,t) = 0 & \forall t \\ V(0,t) \to S \quad as & S \to \infty \\ V(S,T) = max(S_T - K, 0) \end{cases}$$

Let us now change the variables such that :

$$\begin{cases} S = Ke^x \\ t = T - \frac{2\tau}{\sigma^2} \\ V(S,T) = Kv(x,\tau) \end{cases}$$

And we can use the chain rules to transform our equation :

$$\begin{aligned}
\frac{\partial V}{\partial t} &= K\frac{\partial v}{\partial \tau}\frac{\partial tau}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} \\
\frac{\partial V}{\partial S} &= K\frac{\partial v}{\partial x}\frac{dx}{dS} = \frac{K}{S}\frac{\partial v}{\partial x} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial v}{\partial x}\right) \\
&= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}\frac{dx}{dS} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

Now, we can substitute this into the Black-Scholes PDE and what do we get ?

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (c-1)\frac{\partial v}{\partial x} - kv$$

for $x \in]-\infty; \infty[$ and $\tau > 0$, and where we set $c = \frac{2r}{\sigma^2}$. The terminal condition can now be seen as an initial condition : $V(S,T) = max(S_T - K, 0)$ becomes $v(x,0) = max(e^x - 1, 0)$.

If we make another transformation :

$$v(x,\tau) = e^{-\frac{1}{2}(c-1)x - \frac{1}{4}(c+1)^2\tau}u(x,\tau)$$

which is a common method to parabolic equations. Our equation now becomes :

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

which is exactly the heat equation. We still have our initial condition which is now :

$$u_0(x) = u(x,0) = max\left(e^{\frac{1}{2}(c+1)x} - e^{\frac{1}{2}(c-1)x}, 0\right)$$

2.2.3 Solving the heat equation via Fourier transform

This is a very easy way to solve our heat equation. Let us call $\hat{u} = \mathcal{F}(u)$ the Fourier transform of u :

$$\hat{u} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}dx$$

We can now take the Fourier transform of our equation : $\mathcal{F}(u_\tau) = \mathcal{F}(u_{xx})$ which gives $\hat{u}_\tau = -\omega^2\hat{u}$ and the solution is : $\hat{u}(\omega,\tau) = \hat{u}_0(\omega)e^{-\omega^2\tau}$ where $\hat{u}_0 = \hat{u}(x,0)$.

Before going further we need some properties of the Fourier transform :

$$\begin{aligned}
\mathcal{F}\left(\frac{1}{\sqrt{2\tau}}e^{-\frac{x^2}{4\tau}}\right) &= e^{-\omega^2\tau} \\
\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)dy\right) &= \hat{f}(\omega)\hat{g}(\omega)
\end{aligned}$$

and we can now go back to our original problem with the following inverse Fourier transform :

$$\begin{aligned}
u(x,\tau) &= \mathcal{F}^{-1}(\hat{u}) \\
&= \mathcal{F}^{-1}\left(\hat{u}_0(\omega)e^{-\omega^2\tau}\right) \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy
\end{aligned}$$

2.2.4 Pricing a call again

We can now plug that into our previous equations and we can work out the price of our call :

$$\begin{aligned}
u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} max(e^{\frac{1}{2}(c+1)y} - e^{\frac{1}{2}(c-1)y}, 0)e^{-\frac{(x-y)^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} (e^{\frac{1}{2}(c+1)y} - e^{\frac{1}{2}(c-1)y})e^{-\frac{(x-y)^2}{4\tau}}dy \\
&= I_1 - I_2
\end{aligned}$$

where :

$$I_1 = \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} e^{\frac{1}{2}(c+1)y}e^{-\frac{(x-y)^2}{4\tau}}dy$$

and I_2 is the same after replacing $c+1$ with $c-1$. Let us work a little on I_1 :

$$\begin{aligned}
I_1 &= \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} e^{-\frac{y^2-2xy+x^2-2(c+1)\tau y}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} e^{-\frac{(y-(x+(c+1)\tau))^2}{4\tau}}e^{-\frac{x^2}{4\tau}}e^{\frac{(x+(c+1)\tau)^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{\frac{x+(c+1)\tau}{\sqrt{2\tau}}}^{\infty} e^{-\frac{u^2}{2}}\sqrt{2\tau}e^{\frac{x(c+1)}{2}}e^{\frac{1}{4}(c+1)^2\tau}dy \\
&= \mathcal{N}(d_1)e^{\frac{1}{2}(c+1)x+\frac{1}{4}(c+1)^2\tau}
\end{aligned}$$

after setting $u = \frac{y-(x+(c+1)\tau)}{\sqrt{2\tau}}$ and where :

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(c+1)\sqrt{2\tau}$$

We now need to substitute these terms to get our result as a function of our initial parameters :

$$v(x,\tau) = e^x \mathcal{N}(d_1) - e^{-c\tau}\mathcal{N}(d_2)$$

and :

$$C(S,t) = K \times \frac{S}{K} \times \mathcal{N}(d_1) - Ke^{\frac{2r}{\sigma^2}\times\frac{\sigma^2}{2}(T-t)}\mathcal{N}(d_2)$$

which gives us :

$$C(S,t) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

where :

$$\begin{aligned} d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(c+1)\sqrt{2\tau} \\ &= \frac{ln(\frac{S}{K})}{\sigma\sqrt{T-t}} + \frac{1}{2}\left(\frac{2r+\sigma^2}{\sigma^2}\right)\sigma\sqrt{T-t} \\ &= \frac{ln\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{aligned}$$

and :

$$d_2 = \frac{ln\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

And we finally found back our Black-Scholes price which is nice.

3 Binary Options

3.1 Introduction

We are now going to price a binary, or digital, call options, which is not more difficult than the price of a call option, it is even easier. The payoff of a binary option is a Heaviside function :

$$C^d(S_T,T) = \begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K \end{cases}$$

3.2 Direct calculation

Again, the price of our option is the discounted expected payoff under the risk-neutral probability :

$$C^d(S_t,t) = \mathbf{E}^{\mathbb{Q}}[e^{-r(T-t)}\mathbf{1}_{S_T\geq K}] = e^{-r(T-t)}\mathbb{P}^{\mathbb{Q}}(S_T \geq K)$$

such that we just need to work out one probability computation :

$$\begin{aligned}
\mathbb{P}^{\mathbb{Q}}(S_T \geq K) &= \mathbb{P}^{\mathbb{Q}}\left(ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t) > 1\right) \\
&= \mathbb{P}^{\mathbb{Q}}\left(X > -\frac{ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \mathcal{N}(-d_2) = \lceil_\in
\end{aligned}$$

where $X \rightsquigarrow \mathcal{N}(0,1)$ because $W_t \rightsquigarrow \mathcal{N}(0,t)$ and where :

$$d_2 = \frac{ln\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Then the price of our binary option is :

$$C^d(S_t,t) = e^{-r(T-t)}\mathcal{N}(d_2)$$

3.3 PDE approach

We can also work out the price of this option by our PDE approach, which is now easier than before since we have already derived most of the formulas. We start from the solution of the heat equation, but the final payoff $\mathbf{1}_{S_T>K}$ is now $\mathbf{1}_{x>0}$ since $x = ln\left(\frac{S_t}{K}\right)$ then $u_0(x) = \frac{1}{K}\mathbf{1}_{x>0}e^{\frac{1}{2}(c-1)x}$. We follow the same calculation as before :

$$\begin{aligned}
u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\frac{1}{K}\int_0^{\infty} e^{\frac{1}{2}(c-1)y}e^{-\frac{(x-y)^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\frac{1}{K}\int_0^{\infty} e^{\frac{x^2}{4\tau}}e^{\frac{(x+(c-1)\tau)^2}{4\tau}}e^{-\frac{(y-(x+(c-1)\tau))^2}{4\tau}}dy \\
&= \frac{1}{2\sqrt{\pi\tau}}\frac{1}{K}\int_{-\frac{x+(c-1)\tau}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{4}(c-1)^2\tau}e^{\frac{1}{2}(c-1)x}e^{-\frac{u^2}{2}}\sqrt{2\tau}dy \\
&= \frac{1}{K}(1-\mathcal{N}(-d_2))e^{\frac{1}{4}(c-1)^2\tau}e^{\frac{1}{2}(c-1)x}
\end{aligned}$$

such that :

$$v(x,\tau) = \frac{1}{K}\mathcal{N}(d_2)e^{-c\tau}$$

and :

$$C^d(S_t, t) = e^{-r(T-t)}\mathcal{N}(d_2)$$

and once again we found our price with the PDE method. Let us note that we could have guessed this result before since $\mathcal{N}(d_2)$ is the probability that the option is in the money, so our result is just the discounted risk-neutral probability that our stock ends above K at time T.

4 Quanto Options

Conclusion