AN INTRODUCTION TO FINANCIAL MATHEMATICS

PART 2 – BINOMIAL TREES AND RISK NEUTRAL PRICING

Simon Leger
simon.leger@nyu.edu

Yann Renoux
yann.renoux@nyu.edu

Binomial pricing

Problem

You suspect that Delta airlines will merge with Northwest airlines in the coming month. Delta stock is trading at 0.85$. There is a 60% chance that the merger will occur, in which case, the stock will be worth 1.2$. There is a 40% chance that the merger will not occur, in which case, the stock will continue its downward plunge to 0.3$. If r = 0.02, what is the value of a call with strike price K = 1$ and maturity one month i.e. T = 1/12 ? What is the value of this option ?

Introduction to Binomial model

To solve this kind of problem, we introduce a new tool : the binomial tree. It allows us to model the evolution of the stock price by saying that from one period to the next, the price can either go up or down from a certain percentage and with a certain probability :

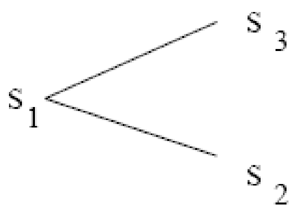

FIG. 1 – Basic binomial tree

We will note S_i the price of the stock and f_i the price of the claim at the state i.

Notions

What is e^{-rT} ?

e^{-rT} is what we call *discount factor*. This is actually the price now of one dollar at time T under interest rate r. So, to evaluate the price of a claim paying at time T we need to discount the payoff by e^{-rT}.

What is *replication* ?

Replication is the fact that you can reproduce a claim by a combination of other claims, usually the stock and the bond. If at time T_0 we can replicate a claim at time T_1 then the price of the claim and of the replicating portfolios at time T_0 must be equal, otherwise there is an arbitrage opportunity.

What is *non arbitrage* ?

Non arbitrage is a common assumption in finance, which means that now you can not with zero money construct a portfolio such that your outcome will be surely positive and there is a strictly positive probability that it will be strictly positive. In simpler words : you can not make surely money in the market.

How to solve the problem

Once we have decided what to choose for S_2 and S_3, we need to know the associated probabilities. They are found by doing this calculation. We try to replicate our claim by a combination of stock and bond, we have the following portfolio in state 1 :

$$\phi s_1 + \psi e^{-rT}$$

such that in the following state our portfolio has the same value than the claim :

$$\begin{cases} \phi s_2 + \psi \times 1 = f_2 \\ \phi s_3 + \psi \times 1 = f_3 \end{cases}$$

which gives us conditions on ϕ, ψ. After solving this system, we get :

$$\begin{cases} \phi = \frac{f_3 - f_2}{s_3 - s_2} \\ \psi = \frac{s_3 f_2 - s_2 f_3}{s_3 - s_2} \end{cases}$$

We have at time 2 a portfolio which has the same value as the claim, so by non arbitrage it must have the same value at time 1 :

$$V(f) = \phi s_1 + \psi e^{-rT} = \frac{f_3 - f_2}{s_3 - s_2} s_1 + \frac{s_3 f_2 - s_2 f_3}{s_3 - s_2} e^{-rT}$$

and by rearranging the above equation and setting :

$$q = \frac{s_1 e^{rT} - s_2}{s_3 - s_2}$$

we get the following result :

$$V(f) = e^{-rT} \left[(1-q) f_2 + q f_3 \right]$$

Now that we have the theory we can move on to our initial problem.

We can now price our option by calculating the following : q, f_3, f_2, e^{-rT} and we get :

$$\begin{cases} q = \frac{0.85 e^{0.02/12} - 0.3}{1.2 - 0.3} \simeq 0.613 \\ e^{-0.02/12} \simeq 0.9983 \\ f_3 = (1.2 - 1)_+ = 0.2 \\ f_2 = (0.3 - 1)_+ = 0 \end{cases}$$

From this we can get the price of our claim :

$$V(f) = e^{-rT} \left[(1-q) f_2 + q f_3 \right] \simeq 0.9983 \left[(1 - 0.613) \times 0 + 0.613 * 0.2 \right] \simeq 0.122$$

This price can obviously be improved by doing this with more steps which would improve the precision by exploring more states of the world, since this is pretty hard to imagine a strict binomial distribution for a stock price.

Continuous time assumptions

To jump from this to continuous time models, we need to assume a diffusion process for the stock price, ie according to what conditions the stock price evolves. One of the main common process for stocks is the Black Scholes process, in which the stock follows the equation :

$$dS_t = \mu S_t dt + \sigma S dW_t$$

where μ and σ can be considered functions of the stock itself or of the time.

What does this equation means ?

Well, it basically assumes that the stock has a certain drift μ that drives the stock price but there is a noise around it, which is called the volatility of the stock, and it appears that this equation models pretty well the evolution of stock prices even if more complex models have been written after this.

For exemple, nobody uses this model for short term interest rates, for economic reasons for exemple. Indeed, the main goal of Fed or European central bank is to keep the rates in a certain range, and usually rates are supposed to follow mean reverting process, which means that there is some power that keeps the interest rates around a certain level, and there is still a noise around this.

Other products

We have seen a basic derivatives product in the previous section, but much more complex products do exist and trade in the market, we will give just a few exemples to see what can be done :

- A *barrier option* is an option which pays 1 if the stock reaches a certain level and 0 otherwise
- A *quanto* option is an typical option, except that the stock is traded in a foreign currency, so there is another source of risk, and very often they are correlated.
- An *american* option can be any kind of options, the only difference is that the payoff can be exercised at any time.
- A *rainbow* option is an option whose payoff depends on multiple assets which are usually correlated.

To price very complicated options, there is very often no closed formulas and we have to use numerical methods to get a price. For example, there is no formula for the price of some rainbow options and we need to run *Monte Carlo* simulations to get a price.

What is a *Monte Carlo* simulation ?

Instead of calculating the price, we simulate many many possibilities according to the diffusion equation we have chosen and look at the payoff over all the possibilities and we average them to get a good approximation of the price. This can be however sometimes pretty tough to do especially when dealing with multiple correlated assets when you have to simulate them together, but this can be done and is a good way to get a price when you dont know what else to do !