Macro Theory II, Spring 2009

Homework 1 (due April 1)

Problem 1

Question 1. Prove that every monotone function is measurable.

Question 2. Show that if f is measurable, then the truncation of f :

$$f^a(x) = \begin{cases} a \text{ if } f(x) > a \\ f(x) \text{ if } f(x) \leq a \end{cases}$$

is also measurable.

Question 3. Can you give an example of a non-measurable f such that f^2 is measurable?

Question 4. Explain, in words, what a measurable set is. Why is this property of sets useful?

Problem 2

Consider an economy with a continuum of firms of mass 1, indexed by i. Each firm produces output using capital according to

$$y_{it} = a_{it} k_{it}^{\alpha}$$

where $\alpha \in (0,1)$ is a parameter and a_{it} is the firm's productivity at date t. Assume

$$\log a_{it} = \rho \log a_{it-1} + \varepsilon_{it}, \varepsilon_{it} \sim iidN(0, \sigma^2)$$

This says productivity is uncorrelated across firms so there is no aggregate uncertainty in this economy. Firms choose how much to invest one period in advance, i.e., prior to the realization of ε_{it}, in order to

$$\max_{k_{it+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left(a_{it} k_{it}^{\alpha} - \left(k_{it+1} - (1-\delta) k_{it}\right)\right)$$

where $k_{it+1} - (1-\delta) k_{it}$ is the investment at date t and δ is the rate at which capital depreciates. A firm's initial stock of capital is k_{i0} and productivity is a_{i0}.

Question 1: Derive an aggregate production function $Y_t = F(K_t)$ for this economy relating total output $Y_t = \int_0^1 y_{it} di$ to the aggregate stock of capital $K_t = \int_0^1 k_{it} di$.

Question 2: Assume next that the government levies a firm-specific marginal tax rate τ_i. In other words, the firm now maximizes

$$\max_{k_{it+1}} \sum_{t=0}^{\infty} \beta^t \left((1-\tau_i) a_{it} k_{it}^{\alpha} - \left(k_{it+1} - (1-\delta) k_{it}\right)\right)$$

Derive an aggregate production function for this economy.

Question 3: Assume next that firms are subject to quadratic costs of adjusting capital. In other words, the firm's problem is

$$\max_{k_{it+1}} \sum_{t=0}^{\infty} \beta^t \left(a_{it} k_{it}^{\alpha} - \left(k_{it+1} - (1-\delta) k_{it}\right) - \lambda \left(k_{it+1} - (1-\delta) k_{it}\right)^2\right)$$

Does this economy admit aggregation? If your answer is yes, derive an aggregate production function. If your answer is no, discuss why.

Problem 3

Consider a version of the neoclassical growth model where the economy is inhabited by two types of agents $i = 1, 2$ with measure μ_i where $\mu_1 + \mu_2 = 1$. Agents of type i solve

$$\begin{aligned} & \max_{\{c_{it}, n_{it}\}} \sum_{t=0}^{\infty} \beta^t \frac{\left(c_{it}^{\alpha} \left(1 - h_{it}\right)^{1-\alpha}\right)^{1-\gamma}}{1-\gamma} \\ & s.t. \\ c_{it} + a_{i,t+1} &= a_{i,t} \left[1 + r_t \left(1 - \tau_t\right)\right] + w_t \varepsilon_i h_{it} + T^0, \\ & a_{i0} \text{ given for } i = 1, 2 \end{aligned}$$

where ε_i are efficiency units of labor endowed to type i, τ_t is a capital-income tax rate and T^0 are lump-sum transfers. The representative firm produces with CRS technology $F(K_t, N_t)$ where N_t is aggregate labor input in efficiency units. The government budget constraint is balanced in every period, i.e.,

$$T^0 = \tau_t K_t r_t$$

Finally, the aggregate resource constraint is

$$C_t + K_{t+1} = F(K_t, N_t) + (1 - \delta) K_t.$$

1) Show that in steady-state the distribution of wealth is indeterminate.

2) Does this economy admit a representative agent formulation?

Suppose that the economy is, initially, in a particular steady-state with transfers T^0 and imagine that the government raises transfers to $T^1 > T^0$.

3) Does the new set of steady-states with higher transfers display higher or lower capital stock? Draw the equilibrium dynamics of this economy between steady-states in the space of wealth holding (a_1, a_2) for the two agents. Is the final steady-state uniquely determined? How would you compute the dynamics?

Problem 4

Consider a pure exchange economy, where time is discrete, indexed by $t = 0, 1, 2, ...$ and continues forever. The economy is populated by 2 individuals $i = 1, 2$ with logarithmic period utility $\ln\left(c_t^i\right)$ and discount factor equal to $\beta \in (0, 1)$ who trade a non-storable consumption good c_t. Agents have deterministic endowment streams $\left\{e_t^i\right\}_{t=0}^{\infty}$ of the consumption good given by

$$e_t^i = \begin{cases} 0 \quad if \;\; (t+i) \text{ is even} \\ \\ 2 \quad if \;\; (t+i) \text{ is odd} \end{cases}$$

Agents behave competitively. All markets open at time zero and contracts are traded specifying how many units of consumption good will be exchanged at each time t between the two agents.

1) Define an Arrow-Debreu competitive equilibrium and verify that the first Welfare

Theorem holds.

2) Solve for the Arrow-Debreu equilibrium, i.e., characterize the equilibrium sequences of prices and allocations of consumption good among agents.

3) Define a Pareto optimal allocation for this economy. Write down the Negishi Planner's problem where the Planner gives weight α to agent 1 and $(1-\alpha)$ to agent 2. Solve the Planner's problem for arbitrary weights $\alpha \in (0,1)$.

4) Characterize the transfer function that allows to map any Pareto optimum, for given weights $(\alpha, 1-\alpha)$, into a competitive equilibrium with transfers among agents and determine the specific value for the planner weights that select exactly the competitive equilibrium you have computed in point 2).