Rational Inattention

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Motivation

- Various macroeconomic time series are very smooth and persistent ("stickiness")
- Generalize dynamic choice with information processing constraints that produce "smoothness"
- Roadmap
  1. General Static problem, similarities/differences with HS
  2. LQG static problem: "Endogenous noise"
  3. Simple LQ Permanent Income example (time permitting)

Main Tool to Measure Information Capacity: *Mutual Information*

$$
\begin{aligned}
I(X;Y) &\equiv D_{KL}(p(X,Y) \mid p(X)\,p(Y)) = \sum_{x,y} p(x,y) \ln \frac{p(x,y)}{p(x)\,p(y)} \\
&= E_{p(X)\cdot p(Y)} m(X,Y) \ln m(X,Y)
\end{aligned}
$$

where $m(x,y) \equiv \frac{p(x,y)}{p(x)p(y)}$. $I(X;Y) = 0$ iff independent

- *Interpretation* of mutual information:Average reduction of uncertainty of X when I observe Y

$$
\begin{aligned}
I(X;Y) &= H(Y) - H(Y \mid X) = H(X) - H(X \mid Y) \\
&= \sum_{y} p(y) \ln \frac{1}{p(y)} - \sum_{x} p(x) \sum_{y} p(y \mid x) \ln \frac{1}{p(y \mid x)}
\end{aligned}
$$

General Static Problem (Joint Control/Estimation)

- $U(x,y)$: convex return function X : state , Y: action. X is observed with finite capacity$\Rightarrow$ Y cannot be function of X.

- Choice variable: $p(x,y)$. Since $p(x)$ given$\Rightarrow$ choice variable $q(y \mid x)$

- R.I. Problem:

$$\min_{q(y|x)} \sum_x p(x) \sum_y q(y \mid x) U(x,y)$$

s.t.

$$\begin{aligned} \sum_y q(y \mid x) &= 1 \quad : \lambda(x) p(x) \\ I(X,Y) &\leq \kappa \quad : \mu \end{aligned}$$

F.O.C. (for interior solution)

$$\underbrace{p(x)\,U(x,y)}_{\text{Marginal Benefit}} + \underbrace{\lambda(x)\,p(x) + \mu \frac{\partial I(X,Y)}{\partial q(y \mid x)}}_{\text{Marginal Cost in capacity terms}} = 0$$

- But $\frac{\partial I(X,Y)}{\partial q(y|x)} = p(x) \ln m(x,y)$
- Optimum with a HS formulation

$$m(x,y) = \frac{q(y \mid x)}{p(y)} = \frac{\exp\left(-\frac{U(x,y)}{\mu}\right)}{\sum_y p(y) \exp\left(-\frac{U(x,y)}{\mu}\right)}$$

- Interpretation of μ: Large $\mu \Rightarrow$small capacity (large noise)$\rightarrow$independence case

- Tempting to create a "multiplier" formulation with value function

$$V(\mu, p(x)) = -\mu \sum_x p(x) \ln \sum_y p(y) \exp\left(-\frac{U(x,y)}{\mu}\right)$$

- But it may be misleading: With constraint formulation $\mu = \mu(\kappa, p(x))$. Thus it changes when we change $p(x)$ in C.S. exercises

LQG example

- $U(x,y) = (y-x)^2$, $X \sim N\left(\mu_x, \sigma_x^2\right)$, $\theta \equiv \frac{\mu}{2}$

- Then $Y\mid_X \sim N\left(\mu_{Y|X}, \sigma^2_{Y|X}\right)$ with

$$\begin{aligned}\frac{1}{\sigma^2_{Y|X}} &= \frac{1}{\sigma^2_Y} + \frac{1}{\theta} \Rightarrow \ \sigma^2_{Y|X} = \frac{\theta\sigma^2_Y}{\theta + \sigma^2_Y} \\ \mu_{Y|X} &= (1-r)\,\mu_Y + rx, \quad r = \frac{\sigma^2_Y}{\sigma^2_Y + \theta}\end{aligned}$$

and $\mu_Y = \mu_X$, $\sigma_{XY} = \sigma^2_Y$

- So the solution behaves "as if"

$$X = Y + n$$

where $n \sim N(0, \theta)$: noise independent of Y.

- From the Information Capacity constraint we have

$$I(X;Y) = \frac{1}{2} \ln \frac{\sigma_Y^2}{\sigma_{Y|X}^2} = \frac{1}{2} \ln \frac{\sigma_X^2}{\theta} = \kappa$$

Thus

$$\lambda \equiv \frac{\sigma_X^2}{\theta} = \exp(2\kappa)$$

The "signal to noise" ratio is endogenous and depends only on κ !

- If $\kappa \to 0$, then $\theta \to \sigma_X^2 \Rightarrow \sigma_Y^2 \to 0$. So optimal $Y \to \mu_X$

- If $\kappa \to \infty$, then $\theta \to 0 \Rightarrow \sigma_Y^2 \to \sigma_X^2$. and optimal $Y \to X$ (case without capacity constraints)

- In the LQG we have an example of signal extraction with endogenous noise.

Simple Permanent Income Example

- Problem without information constraints

$$\max_{c_t} E \sum_{t=0}^{\infty} \beta^t \left(C_t - \frac{1}{2} C_t^2 \right)$$

$$\text{s.t. } W_{t+1} = R\left(W_t - C_t\right) + Y_{t+1}$$

- Y_t iid $N\left(\bar{Y}, \omega^2\right)$, $\beta R = 1$. Thus state for recursive formulation: W_t

- Exploit the results of the previous example (LQG). Assume a "signal" structure, which will be chosen optimally given the capacity constraint.

- Consumption C_t will be measurable wrt to the information set generated by the signals: $\mathcal{I}_t = \left\{ s_{0,} s_1, ..., s_t \right\}$

- Why? In order to use standard results of Signal Extraction Problems like Certainty Equivalence. In the deterministic case

$$C_t = (1 - \beta) W_t + \beta \bar{Y}$$

  Now

$$C_t = (1 - \beta) \hat{W}_t + \beta \bar{Y}$$

  where $\hat{W}_t \equiv E(W_t \mid \mathcal{I}_t)$.

- Proceed with the optimal choise of signal: $W_t \mid \mathcal{I}_t \sim N\left(\hat{W}_t, \sigma_t^2\right)$, where $\sigma_t^2 \equiv Var_t(W_t)$. By budget constraint and optimal consumption we have

$$
\begin{aligned}
E_t W_{t+1} &= \hat{W}_t \\
Var_t\left(W_{t+1}\right) &= R^2 \sigma_t^2 + \omega^2
\end{aligned}
$$

- Conditional Mutual information : Signal S_t carries as much information as C_t

$$
\begin{aligned}
I\left(W_{t+1}; S_{t+1} \mid \mathcal{I}_t\right) &= H\left(W_{t+1} \mid \mathcal{I}_t\right) - H\left(W_{t+1} \mid \mathcal{I}_{t+1}\right) \\
&= \frac{1}{2} \ln \frac{R^2 \sigma_t^2 + \omega^2}{\sigma_{t+1}^2} = \kappa
\end{aligned}
$$

- Find s.s.

$$
\bar{\sigma}^2 = \frac{\omega^2}{e^{2\kappa} - R^2}
$$

- In the s.s. our agent behaves as if

$$S_t = W_t + \xi_t$$

Also from the updating equation for the conditional variance we have

$$Var_t\left(W_t\right) = \frac{Var_{t-1}(W_t)Var_{t-1}\xi_t}{Var_{t-1}(W_t) + Var_{t-1}\xi_t}$$

- Thefore we can solve for $Var_{t-1}\xi_t$

$$Var\xi_t = \frac{\bar{\sigma}^2\left(R^2\bar{\sigma}^2 + \omega^2\right)}{\left(R^2 - 1\right)\bar{\sigma}^2 + \omega^2}$$

Remark:If we weren't at s.s. the variance of the noise would be time-varying at

$$Var_{t-1}\xi_t = \frac{R^2\sigma_{t-1}^2 + \omega^2}{e^{2\kappa} - 1}$$