Market Survival in the Economies with Heterogeneous Beliefs

Viktor Tsyrennikov

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Abstract

This works aims analyzes market survival of agents with incorrect beliefs. A model with heterogeneous beliefs is well suited for analysis of asset prices and economic volatility. However, I argue that a successful model of asset prices cannot be studied separately from market survival.

1 Introduction

People's ability to make accurate predictions about the future has been solidly built into classical rational expectations models. However, in reality economic forecasts may be very imprecise. For example, the Congressional Budget Office has made on average a 1.2% mistake when predicting 2-year growth rate of the U.S. real GDP over the period from 1992-2003. One might argue that these biases are politically motivated. The popular "Blue Chip Consensus Forecast" publishes forecasts of the 50 largest market players. For example, in March 2005 the average opinion was that the U.S. economy would grow at 3.8%, with top ten average being 4.0% and bottom ten average being 3.0%. It seems that there is a lot of disagreement about the evolution of the U.S. economy and it also seems that these disparities persist over time.

Friedman (1953) was the first to point out that agents with incorrect beliefs would loose their wealth and would be driven out of the market.

Thus, convergence to rational expectations equilibrium obtains. Blume and Easley (1992) show that in the economy in which agents use the same savings rule agents that maximize expected logarithm of their wealth will eventually dominate the market.

Sandroni (2000) shows that in the standard Lucas model (*i.e.* is when saving decision is endogenous) only agents able to make more accurate predictions survive in the market. Hence, beliefs converge to rational expectations equilibrium as agents making inaccurate predictions are driven out of the market.

Yan (2006) shows that even though agents are driven out of the market, inaccurate predictor's wealth decreases at a very slow rate – in the example agents loose only half of their wealth in 400 years. Yan (2006) assumes that endowment stream is a Brownian motion process and agents form wrong beliefs about its drift parameter. In other words, the agents have incorrect perception of the economy's growth potential.

In this work I analyze a sequence of models and try to understand how it is possible that agents with incorrect beliefs survive in the market. A model in with heterogenous beliefs that well replicates behavior of asset prices can be used to study the effect of Tobin's tax on financial market volatility and trading volume.

I analyze the discrete time environment with aggregate uncertainty being governed by a Markov(1) process. Heterogeneity of beliefs considered in this paper is more general than that in, for example, Veronesi (2000). There agents have wrong beliefs about the drift parameter of the aggregate endowment, but not diffusion. In the present model agents may have a completely misspecified distribution, that is a wrong Markov transition probability matrix. Of course, in the simulation agents beliefs are marginally different from the true data generating process. Small differences in beliefs are capable of generating significant fluctuations in individual consumption and, hence, asset prices.

In the presence of agents with heterogeneous beliefs, individual consumption is more volatile than the aggregate consumption. Thus the econometrician using aggregate consumption instead of individual quantities underestimates the price of risk and overestimates risk free rate.

Further, the model can be also estimated using non-parametric techniques – it is possible to estimate the latent incorrect beliefs that explain asset prices dynamics.

2 Economic Environment

We consider the economy in which markets are open at each date and agents can trade the full set of Arrow securities, *i.e.* markets are complete. This economy is studied in great depth in Ljungqvist and Sargent (2004), chapter 8. It is an endowment economy, and the only source of uncertainty is stochastic endowment. Unlike in the previous analyzes, agents no longer share the same *objective* probability distribution over the sequences of states. On the opposite, agents have different *subjective* probability distributions over histories of the state. This work does not intend to explain how rational agents could come up with different probability distributions, but rather explains the consequences for the aggregate economy if they did.

To motivate why there may exist (small) differences in the individual probability distributions, the reader may think of agents as econometricians trying to estimate the true data generating process (DGP). Having observed a finite history of the state, it is not possible to absolutely distinguish any two data generating processes. Instead econometricians make probabilistic statements like 'With confidence 95% the estimated process 1 is the same as the estimated process 2.' It will be insured that, given the observed history, agents will be highly confident in their subjective probability distribution. In other words, the agent will not be able to distinguish his subjective distribution from the true one at, say, 95% the confidence level.

The reader may also think of the agents as robust decision makers. Such decision makers act as if the probability distribution is the worst possible in a certain class of distributions. The higher the preference for robustness, the worse probability distribution (more 'different' from the true distribution) agent will use.

As alternative explanation for why differences in the subjective probability distributions could persist, it may be assumed that learning is costly.

2.1 Preferences and Endowments

Time is discrete and is indexed by $t \in N$, where N is the set of natural numbers. Let $s_t \in S = \{1, 2, .., n\}$ denote realization of the state of the economy at date t. States are ordered from the worst state 0 to the most favorable state n. History $\{s_0, s_1, .., s_t\}$ of the state at time t is customary denoted by s^t.

Assumption 1. *Process s_t follows a first-order Markov process.*

The true transition probability from state $s_t = j$ to state $s_{t+1} = k$ is given by $\pi(j|k)$. Unlike in the work of Harrison and Kreps (1978), the true distribution plays a crucial role in the present analysis. Closeness of the subjective distribution to the true distribution will be related to the agent's welfare.

There are two (classes of) infinitely lived agents indexed by $i \in \{1, 2\}$. Agents form their own subjective beliefs about the evolution of uncertainty in the economy. Transition probability from state j to state k perceived by agent i is denoted by $\hat{\pi}^i(j|k)$. Agents know each other's transition probabilities, but, they do not know the true distribution. Information $\mathcal{I}_t$ available to each agent in the economy at date t is the history of the past state realizations, $\mathcal{I}_t = s^t$.

Assumption 2. *Agents' beliefs are mutually absolutely continuous, i.e.* $\hat{\pi}^i(s^t) > 0 \Leftrightarrow \hat{\pi}^j(s^t) > 0, \forall s^t, i, j$.

Both agents are risk averse and rank different consumption streams $c^i := \{c_t^i(s^t)\}_{t=0}^{\infty}$ according to standard time separable utility

$$W(c^i|s_0) := \sum_{t=0}^{\infty} \sum_{s^t} \hat{\pi}^i(s^t|s_0)\beta^t u(c_t^i(s^t)), \quad u' > 0, u'' < 0. \tag{2.1}$$

Each agent i receives stochastic endowment $e_t^i(s_t)$ at date t and history s^t.

2.2 Competitive Equilibrium

We consider the economy with *sequential trading* in which markets are open in each date. The whole analysis could also be done for the economy with date-0 trading protocol. The two setups are equivalent if natural borrowing limits are imposed on the agents in the economy with sequential markets. We will exploit this equivalence to devise a computational algorithm.

The essential ingredient of sequential markets are one-period-ahead contingent claims or Arrow securities. At each date t and history s^t there is a full set of Arrow securities available for trading – one for each possible realization of the state one period from now. Thus, markets are complete. Let the date-t price of Arrow security paying one unit of good in state s_{t+1} contingent on history s^t be $Q_t(s_{t+1}|s^t)$.

Agent i's problem is to choose consumption stream $\{c_t^i(s^t)\}$ and portfolio of Arrow securities $\{a_{t+1}^i(s_{t+1}, s^t)\}$ to maximize $W(s_0)$ subject to a sequence of budget constraints

$$c_t^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s^t) a_{t+1}^i(s_{t+1}, s^t) = e_t^i(s_t) + a_t^i(s_t), \quad \forall t, s^t, \tag{2.2}$$

and natural borrowing limits, $a_{t+1}^i(s_{t+1}, s^t) \geqslant -A_{t+1}^i(s^{t+1})$.[1]

DEFINITION. *Allocation* is a list $\{c^i, a^i\}_{i=1}^2$, where $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$ and $a^i = \{a_{t+1}^i(s_{t+1}, s^t)\}_{t=0}^\infty$.

DEFINITION. *Feasible allocation* is an allocation $\{c^i, a^i\}_{i=1}^2$ such that goods market and financial market are cleared at each date t and every history s^t

$$\text{Goods market}: \quad c_t^1(s^t) + c_t^1(s^t) = e_t^1(s^t) + e_t^2(s^t), \tag{2.3}$$

$$\text{Financial market}: \quad a_{t+1}^1(s_{t+1}, s^t) + a_{t+1}^2(s_{t+1}, s^t) = 0, \quad \forall\ s_{t+1}. \tag{2.4}$$

DEFINITION. *Price system* a sequence of non-negative functions $\{Q_t(s_{t+1}, s^t)\}_{t=0}^\infty$.

DEFINITION. *Competitive equilibrium* is a feasible allocation $\{c^i, a^i\}_{i=1}^2$ and price system $\{Q_t(s_{t+1}, s^t)\}_{t=0}^\infty$ such that

– (c^i, a^i) solves agent i's problem given the price system;

– natural borrowing limits satisfy recursive relation

$$A_t^i(s^t) = y_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A_{t+1}^i(s^{t+1}).$$

2.3 Equilibrium Characterization

It follows from the agent i's problem that optimal consumption plan must satisfy

$$Q_t(s_{t+1}|s^t) = \beta \hat{\pi}^i(s_{t+1}|s^t) \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))}, \tag{2.5}$$

which uses subjective transition probability, but otherwise is a standard Euler equation. Since both agents buy and sell Arrow securities at the same price

$$\beta \hat{\pi}^1(s_{t+1}|s^t) \frac{u'(c^1(s^{t+1}))}{u'(c^1(s^t))} = \beta \hat{\pi}^2(s_{t+1}|s^t) \frac{u'(c^2(s^{t+1}))}{u'(c^2(s^t))}.$$

[1] Natural borrowing limits are well explained in Ljungqvist and Sargent (2004), pp. 224-5.

Rewrite the above equation to get

$$\frac{\hat{\pi}^1(s_{t+1}|s^t)}{\hat{\pi}^2(s_{t+1}|s^t)} = \frac{u'(c^2(s^{t+1}))/u'(c^2(s^t))}{u'(c^1(s^{t+1}))/u'(c^1(s^t))}. \tag{2.6}$$

According to the above equation, if preferences are homothetic, optimal consumption of the agent that puts higher probability on state s_{t+1} tomorrow grows faster. Immediate implication is that in equilibrium *agents imperfectly insure each other* – an equilibrium outcome that usually requires some sort of market incompleteness.

Had the agents shared the same transition distribution $\bar{\pi}$ – not necessarily equal to the objective π – they would perfectly insure each other. However, the resulting allocation, although Pareto optimal, would be different from the one in which all agents use the true transition probabilities.

2.4 CRRA Preferences

The assumption of CRRA preferences makes problem more tractable.

Assumption 3.

$$u(c) = \frac{c^{1-1/\gamma} - 1}{1 - 1/\gamma}, \gamma > 0.$$

Denote the aggregate endowment $\bar{e}_t(s^t) := e^1_t(s^t) + e^2_t(s^t)$ and rewrite equation (2.6)

$$\begin{aligned} l(s_{t+1}|s^t) := \frac{\hat{\pi}^1(s_{t+1}|s^t)}{\hat{\pi}^2(s_{t+1}|s^t)} &= \Big[\frac{c^1_{t+1}(s^{t+1})/c^1_t(s^t)}{c^2_{t+1}(s^{t+1})/c^2_t(s^t)}\Big]^{1/\gamma} \\ &= \Big[\frac{c^1_{t+1}(s^{t+1})}{c^1_t(s^t)} \cdot \frac{\bar{e}_t(s^t) - c^1_t(s^t)}{\bar{e}^1_{t+1}(s^{t+1}) - c^1_{t+1}(s^{t+1})}\Big]^{1/\gamma}. \end{aligned} \tag{2.7}$$

The ratio of the two consumption growth rates is an increasing function of the likelihood ratio $l := \hat{\pi}^1/\hat{\pi}^2$. The agent that assigns higher probability to state s_{t+1} saves more today to buy Arrow securities that pay out in that same state and in this way insures herself (or, at least, she thinks that she does).

Equation 2.7 is a first order non-linear difference equation in consumption of the first agent. Thus, had we known consumption of the agent at date 0 it would be possible to compute all the equilibrium quantities. This observation will be used to devise a computational algorithm. Before, however, equation (2.7) has to be analyzed. To this end, solve it for $c^1_{t+1}(s^{t+1})$ as a function of $c^1_t(s^t)$

$$c^1_{t+1}(s^{t+1}) = \bar{e}^1_{t+1}(s^{t+1}) \frac{[l(s_{t+1}|s^t)]^\gamma c^1_t(s^t)}{\bar{e}^1_t(s^t) - c^1_t(s^t) + [l(s_{t+1}|s^t)]^\gamma c^1_t(s^t)}. \tag{2.8}$$

To understand the above relation consider the following special cases

$$c^1_{t+1}(s^{t+1}) = \begin{cases} \bar{e}_{t+1}(s^{t+1}), & \text{if } l(s_{t+1}|s^t) = \infty \quad (\hat{\pi}^2(s_{t+1}|s^t) = 0) \\ 0, & \text{if } l(s_{t+1}|s^t) = 0 \quad (\hat{\pi}^1(s_{t+1}|s^t) = 0) \\ c^1_t(s^t)\bar{e}_{t+1}(s^{t+1})/\bar{e}_t(s^t), & \text{if } l(s_{t+1}|s^t) = 1 \quad (\hat{\pi}^1(s_{t+1}|s^t) = \hat{\pi}^2(s_{t+1}|s^t)) \end{cases}.$$

Finally, figure 1 plots the relation between $c^1_{t+1}(s^{t+1})$ and $c^1_t(s^t)$.

Figure 1: Relation between $c_{t+1}(s^{t+1})$ and $c_t(s^t)$

$c_{t+1}(s^{t+1})$

$\bar{e}_{t+1}(s^{t+1})$

$l > 1$

$l = 1$

$l < 1$

0

$\bar{e}_t(s^t)$ $c_t(s^t)$

Unfortunately, it is not possible to characterize equilibrium any better and thus we need to solve the model numerically. Luckily, there exists a simple computational strategy described in the following section.

3 Computational Algorithm

Let *pseudo-optimal allocation* be a feasible allocation that satisfies (2.7), a relation that was derived from the agents' Euler equations and goods market clearing condition. Such allocations might not be optimal as they are not required to satisfy individual budget constraint (2.2). The set of such allocations is a compact set in $\mathbf{R}$, which makes this set very easy to analyze. To see this, note that any pseudo-optimal allocation can be indexed by the initial consumption of the agent 1, $c_0^1 \in [0, \bar{e}_0]$. The rest of this section describes how to choose a pseudo-optimal allocation that is optimal, that is it satisfies (2.2).

Let c_0^1 be the consumption of the agent 1 at date 0. The idea is to fix c_0 and compute the consumption stream recursively using equation (2.7) and pricing system using (2.5). Then it is possible to compute the values of the consumption and the endowment streams. The value of c_0^1 that equates the values of the consumption and endowment streams identifies optimal allocation.

The key relation is the formula for pricing kernel

$$\begin{aligned} Q_t(s_{t+1}|s^t) &= \beta\hat{\pi}^1(s_{t+1}|s^t)\Big[\frac{c_t(s^t)}{c^1(s^{t+1}))}\Big]^{1/\gamma} \\ &= \beta\hat{\pi}(s_{t+1}|s^t)\Big[\frac{\bar{e}_t(s^t) + (l(s_{t+1}|s^t)^\gamma - 1)c_t(s^t)}{\bar{e}_{t+1}(s^{t+1})}\Big]^{1/\gamma}, \end{aligned} \tag{3.1}$$

which depends only on $c_t^1(s^t)$. Current consumption is an additional state variable that is needed to price streams of payoffs.

Assumption 4. *Agents' endowment depends only on the last realization of the state,* $e_t^i(s^t) = e^i(s_t)$

There are setups in which assumption 4 is violated. One is an economy with stochastic growth rates, $e_t^i(s^t) = e_0^i(s_0)g(s_1)g(s_2)..g(s_t)$. To handle such setups the state space has to be augmented with the last realization of endowment.

We need to compute the difference between the value of a consumption and endowment stream. Let $P(c_0^1, s_0)$ be the difference between equilibrium value of a consumption and endowment stream of the agent 1 when her consumption starts with c_0^1 in state s_0. Then P solves the following Bellman equation

$$
\begin{aligned}
P(c_0^1, s_0) &= c_0^1 - e^1(s_0) + \sum_{s_1} Q(s_1, c_0^1|s) P(c_1^1(s_1), s_1) \\
&= c_0^1 - e^1(s_0) + \beta \sum_{s_1} \pi(s_1|s_0) \Big[\frac{\bar{e}(s_0) + (l(s_1|s_0)^\gamma - 1)c_0}{\bar{e}(s_1)} \Big]^{1/\gamma} \times \\
&\quad \times P\Big(\frac{c_0^1 \bar{e}(s_1)}{\bar{e}(s_0) + (l(s_1|s_0)^\gamma - 1)c_0^1}, s' \Big). \qquad (3.2)
\end{aligned}
$$

Although equation (3.2) looks complicated, it can be, in fact, easily solved. Note also that the above equation does not involve optimization.

Computational algorithm is going to have only two steps.

1. Solve for P defined by equation (3.2) on $(0, \bar{e}(s_0))$;
2. Choose c_0^1 such that $P(c_0^1, s_0) = 0$.

4 Examples

It is an unrealistic but well known setup. State space has only two elements and $e(s_1) = e < 1, e(s_2) = 1 - e$. Endowments of the two agents are perfectly negatively correlated. Initial state is s_1 in which agent 1 receives endowment e. Aggregate endowment $\bar{e}$ is constant and equals unity in each state.

For simplicity it is assumed that $\gamma = 1$, *i.e.* preferences are logarithmic. Discount factor β is equal to 0.95 and endowment in bad state e is 0.25.

4.1 Benchmark

Naturally, in the the benchmark economy agents know the true transition probabilities. In this setup consumption of both agents is constant across time and states. Consumption of the agent 1 is equal to

$$
c^1 = (1-\beta)\Big(e + \frac{\beta}{1-\beta} 0.5\Big) = (1-\beta)e + \beta 0.5 = 0.4875. \qquad (4.1)
$$

Consumption of the agent 2 is equal to $c^2 = 1 - c^1 = 0.5125$.

4.2 Example 1

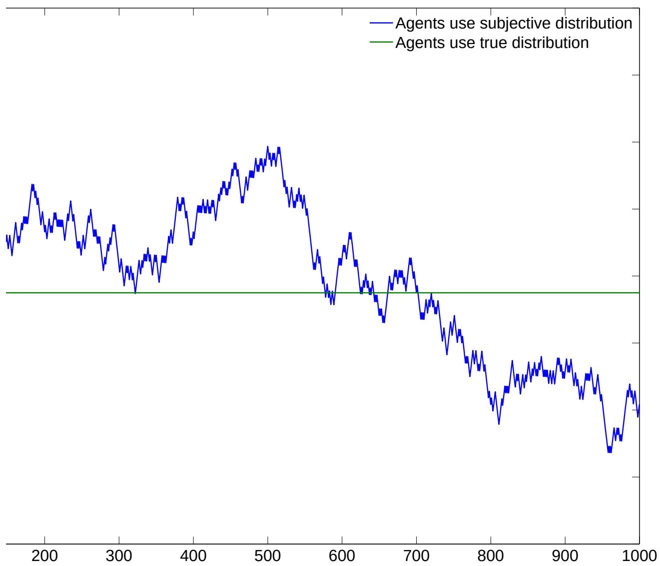
The true transition matrix, and transition matrices 'estimated' by agents 1 and 2 are respectively

$$\pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \hat{\pi}_1 = \begin{bmatrix} 0.5 - x & 0.5 + x \\ 0.5 - x & 0.5 + x \end{bmatrix}, x > 0, \hat{\pi}_2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \tag{4.2}$$

Thus, both agents think that their endowments are i.i.d. as they indeed are. Agent 2 assigns equal probability to each state and her transition matrix coincides with the true one. Agent 1, however, wrongly assigns higher probability to state 2.

Figure 2 plots simulated path of the agent 1's consumption (the blue line). Had the agent 1 used the same transition matrix as the agent 2, consumption path would be constant and equal to 0.4875 (the green line).

Figure 2: Simulated consumption path of agent 1



4.3 Martingale property of consumption

Had agent 1 had the same 'estimate' of transition probability matrix π as agent 2 there would be complete risk sharing and individual consumption would inherit the properties of the aggregate endowment. Here situation changes, however. As pointed out in earlier studies, agent 1 is penalized for using wrong transition probabilities when forming her consumption plan – her consumption and hence wealth is drifting towards zero.

To see this, let $\lambda_t(s^t) := c_t^1(s^t)/e_t(s^t)$ be the share of agent 1's consumption in the aggregate endowment. Then according to equation (2.8)

$$\lambda_{t+1}(s^{t+1}) = \lambda_t(s^t)\frac{l(s^{t+1}|s^t)^\gamma e_t(s^t)}{e_t(s^t) - c_t(s^t) + l(s^{t+1}|s^t)^\gamma c_t(s^t)}.$$

Taking conditional expectation gives

$$\begin{aligned} E[\lambda_{t+1}(s^{t+1})|s^t] &= \lambda_t(s^t)E\Big[\frac{l(s^{t+1}|s^t)^\gamma e_t(s^t)}{e_t(s^t) - c_t(s^t) + l(s^{t+1}|s^t)^\gamma c_t(s^t)}\Big|s^t\Big] \\ &< \lambda_t(s^t)\frac{(E[l(s^{t+1}|s^t)|s^t])^\gamma e_t(s^t)}{e_t(s^t) - c_t(s^t) + (E[l(s^{t+1}|s^t)|s^t])^\gamma c_t(s^t)} \\ &= \lambda_t(s^t), \end{aligned}$$

where the last line uses $E[l(s^{t+1}|s^t)|s^t] = 1$. Thus the share of agent 1's consumption process in the aggregate endowment is a *super-martingale.* This is a positive super-martingale; thus, it must converge according to Doob's martingale convergence theorem. It is not proved here, but I anticipate that the limit is $c_1 = 0$. Note also that $\lambda = 0$ is an absorbing boundary.

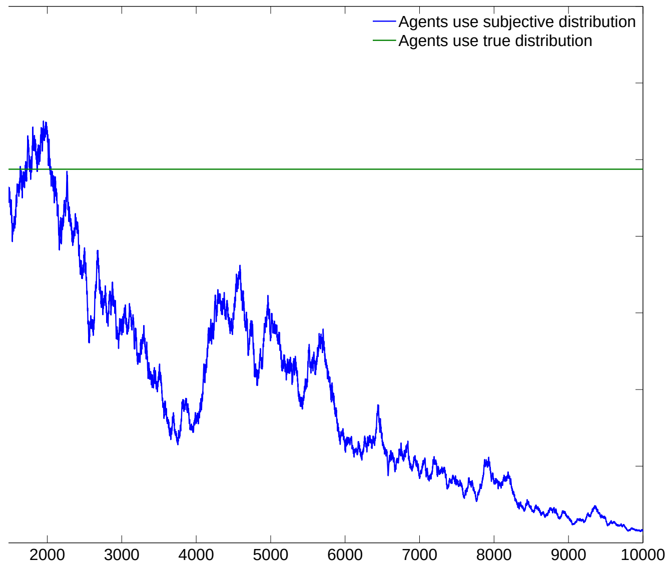
Figure 3 plots the continuation of the consumption path drawn in figure 2. As shown above consumption of agent 1 is drifting towards zero.

4.4 Implications for asset pricing

In this section we show how using aggregate consumption may lead to spurious conclusions. Note that in the example aggregate endowment $\bar{e}$, and thus aggregate consumption $\bar{c}$, is constant and equals 1. Econometrician that observes only aggregate consumption would compute pricing kernel as

$$\hat{Q}_{t+1}(s_{t+1}|s^t) = \beta\frac{u'(\bar{c}_{t+1}(s^{t+1}))}{u'(\bar{c}_t(s^t))} = \beta. \tag{4.3}$$

Figure 3: Simulated consumption path of agent 1: long horizon



However, the true pricing kernel is given by (3.1). So the error that econometrician makes is

$$\frac{\hat{Q}(s_{t+1}|s^t)}{Q(s_{t+1}|s^t)} = \begin{cases} (1 - 2xc_t^1(s^t)), & \text{if } s_{t+1} = 1 \\ (1 + 2xc_t^1(s^t)), & \text{if } s_{t+1} = 2 \end{cases} .$$

In other words, the econometrician *underestimates* pricing kernel's variance and hence risk premium implied by the model. Similarly,

$$E[Q_{t+1}(s_{t+1}|s^t)\ |s^t] = \beta \frac{1 - 4x^2 c_t^1(s^t)}{1 - 4x^2} > \beta,$$

not $E[\hat{Q}_{t+1}(s_{t+1}|s^t)|s^t] = \beta$ as econometrician would get. Thus, econometrician also *overestimates* the risk free rate. The quantitative significance of the error has to be addressed numerically.

4.5 Example 2

In this section I analyze situation in which both agent have wrong but the same estimates of the transition probabilities. Thus,

$$\hat{\pi}_1 = \hat{\pi}_2 = \begin{bmatrix} 0.5 - x & 0.5 + x \\ 0.5 - x & 0.5 + x \end{bmatrix}.$$

In this case consumption of both agents is constant across time and different states. Consumption of agent 1 is equal to

$$c^1 = (1 - \beta)e + \beta 0.5 + \beta x(1 - 2e) = 0.49225 > 0.4875.$$

Thus agent 1 gets higher consumption than in the benchmark case. This happens because all agents think that bad state 1 is less likely relative to the benchmark case; hence prices are twisted so that Arrow securities paying off in bad state are cheaper. This benefits the poor agent 1 who starts off in the bad state because it is cheaper now to insure against bad state in the future.

The correction to agent 1's consumption is $\beta x(1 - 2e) > 0$. It is proportional to x – estimation error of transition probabilities. The larger the error the better for agent 1. The term $(1 - 2e)$ determines direction of the effect: had e been larger than 0.5 state 1 would become a good state and agent 1 would be worse off in the new setup.

5 Learning

It is both fortunate and unfornutate that learning is exogenous to the model, *i.e.* learning dynamics can be solved for separately and then used as a driving process. This fact was used in Veronesi (2000) who built a model with heterogeneous beliefs and learning. To preclude complete learning of the uncertainty Veronesi (2000) assumes that the underlying process for the state switches between one of the few regimes.

Generally solution to the learning problem can be written as $\hat{\pi}^i(s^t) = \hat{\pi}^i_t(s^t)$. To be explicit, suppose that prior distribution over $\hat{\pi}$ is $\beta(1, 1)$. Then posterior distribution after observing k occurrences of state 1 along history s^t is $\beta(1 + k, 1 + t - k)$. It is now possible to solve the model with learning taking the stochastic process for beliefs as exogenous. This is not taken here however. Instead I assume, a much simpler deterministic evolution of $\hat{\pi}^i(s^t)$

$$\hat{\pi}^i(s^t) = x/t \tag{5.1}$$

to emulate learning in the model.

6 Endogenous Labor Supply

Preferences are separable in consumption and labor

$$\sum_{t=0}^{\infty}\sum_{s^t}\hat{\pi}^i(s^t|s_0)\Big[u(c_t^i(s^t)) - v(l_t^i(s^t))\Big], u' > 0, u'' < 0, v' > 0, v'' > 0. \quad (6.1)$$

Firm owns production technology

$$y_t(s^t) = z_t(s^t)l_t^f(s^t), \quad (6.2)$$

where z_t is the aggregate productivity and l^f is the labor input. Productivity z_t follows a discrete first-order Markov process. Firm maximizes present discounted value of its profits

$$\max_{\{l_t(s^t)\}}\sum_{t=0}^{\infty}\sum_{s^t}q_t^0(z_t(s^t))\Big[z_t(s^t)l_t^f(s^t) - w_t(s^t)l_t^f(s^t)\Big], \quad (6.3)$$

which is equivalent to maximizing period by period, state by state profits. For the firm to make finite profit, wage has to satisfy $w_t(s^t) = z_t(s^t)$ at each date and state.

Agent i chooses $\{c_t(s^t), l_t(s^t)\}$ to maximize her life-time utility given by (6.1) subject to a sequence of budget constraints

$$c_t^i(s^t) + \sum_{s_{t+1}} q_{t+1}^t(s^{t+1})a_{t+1}^i(s^{t+1}) = a_t^i(s^t) + l_t^i(s^t)w_t(s^t), \forall t, s^t. \quad (6.4)$$

First order necessary optimality conditions are

$$q_{t+1}^t(s^{t+1}) = \beta\hat{\pi}^i(s^{t+1}|s^t)\frac{u'(c_{t+1}^i(s^{t+1})}{u'(c_t^i(s^t))} \quad (6.5)$$

and

$$v'(l_t^i(s^t)) = w_t(s^t)u'(c_t^i(s^t)). \quad (6.6)$$

As long as there is the complete set of Arrow securities traded at each date (*i.e.* financial markets are complete) the following relation holds

$$\frac{\hat{\pi}^1(s^{t+1}|s^t)}{\hat{\pi}^2(s^{t+1}|z_t)} = \frac{u'(c_{t+1}^1(s^{t+1}))/u'(c_t^1(s^t))}{u'(c_{t+1}^2(s^{t+1}))/u'(c_t^2(s^t))}. \quad (6.7)$$

Thus, if u is homothetic, complete idiosyncratic risk sharing is not optimal.

6.1 Example

Assume that $u(c) = ln(c)$ and $v(l) = l^2/2$. Then, as before,

$$\frac{\hat{\pi}^1(s^{t+1}|s^t)}{\hat{\pi}^2(s^{t+1}|s^t)} = \frac{c_t^1(s^t)/c_{t+1}^1(s^{t+1})}{c_t^2(s^t)/c_{t+1}^2} = \frac{c_t^1(s^t)}{c_{t+1}^1(s^{t+1})}\frac{y_{t+1}(s^{t+1}) - c_{t+1}^1(s^{t+1})}{y_t(s^t) - c_t^1(s^t)}. \quad (6.8)$$

The difference with the previous setup is that now y_t is endogenous. The optimal choice of employment must satisfy

$$l_t^i(s^t) = \frac{w_t(s^t)}{c_t^i(s^t)} = \frac{z_t(s^t)}{c_t^i(s^t)}. \quad (6.9)$$

The next step is to express aggregate output as a function of agent 1's consumption. To this end

$$y_t(s^t) = z_t(s^t)l_t^f(s^t) = z_t(s^t)(l_t^1(s^t) + l_t^2(s^t)) = [z_t(s^t)]^2\Big[\frac{1}{c_t^1(s^t)} + \frac{1}{c_t^2(s^t)}\Big].$$

After using feasibility condition $y_t(s^t) = c_t^1(s^t) + c_t^2(s^t)$, the above equation can be simplified to

$$y_t(s^t) - c_t^1(s^t) = \frac{[z_t(s^t)]^2}{c_t^1(s^t)}. \quad (6.10)$$

Then equation (6.8) becomes

$$l(s^{t+1}|s^t) = \frac{\hat{\pi}^1(s^{t+1}|s^t)}{\hat{\pi}^2(s^{t+1}|s^t)} = \Big[\frac{c_t^1(s^t)}{c_{t+1}^1(s^{t+1})}\frac{z_{t+1}(s^{t+1})}{z_t(s^t)}\Big]^2.$$

which can be easily solved for $c_{t+1}^1(s^{t+1})$:

$$c_{t+1}^1(s^{t+1}) = c_t^1(s^t)\frac{z_{t+1}(s^{t+1})}{z_t(s^t)}[l(s^{t+1}|s^t)]^{-1/2}. \quad (6.11)$$

In the setup in which agents have the same, possibly incorrect, beliefs and $l(s^{t+1}|s^t) = 1$ and complete idiosyncratic risk-sharing results. However, in the present environment with heterogeneous beliefs consumption will be more volatile than the aggregate income to reflect changing 'optimism' about future states.

6.2 Properties of Individual Consumption

The question asked in this section is whether consumption of agent 1, who is endowed with incorrect beliefs, is a sub-martingale as in the previous model with exogenous endowment.

Expected value of future consumption of agent 1 is

$$E[c^1_{t+1}(s^{t+1})|s^t] = \frac{c_t(s^t)}{z_t(s^t)} E\Big[z_{t+1}(s^{t+1})\sqrt{l(s^{t+1}|s^t)}\ \Big| s^t \Big].$$

Thus, $E[c^1_{t+1}(s^{t+1})|s^t] < c_t(s^t)$ only if $E[z_{t+1}(s^{t+1})\sqrt{l(s^{t+1}|s^t)}|s^t] < z_t(s^t)$.

Let us now analyze the simple two-state example studied in example 1. Given the specification of the agents' beliefs in equation (4.2), we have

$$\begin{aligned} E\Big[z_{t+1}(s^{t+1})\sqrt{l(s^{t+1}|s^t)}\ \Big| s^t \Big] &= 0.5(z(1)/\sqrt{1-2x} + z(2)/\sqrt{1+2x}) \\ &\approx 0.5(z(1)(1+x) + z(2)(1-x)) + o(x). \end{aligned}$$

It can be shown that there exists a small $\bar{x} > 0$ such that[2]

$$z(1) < E\Big[z_{t+1}(s^{t+1})\sqrt{l(s^{t+1}|s^t)}\ \Big| s^t \Big] < z(2),\ \forall |x| < \bar{x}. \tag{6.12}$$

It is important to see how endogenous labor supply saved agent 1 from impoverishment. Firstly, note that agent i's income is negatively related consumption

$$y^i_t(s^t) = w_t(s^t) l^i_t(s^t) = [z_t(s^t)]^2 / c^i_t(s^t) \tag{6.13}$$

because when agent's consumption is low she chooses to work more according to (6.9). Note also that this reasoning does not depend on agent being over-optimistic or over-pessimistic but rather being wrong. Suppose agent 1 underestimates probability of state $j \in \{1, 2\}$. Then she must save too little for state j because she thinks it is less likely to occur than it truly is. Then she must also work more in state j and that would be fine according to her calculations if the state j happened less frequently.

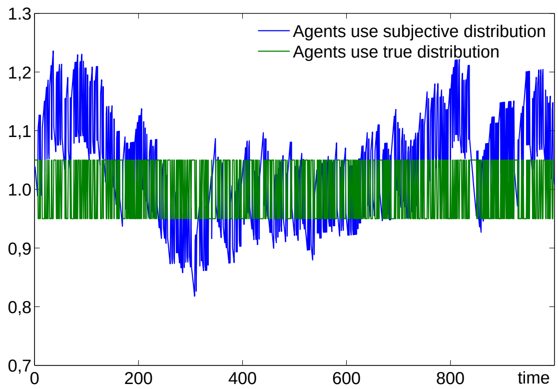
6.3 Simulation Results

For the simulation $z(1) = 0.95, z(2) = 1.05$ is chosen. As expected, consumption in the model with heterogeneous beliefs is more volatile than in

[2] It can be verified that $\bar{x} = ...$ has the desired property.

the benchmark economy. Unlike in example 1 (see figure **??**) consumption of agent 1 is not longer heading south-west but rather fluctuates around the consumption path in the benchmark economy.

Figure 4: Simulated consumption path of agent 1



6.4 Asset Prices

[**TBW**]

6.5 Estimation

[**TBW**]

7 Conclusion

This paper studies survival in the financial markets and asset prices behavior. It is shown that financial survival is a necessary part of the asset prices analysis. If there is a risk that agent's wealth is (stochastically) decreasing over time, it is important to find conditions under which a stationary wealth distribution exists. If agents with incorrect beliefs are driven out of the market and convergence to rational expectations equilibrium obtains then such a model cannot provide us with insight about dynamics of asset prices.

I show that in the economy with simple production agents are not impoverished. This happens because agents are working harder. This is in line with the result obtained by Yan (2006) who finds that convergence to rational expectations equilibrium in the exchange economy is very slow. Hence, very little wealth is lost every period and it is not difficult to make up the lost by working harder.

In the presence of agents with heterogeneous beliefs, individual consumption is more volatile than the aggregate consumption. Thus the econometrician using aggregate consumption instead of individual quantities underestimates the price of risk and overestimate risk free rate.

References

Blume, E. and Easley, D. (1992), 'Evolution and market behavoiur', *Journal of Economic Theory* **58**, 9–40.

Friedman, M. (1953), *Essays in Positive Economics.*

Harrison, J. M. and Kreps, D. M. (1978), 'Speculative investor behavior in a stock market with heterogeneous expectations', *Quarterly Journal of Economics* **92**(2), 323–336.

Ljungqvist, L. and Sargent, T. (2004), *Recursive Macroeconomic Theory.*

Sandroni, A. (2000), 'Do markets favor agents able to make accurate predictions', *Econometrica* **68**(6), 1303–1341.

Veronesi, P. (2000), 'How does information quality affect stock returns?', *Journal of Finance* **55**(2).

Yan, H. (2006), 'Natural selection in financial markets: Does it work?', *wp* .