1 Heterogeneity in the neoclassical growth model with complete markets

Before we talk about heterogeneity, let's briefly review the representative agent framework.

1.1 Representative agent framework

Consider an economy in populated by a continuum of identical consumers and a continuum of identical firms. Let's normalize their measure to 1 so that averages and sums are equivalent. (this is an innocuous normalization). Let us also assume away uncertainty. Markets are perfectly competitive.

Firms are indexed by $j \in [0,1]$. They have access to a constant returns to scale production technology that transforms capital k_j and labor l_j into output

$$y_{jt} = F(k_{jt}, l_{jt})$$

Firms rent capital and labor from households each period at prices r_t and w_t respectively and sell output to the consumers (the price of output is normalized to 1). Thus firms solve

$$\max_{k_{jt}, l_{jt}} F(k_{jt}, l_{jt}) - r_t k_{jt} - w_t l_{jt}$$

which gives

$$\begin{aligned} w_t &= F_2(k_{jt}, l_{jt}) \\ r_t &= F_1(k_{jt}, l_{jt}) \end{aligned}$$

Consumers (indexed by $i \in [0,1]$) maximize life-time utility and face a consumption-savings tradeoff. Consumers save via investment, x, in capital, k, where k denote the capital holdings of an individual consumer. Consumers derive income from renting capital and labor to firms. The supply of labor is normalized to 1. The consumer's problem is to

$$\max_{x_{it}, c_{it}} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$
$$s.t.$$
$$c_{it} + x_{it} = r_t k_{it} + w_t$$
$$k_{it+1} = (1-\delta) k_{it} + x_{it}$$

Let us conjecture that the aggregate stock of capital in this economy, $K = \int_0^1 k_i di = \int_0^1 k_j dj$ is sufficient to describe the aggregate state of this economy. In other words, we postulate that aggregate prices and quantities are functions of K alone. We will verify this conjecture immediately.

We can write consumer $i's$ problem recursively as

$$\begin{aligned} v_i(k, K) &= \max_x u\left(c\right) + \beta v_i\left(k', K'\right) \\ s.t.\ c &= r(K)k + w(K) - x \\ k' &= \left(1 - \delta\right) k + x \\ K' &= \left(1 - \delta\right) K + X(K) \end{aligned}$$

Consumers take as given the price functions $w(K), r(K)$ as well as $X(K)$. Let $x_i(k, K)$ be the policy function that gives a consumer's optimal decision rule for capital accumulation.

A recursive competitive equilibrium in this economy is a value function $v_i\left(k, K\right)$ for each consumers, policy function $x_i(k, K)$, an aggregate policy function $X(K)$ and factor prices, $r(K)$ and $w(K)$ that satisfy:

1) household's problem, 2) firm profit maximization, 3) consistency of aggregate and individual decision rules: $X(K) = \int_0^1 x_i(k_i, K)di$, 4) law of motion for the aggregate capital stock $K' = \left(1 - \delta\right) K + X(K)$.

We can now verify our original conjecture that K alone is a sufficient state variable. To see this, notice that all firms are identical so that $k_j = K, l_j = 1$ so that $w_t = w(K_t)$ and $r_t = r(K_t)$ from the firm's FOCs. Also, because consumers decision rules and initial capital holdings are the same $X(K) = \int_0^1 x_i(k_i, K)di = x(K, K)$, where we have used the fact that all consumers start with the same initial stock of capital and have identical policy rules and thus $k_i = K$.

Thus all we need to keep track of in this economy is the average stock of capital, K, and not how this capital is distributed across agents. In fact, most expositions of the economy above drop the j and i subscripts altogether, although it is understood that the economy is populated by a large number of identical agents.

More generally, we will say that an economy **admits aggregation** if aggregate prices and quantities do not depend on the distribution of individual quantities across agents. In the economy above this follows trivially because all agents are identical.

Consider what happens however if we assume that households differ in their stock of capital in any given period and let μ denote the measure of k across households. In this case aggregates are function of $\mu : w(\mu), r(\mu)$ and the law of motion of μ becomes part of the description of an equilibrium $\mu'(\mu)$. (for simplicity let's assume $x_i(k, \mu) = x(k, \mu)$ nevertheless). Then, e.g., the equilibrium condition 3 above would read:

$$X(\mu) = \int_0^1 x_i(k_i, \mu)di = \int_0^1 x(k_i, \mu)di = \int x(k, \mu)d\mu(k)$$

(although this condition would not be sufficient as we would have to characterize the evolution of μ). Thus in principle the aggregate stock of capital $K = \int_0^1 k_i di = \int k d\mu(k)$ may not be a sufficient object that summarizes the state of the economy (one important exception is of course if x is affine in k).

Before we move on to studying economies in which the measure of agents over individual states is necessary to characterize equilibria, we first study economies in which agents are no longer identical, but assumptions on preferences/technology nevertheless ensure that aggregates prices and quantities only depend on the aggregate stock of capital (or wealth etc.) and not on the entire distribution of capital (wealth etc.) across agents. Although there is no representative agent in this economies, we will show that we can still define a fictitious "representative agent" that behaves, in equilibrium, as the sum of all individual consumers.

1.2 An aggregation result (Mas-Collel et. al. 4B)

Consider a static economy with I consumers and L goods and assume that each consumer has wealth a_i and standard preferences (potentially different for each consumer) over consumption bundles that lead to demand functions of the form $c_i(p, a_i)$ where p is a vector of prices. Let aggregate demand

$$c(p, a_1, ..., a_I) = \sum_i c_i(p, a_i)$$

As above, computing aggregate demand (and thus equilibrium prices) requires knowing in principle the entire distribution of wealth. We next ask, under what conditions can we write

$$C(p, \sum_i a_i) = \sum_i c_i(p, a_i) \text{ for every } p, \ (a_1, ..., a_I)$$

For this equation to hold, it must the be the case that for any $(a_1, ..., a_I)$ and $(a'_1, ..., a'_I)$ such that $\sum_i a_i = \sum_i a'_i$ we have $\sum_i c_i(p, a_i) = \sum_i c_i(p, a'_i)$.

To see what this implies, start from a given distribution $(a_1, ..., a_I)$ and consider a differential change in wealth $(da_1, ..., da_I)$ such that $\sum_i da_i = 0$. Assuming differentiability of demand functions we have

$$\sum_i \frac{\partial c_i^l(p, a_i)}{\partial a_i} da_i = 0 \text{ for every } l.$$

where c_i^l is the demand for good l by consumer i. The latter equality is satisfied for all redistributions if and only if

$$\frac{\partial c_i^l(p, a_i)}{\partial a_i} = \frac{\partial c_j^l(p, a_j)}{\partial a_j}$$

This says that we need the wealth effect at p to be the same for all consumers at all levels of wealth. A special examples were this property holds if preferences are homothetic ($u(x) = F(f(x))$ where $f(x)$ is homogeneous of degree one and F is a monotone increasing function).

More generally, Proposition 4.B.1 in Mas-Collel et. al. shows that a necessary and sufficient condition for the property above to be satisfied is that preferences admit an indirect utility function

$$v_i(p, a_i) = a_i(p) + b(p)a_i$$

Using Roy's identity one can easily verify that demand functions are affine in a_i (which establishes sufficiency). The indirect utility above is said to be of the Gorman form and preferences refered to as quasi-homothetic preferences: these are preferences for which the wealth-expansion path is linear (Engel curves are affine in wealth). We next turn to an application of this result to the neoclassical growth model.

1.3 Heterogeneity in endowments

We begin by studying a version of the neoclassical growth model with heterogeneity where consumers are different only in terms of their initial endowments of wealth.

1.3.1 The economy

Demographics– The economy is inhabited by N types of infinitely lived agents, indexed by $i = 1, 2, ..., N$. Denote by μ_i the number of agents i and normalize the total number of agents to one, so that averages and aggregates are the same.

Preferences– Preferences are time separable, defined over streams of consumption, given by

$$U = \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right),$$

where the period utility function u belongs to one of the following three classes: log, power, exponential, i.e.

$$u(c) = \begin{cases} \ln(c - \bar{c}) & \text{with } \bar{c} + c > 0, \quad \bar{c} \geq 0 \\ \frac{(c-\bar{c})^{1-\sigma}}{1-\sigma} & \text{with } \bar{c} + c > 0, \quad \bar{c} \geq 0 \\ -\sigma \exp(-\sigma c) & \end{cases} \tag{1}$$

We impose $\bar{c} \geq 0$ for log and power utility to allow for a subsistence level for consumption. Notice that these preferences all satisfy quasi-homotheticity (homotheticity with $\bar{c} = 0$), so we could have directly applied the Gorman result. But it is instructive to proceed from scratch

Markets and property rights– There are spot markets for the final good (which can be used for both consumption and investment) and complete financial markets, i.e. there are no constraints on transfers of income across periods. We will describe a time 0 trading arrangement in which all claims are traded at time 0. See L.S Chapter 8 for equivalence of allocations under this arrangement with that under which agents trade one-period securities each period.

We will let the initial level of wealth, at date 0, differ across agents. Let a_{i0} be the individual wealth of type i at time 0. Then, $a_{i0} = s_{i0}A_0$, where s_{i0} is the share of the firm-value owned by consumer i at time 0.

Technology and firm's problem– The aggregate production technology is $Y_t = f(K_t)$ with f strictly increasing, strictly concave and differentiable. The representative firm owns physical capital and makes the investment decision by solving the problem

$$A_0 = \max_{\{I_t\}} \sum_{t=0}^{\infty} \frac{p_t}{p_0} \left[f(K_t) - I_t\right] \tag{2}$$

$$s.t.$$

$$K_{t+1} = (1-\delta) K_t + I_t,$$

A_0 is the value of the firm, i.e. the present value of future profits, where profits at date t are valued at $\frac{p_t}{p_0}$, the relative price of consumption at dates t and 0.The firm's first order conditions are clearly

$$1 = \frac{p_{t+1}}{p_t}\left[f'\left(K_{t+1}\right) + (1-\delta)\right]. \tag{3}$$

Household's problem– Given complete markets, the maximization problem of household i at time t can therefore be stated as:

$$\max_{\{c_{it}\}} \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t c_{it} \leq p_0 a_{i0} \tag{4}$$

Solution– Consider the log-preferences case. From the FOC of the household problem at time t, we have:

$$\beta^t u'\left(c_{it}\right) = \lambda_i p_t \quad \Rightarrow \quad \beta^t \left(\frac{1}{c_{it} - \bar{c}}\right) = \lambda_i p_t \quad \Rightarrow \quad c_{it} = \frac{\beta^t}{\lambda_i p_t} + \bar{c}, \tag{5}$$

where λ_i is the Lagrange multiplier on the consumer's budget constraint. Notice also

$$\frac{u'\left(c_{it}\right)}{\beta u'\left(c_{it+1}\right)} = \frac{p_t}{p_{t+1}}$$

Substituting the FOC into the budget constraint of (4), we can derive an expression for the multiplier λ_i:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t \left(\frac{\beta^t}{\lambda_i p_t} + \bar{c}\right) &= p_0 a_{i0} \\
\frac{1}{\lambda_i (1-\beta)} + \bar{c} \sum_{t=0}^{\infty} p_t &= p_0 a_{i0} \\
\frac{1}{\lambda_i} &= (1-\beta)\left[p_0 a_{i0} - \bar{c} \sum_{t=0}^{\infty} p_t\right]
\end{aligned} \tag{6}$$

Let's now substitute the expression on the last line into $c_{it} = \frac{\beta^t}{\lambda_i p_t} + \bar{c}$ in order to solve explicitly for c_{it} :

$$\begin{aligned}
c_{it} &= \frac{\beta^t}{p_t}\left[(1-\beta) p_0 a_{i0} - (1-\beta) \bar{c} \sum_{\tau=0}^{\infty} p_\tau\right] + \bar{c} \\
&= -\bar{c}\left[(1-\beta) \beta^t \sum_{\tau=0}^{\infty} \left(\frac{p_\tau}{p_t}\right) - 1\right] + (1-\beta) \beta^t \frac{p_0}{p_t} a_{i0}
\end{aligned} \tag{7}$$

Finally, let a_{it} denote the period t wealth of consumer i expressed in units of date t consumption. From the household's budget constraint, the law of motion for wealth is

$$a_{it+1} = \frac{p_t}{p_{t+1}} \left(a_{it} - c_{it}\right) \tag{8}$$

which one can then use to show

$$c_{it} = -\bar{c} \left[(1-\beta) \sum_{\tau=\tau}^{\infty} \left(\frac{p_\tau}{p_t} \right) - 1 \right] + (1-\beta) \, a_{it}$$

or

$$c_i = \Theta\left(p^t, \bar{c}\right) + (1-\beta) \, a_{it}, \tag{9}$$

where $\Theta\left(p^t, \bar{c}\right)$ is a function of the subsistence level and of the whole price sequence $p^t = \{p_t, p_{t+1}, ...\}$. Thus, we have the optimal individual consumption rule

$$\Theta\left(p^t, \bar{c}\right) \equiv -\bar{c} \left[(1-\beta) \sum_{\tau=t}^{\infty} \left(\frac{p_\tau}{p_t} \right) - 1 \right] \tag{10}$$

Thus consumption is, as we should have expected given the discussion above, an *affine function* of asset holdings at time t for each type i. Even though we have only derived it for the log-case, it is easy to check that this representation of the consumption function holds also for the other two classes of preferences (power and exponential utility).

1.3.2 Equilibrium aggregate dynamics

Denote aggregate variables with capital letters. From (9), we derive easily that aggregate consumption only depends on aggregate variables (prices and aggregate wealth), but it is independent of the distribution of wealth. By summing over i on the LHS and RHS of (9) with weights μ_i we arrive at:

$$C_t = \Theta\left(p^t, \bar{c}\right) + (1-\beta) \, A_t. \tag{11}$$

Since equilibrium aggregate consumption C_t has the same form as individual optimal consumption choice c_{it}, it is clear that (11) can be obtained as the solution to the following representative agent problem:

$$\max_{\{C_\tau\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau)$$

$$s.t. \tag{PP}$$

$$\sum_{\tau=t}^{\infty} p_\tau C_\tau \leq p_t A_t$$

which is exactly as in (4) but we have replaced small letters with capital letters. Let's make some further progress on the solution. From the FOCs

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{p_t}{p_{t+1}}. \tag{12}$$

From (12) and the FOC for the firm's problem (3), we obtain the familiar Euler equation of the neoclassical growth model

$$u'(C_t) = \beta u'(C_{t+1}) \left[f'(K_{t+1}) + (1-\delta) \right]. \tag{13}$$

We can state our first important result:

Result 1.1: If preferences are quasi-homothetic and agents are heterogeneous in initial endowments, the aggregate dynamics of the neoclassical growth model with complete markets admit a single-agent representation. Put differently, the dynamics of aggregate quantities and prices are exactly the same as in the standard neoclassical growth model with representative agent.

Steady-state– The dynamics of the economy will converge to the steady-state values of capital stock satisfying the modified golden rule $f'(K^*) = 1/\beta - (1-\delta)$. Note now that in steady-state $p_t/p_{t+1} = 1/\beta$ for all t, hence from the definition of $\Theta(p^t, \bar{c})$ in (**??**) we conclude that $c_i = (1-\beta) a_i$. In other words, in steady-state, the average propensity to save is β, independently of wealth, for every type of household.

To conclude, in the neoclassical growth model with complete markets and where agents have heterogeneous wealth endowments, the dynamics of the aggregate variables do not depend on the evolution of the wealth distribution. But is the inverse statement true? Does the evolution of the wealth distribution across households (i.e., wealth inequality) depend on the dynamics of aggregate variables (prices and quantities)? We show below that the answer is: yes, it does.

1.3.3 Equilibrium dynamics of the wealth distribution

From the law of motion for a household's wealth above we have

$$a_{it+1} = \frac{p_t}{p_{t+1}}(a_{it} - c_{it}) \tag{14}$$

$$\frac{a_{it+1}}{a_{it}} = \left(\frac{p_t}{p_{t+1}}\right)\left(1 - \frac{c_{it}}{a_{it}}\right), \tag{15}$$

which expresses the growth rate of wealth for type i as a function of her consumption-wealth ratio.

By aggregating over types in equation (14), we can obtain an equivalent equation at the aggregate level:

$$A_{t+1} = \frac{p_t}{p_{t+1}}(A_t - C_t)$$

$$\frac{A_{t+1}}{A_t} = \left(\frac{p_t}{p_{t+1}}\right)\left(1 - \frac{C_t}{A_t}\right)$$

We want to establish conditions under which an individual's share of total wealth will grow over time, i.e. $s_{it+1} > s_{it}$. First of all, note that:

$$\frac{s_{it+1}}{s_{it}} > 1 \quad \Leftrightarrow \quad \frac{a_{it+1}}{a_{it}} > \frac{A_{t+1}}{A_t} \quad \Leftrightarrow \frac{c_{it}}{a_{it}} < \frac{C_t}{A_t} \tag{16}$$

Moreover, from equations (9) and (11),

$$\frac{c_{it}}{a_{it}} = \frac{\Theta(p^t, \bar{c})}{a_{it}} + (1-\beta), \quad \text{and} \quad \frac{C_t}{A_t} = \frac{\Theta(p^t, \bar{c})}{A_t} + (1-\beta)$$

and therefore

$$\frac{c_{it}}{a_{it}} < \frac{C_t}{A_t} \quad \Leftrightarrow \quad \frac{\Theta(p^t, \bar{c})}{a_{it}} < \frac{\Theta(p^t, \bar{c})}{A_t} \quad \Leftrightarrow \quad \Theta\left(p^t, \bar{c}\right)(a_{it} - A_t) > 0$$

and, thus, summarizing we have the following equivalence (i.e., "if and only if") condition:

$$\frac{s_{it+1}}{s_{it}} > 1 \quad \Leftrightarrow \quad \Theta\left(p^t\right)(a_{it} - A_t) > 0,$$

which means that whether consumer's i wealth share is increasing or decreasing over time depends on 1) the sign of the constant Θ (equal for everyone) and 2) on her relative position in the distribution. For example, if $\Theta > 0$ then for a consumer whose initial wealth is above average, her share will grow, whereas for a consumer whose initial wealth is below average, her share will fall. And hence the distribution will become more unequal over time. Note that the dynamics of the wealth distribution depend on the entire sequence of prices, hence on the dynamics of aggregate variables in equilibrium.

First of all, in absence of subsistence level, $\bar{c} = 0$ we have $\Theta = 0$, and therefore the neoclassical growth model with heterogeneous endowments has a sharp prediction for the evolution of inequality.

Result 1.2: In the neoclassical growth model with complete markets, homothetic preferences, heterogeneous endowments, but without subsistence level ($\bar{c} = 0$), the wealth distribution remains unchanged along the transition path, i.e. initial conditions in endowments (and inequality) persist forever.

The intuition is that if $\bar{c} = 0$ then $c_i = (1 - \beta) a_i$, so the average propensity to consume, and save, is the same for every agent. Every agent accumulates wealth at the same rate.

In presence of a subsistence level, the dynamics are more interesting. We now determine the sign of Θ, through:

Lemma 1.1 (Chatterjee, 1994): *The common constant term of the consumption function $\Theta(p^t, \bar{c})$ is greater (less) than zero if and only if the economy is converging from below (above) to the steady-state, i.e. if $K_\tau < (>) K^*$.*

Proof: *Suppose the economy grows towards the steady-state, i.e. $K_\tau < K^*$. From equation (13), the sequence $\{f'(K_\tau)\}$ is decreasing and the sequence $\{p_\tau/p_{\tau+1}\}$ will be decreasing towards $1/\beta$. Therefore, the sequence $\{p_{\tau+1}/p_\tau\}$ is increasing towards β, i.e., $p_{\tau+1}/p_\tau \leq \beta$ for all $\tau \geq t$ where the strict inequality holds at least for some τ. It follows that*

$$p_\tau/p_t = (p_\tau/p_{\tau-1})(p_{\tau-1}/p_{\tau-2}) \ldots (p_{t+2}/p_{t+1})(p_{t+1}/p_t) < \beta^{\tau-t}.$$

From the definition of $\Theta(p^t, \bar{c})$ in (7), use the above equation to obtain

$$\Theta\left(p^t, \bar{c}\right) = -\bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right) - 1\right] > -\bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t} - 1\right] = 0$$

QED

The implications for the evolution of the wealth distribution in an economy growing towards the steady-state (the empirically interesting case) are easy to determine, at this point. In the presence of a subsistence level ($\bar{c} > 0$), $\Theta > 0$. From equation (16) this implies that the average propensity to consume (save) declines (increases) with wealth: poor agents must consume proportionately more out of their wealth to satisfy the subsistence level. In other words:

Result 1.3: In the neoclassical growth model with complete markets, homothetic preferences, heterogeneous endowments and subsistence level $\bar{c} < 0$, *as the economy grows towards the steady-state: (i) the wealth distribution becomes more unequal, as rich agents accumulate more than poor agents along the transition path, and (ii) there is no change in the ranking of households in the wealth distribution, i.e., initial conditions in wealth (and consumption) ranking persist forever.*

The main conclusion of this lecture is that in this model there is no economic or social mobility. This is not a good model to understand why some individual are born poor and make it in life, while other are born rich and end up poor as rats. This is just a model of castes.

Robustness– We now discuss how robust this result is to two of the key assumptions made so far in the analysis: 1) all agents have same discount factor β, 2) markets are complete.

- 1. When agents have different discount factors, then none of the results hold any longer. Suppose that $\bar{c} = 0$ to simplify the analysis. Then, from (9)

  $$c_{it} = (1 - \beta_i)\, a_{it},$$

  therefore the average propensity to save out of wealth is higher the more patient is the individual and from (16), wealth grows faster for the more patient individuals. In the limit, in steady-state, the most patient type holds all the wealth, and the distribution becomes degenerate.

  2. In absence of markets and trade (autarky), every consumer has access to her own technology. Each agent i will solve his own maximization problem in isolation

  $$\max_{\{c_{it}\}} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$
  $$s.t.$$
  $$k_{it+1} = (1-\delta)\, k_{it} + f(k_{it}) - c_{it}$$
  $$k_{it} \text{ given}$$

with different initial conditions k_{it}. It is easy to see that, independently of initial conditions, each agent will converge to the same capital stock k^*, hence in the long-run the distribution of wealth is perfectly equal. Interestingly, we conclude that less developed financial markets induce less wealth inequality, in the long-run.

1.3.4 Indeterminacy of the wealth distribution in steady-state

One very important implication is that in steady-state the wealth distribution is indeterminate. From (2), (13) and (9), the set of equations characterizing the steady-state is:

$$\begin{aligned} f'(K^*) &= 1/\beta - (1-\delta), \\ A^* &= \frac{1}{1-\beta}\left[f(K^*) - \delta K^*\right] \\ \sum_{i=1}^{N} \mu_i a_i &= A^*, \\ c_i &= (1-\beta)\, a_i,\ i = 1, 2, ..., N \end{aligned}$$

We therefore have $(N+3)$ equations and $(2N+2)$ unknowns $\left(\{c_i, a_i\}_{i=1}^{N}, K^*, A^*\right)$. In other words, the multiplicity of the steady-state wealth distributions is of order $N-1$.[1]

However, suppose we start from a given wealth distribution at date $t=0$ when the economy has not yet reached its steady-state, then the dynamics of the model are uniquely determined by Results 1.2 and 1.3 and the final steady-state distribution is determined as well. So, let's restate our finding in:

Result 1.4: In the steady-state of the neoclassical growth model with N agents, heterogeneous initial endowments and homothetic preferences, there is a continuum with dimension $(N-1)$ of steady-state wealth distributions. However, given an initial wealth distribution $\{a_{i0}\}_{i=1}^{N}$ at $t=0$, the equilibrium wealth distribution $\{a_{it}\}_{i=1}^{N}$ in every period t is uniquely determined, and so is the final steady-state distribution.

Thus it is important to distinguish "steady-state" from "equilibrium path". In this economy, the equilibrium path is always unique (given initial conditions), but the steady-state is not.

[1] This means that, if $N=1$ (representative agent), the steady-state is unique. If $N=2$, there is a continuum of steady-states of dimension 1, and so on.

2 The Negishi Approach

Negishi (1960) suggested a method to calculate the competitive equilibrium (CE) prices and allocations of complete markets economies (in particular, economies for which the first welfare theorem holds) with heterogeneous households. This method is particularly useful for those economies where aggregation does not go through and, hence we cannot use the representative agent.[2]

From the first welfare theorem, we know that any CE is a Pareto optimum (PO), hence it can be found as the solution to a social planner problem with "some" Pareto weights given to each agent. Suppose we want to compute a particular CE of an economy where agents are initially endowed with heterogeneous shares $\{s_{i0}\}_{i=1}^{N}$ of the aggregate wealth. Can we use the planner problem for this purpose? Negishi showed that the key is to search for the "right" weights given to each type of agent in the social welfare function of the planner. Each set of weights corresponds to a Pareto efficient allocation, the key is to find the set of weights which correspond to our desired CE allocation. As an aside, the concept of social welfare was introduced by Samuelson (1956).

2.1 An Example

Consider our neoclassical growth model of section (1.3) with two types of consumers ($N = 2$). The agent's i problem in the decentralized Arrow-Debreu equilibrium can be written as

$$\max_{\{c_{it}\}} \sum_{t=0}^{\infty} \beta^t u\left(c_{it}\right)$$
$$s.t.$$
$$\sum_{t=0}^{\infty} p_t c_{it} \leq p_0 a_{i0}$$

Let's assign the property rights on capital to the firm, so the firm's problem is exactly the one of the previous section.

From the FOC of the agent of type i, we obtain

$$FOC\left(c_{it}\right) \longrightarrow \beta^t u'\left(c_{it}\right) = \lambda_i p_t,$$

[2] In his original paper, Negishi (1960) used this equivalence result to propose a simple way to show existence of competitive equilibria.

where λ_i is the multiplier on the time zero budget constraint. Thus, putting together the FOC's for the two types:

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\lambda_1}{\lambda_2}. \tag{17}$$

Now, write down the following Negishi planner problem (NP) for our economy

$$\max_{\{c_{1t}, c_{2t}, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 u(c_{1t}) + \alpha_2 u(c_{2t})\right] \tag{NP}$$

$$s.t.$$

$$c_{1t} + c_{2t} + K_{t+1} \leq f(K_t) + (1-\delta) K_t$$

$$K_0 \text{ given}$$

where (α_1, α_2) are the planner's weights for each type of household in the social welfare function.

The FOC's for this problem are

$$FOC(c_{it}) \longrightarrow \alpha_i \beta^t u'(c_{it}) = \theta_t,\ i = 1, 2 \tag{18}$$

$$FOC(K_t) \longrightarrow \theta_t = \beta \theta_{t+1} \left[f'(K_{t+1}) + (1-\delta)\right] \tag{19}$$

where θ_t is the Lagrange multiplier on the planner's resource constraint at time t. Note that putting together the first-order conditions for consumption for the two agents we arrive at

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\alpha_2}{\alpha_1}, \tag{20}$$

which tells us that the planner allocates consumption proportionately to the weight it gives to each consumer (with strictly concave utility).[3]

If we want the NP to deliver the same solution as the CE, we need the PO allocations and the CE allocations to be the same. Given strict concavity of preferences, this implies that, putting together (20) and (17):

$$\frac{\alpha_2}{\alpha_1} = \frac{\lambda_1}{\lambda_2}$$

Hence, the relative weights of the planner must correspond to the inverse of the ratio of the Lagrange multipliers on the time-zero Arrow-Debreu budget constraint for the two agents in the CE.

[3] Note also that the ratio of marginal utility across agents is kept constant in every period (a key features of complete markets allocations, also called *full insurance*).

In particular, for log preferences $u(c_{it}) = \log c_{it}$, we derived in equation (6) that

$$\frac{1}{\lambda_i} = (1-\beta) p_0 a_{i0}$$

therefore we obtain that

$$\frac{\alpha_2}{\alpha_1} = \frac{a_{20}}{a_{10}}.$$

In other words, the weights that ensure that the planner's problem gives the same allocations as the competitive equilibrium are proportional to the initial wealth shares: the planner must deliver more to consumption to the agent who has a large initial share of wealth in the decentralized equilibrium.

Result 1.5: Consider an economy with agents heterogeneous in endowments where the First Welfare Theorem holds. Then, the competitive equilibrium allocations can be computed through an appropriate planner's problem where the relative weights on each agent in the social welfare function are proportional to the relative individual endowments: those agents who initially have more wealth will get a higher weight in the planner's problem.

It is also useful to compare the capital accumulation equation in the planner's problem:

$$\frac{\theta_t}{\theta_{t+1}} = f'(K_{t+1}) + (1-\delta).$$

and compare it to that in the CE:

$$\frac{p_t}{p_{t+1}} = f'(K_{t+1}) + (1-\delta).$$

Hence, we have

$$\frac{\theta_t}{\theta_{t+1}} = \frac{p_t}{p_{t+1}},$$

in other words, the Arrow-Debreu equilibrium prices can be uncovered as the sequence of Lagrange multipliers in the Pareto problem: intuitively, the multipliers gives us the shadow value of an extra unit of consumption and, in the CE, prices signal exactly this type of scarcity. Using p_0 as the numeraire and imposing the normalizations $p_0 = \theta_0 = 1$, the above relationship implies that $p_t = \theta_t$ so equilibrium prices are exactly equal to the shadow prices of consumption in the planner's problem.

In conclusion, we have uncovered a tight relation between weights of the NP problem and initial endowments in the CE and an equivalence between Lagrange multiplier on the resource constraint of the NP problem and prices in the CE. This strict relationship, that we have uncovered for the log utility case, is true more in general.

In general, without restrictions on preferences, one may not have closed form solutions for the λ's in the CE. Then one solves for the λ that ensure that each individual's budget constraint is satisfied (by exploring the equivalence of multiplier in the planner's problem and prices in the CE to guarantee that the allocations the planner assigns to each agent would be affordable in a CE). A detailed algorithm follows:

2.2 General application of the Negishi method

In general, without restrictions on preferences, one may not have closed form solutions for the λ's in the CE, so the algorithm is a little more involved. The objective is to compute the CE allocations for an economy with N types of agents and endowment distribution $\{a_{i0}\}_{i=1}^{N}$. We can describe the algorithm in four steps:

1. In the social planner problem (NP), guess a vector of weights $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_N\}$. The relative weights only matter so one can restrict the guess to $\Sigma_i \alpha_i = 1$.

2. Compute the sequence of allocations $\left\{\{c_{it}\}_{i=1}^{N}, K_t\right\}_{t=0}^{\infty}$ and the implied sequence of multipliers $\{\theta_t\}_{t=0}^{\infty}$ on the resource constraint in each period t. In practice, at every t, one needs to solve the $N+2$ equations

$$\begin{aligned} \alpha_i \beta^t u'(c_{it}) &= \theta_t,\ i = 1, .., N \\ \sum_{i=1}^{N} \mu_i c_{it} + K_{t+1} &= f(K_t) + (1-\delta) K_t \\ \frac{\theta_t}{\theta_{t+1}} &= f'(K_{t+1}) + (1-\delta) \end{aligned}$$

in $N+2$ unknowns $\left(\{c_t^i\}_{i=1}^{N}, K_{t+1}, \theta_{t+1}\right)$. At every t, (K_t, θ_t) are given (recall θ_0 can be normalized to 1), therefore the Negishi method simplifies enormously the computation of the equilibrium: the Negishi solution requires solving, for every time t, a small system of equations. Recall that, instead, to solve for the CE allocations, at every time t one must set the excess demand function to zero and the excess demand function depends on the entire price sequence—an infinitely dimensional object. To understand, take another look at the consumption allocation (9) where

$\Theta(\cdot)$ depends on the entire price sequence from t onward.

Instead of guessing (and iterating over) an infinite sequence of prices, one guesses and iterates over a finite set of weights.

3. Exploit the equivalence between prices p_t and multipliers θ_t to verify whether the time-zero Arrow-Debreu budget constraint of each agent holds exactly at the guessed vector of weight $\boldsymbol{\alpha}$. Specifically, for each agent, compute the implicit transfer function $\tau_i(\boldsymbol{\alpha})$ associated with the assumed vector of weights

$$\tau_i(\boldsymbol{\alpha}) = \sum_{t=0}^{\infty} \theta_t c_{it} - \theta_0 a_{i0}, \text{ for every } i = 1, 2, ..., N \tag{21}$$

and if (21) holds for agent i with a "greater than" sign, it means that the planner is giving too much weight to agent i. So, in the next iteration reduce the weight α^i given to agent i.

4. Iterate over $\boldsymbol{\alpha}$ until you find the vector of weights $\boldsymbol{\alpha}^*$ that sets *every* individual transfer function $\tau_i(\boldsymbol{\alpha}^*)$ to zero. This vector corresponds to the PO allocations that are affordable by each agent in the CE, given their initial endowment, without the need for any transfer across-agents. Thus, we are computing exactly the CE associated to initial conditions $\{a_{i0}\}_{i=1}^{N}$.

See also Ljungqvist-Sargent, section 8.5.3, for a discussion of the Negishi algorithm.

2.3 "Non-Gorman" aggregation

Maliar and Maliar (2001, 2003) make use of the Negishi approach to prove a more general aggregation result. In their model, agents have non-homothetic preferences in consumption and leisure, and are subject to idiosyncratic, but insurable, shocks to labor endowment.

They prove that the strong Gorman aggregation fails, but one can obtain a weaker aggregation result. The aggregate dynamics of the model can still be described by a representative agent (RA). However, the RA's preferences are different from preferences of the individual consumer. There exists a new preference shifter for the RA that depends on the distribution of individual productivity shocks. Therefore, the dynamics of aggregate variables do depend on the distribution of the shocks.

2.3.1 The Model

Demographics– The economy is inhabited by a continuum of infinitely lived agents, indexed by $i \in [0, 1]$.

Uncertainty– Agents are subject to idiosyncratic productivity shocks to skills. Let ε_{it} be the shock of agent i, and suppose shocks are iid, with mean 1, and defined over the set E. This is not necessary, but it simplifies the notation.

Preferences– Preferences are time separable, defined over streams of consumption, given by

$$U = \sum_{t=0}^{\infty} \beta^t u\left(c_{it}, 1 - n_{it}\right).$$

where period utility is given by

$$u\left(c_t, 1 - n_t\right) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + \psi \frac{\left(1 - n_t\right)^{1-\sigma} - 1}{1 - \sigma} \tag{22}$$

and note that preferences are not quasi-homothetic, unless $\sigma = \gamma$. [4]

Markets and property rights–There are spot markets for the final good (which can be used for both consumption and investment) whose price is normalized to one, and complete financial markets, i.e. agents can trade a full set of state-contingent claims. The agent's portfolio is composed by Arrow securities of the type $a_{it+1}(\varepsilon)$ which pay one unit of consumption at time $t + 1$ if the individual's shock is ε and zero otherwise. Let $p_t(\varepsilon)$ the price of this security and $\int p_t(\varepsilon)\, a_{it+1}(\varepsilon)\, d\varepsilon$ the value of such portfolio for agent i. Assume now that households own capital and rent it to firms, although clearly, with complete markets, this is equivalent to the opposite case studied earlier.

Technology and firm's problem– The aggregate production technology is $Y_t = Z_t f\left(K_t, H_t\right)$ with f strictly increasing, strictly concave and differentiable. The representative firm rents capital from households. H_t is aggregate labor input in efficiency units, i.e. $H_t = \int_I \varepsilon_{it} n_{it} di$.

[4] To see this, ignore the dynamics and pretend that the consumer solves $\max u(c, 1 - n)$ s.t. $c + w(1 - n) = a$. The first-order condition reduces to $w = A\frac{(1-n)^{-\sigma}}{c^{-\gamma}}$ or $(1 - n) = w^{-\frac{1}{\sigma}} \psi^{\frac{1}{\sigma}} c^{\frac{\gamma}{\sigma}}$, which when you substitute back into the budget constraint shows that c and thus n are not affine in a.

Household problem– For agent i:

$$\max_{\{c_{it},k_{it+1},a_{t+1}(\varepsilon)\}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_{it}, 1-n_{it}\right) \tag{23}$$

$$s.t.$$

$$c_{it} + k_{it+1} + \int p_t\left(\varepsilon\right) a_{it+1}\left(\varepsilon\right) d\varepsilon = \left(1-\delta\right) k_{it} + w_t \varepsilon_{it} n_{it} + a_{it}\left(\varepsilon_{it}\right)$$

$$k_{i0}, a_{i0} \text{ given}$$

Equilibrium– This is a CM economy. The First Welfare Theorem tells us that the equilibrium is PO, so we can use a social planner problem to characterize the equilibrium by applying the Negishi method. The key, as usual, is to find the right weights that guarantee that allocations are affordable for each agent, given their initial endowments.

Aggregation?– Given that preferences are not homothetic, we know that Gorman's strong aggregation concept will not hold. But can we, nevertheless, obtain a RA whose choices describe the evolution of the aggregate economy? And how the preferences of the RA look like?

Planner's problem:

$$\max_{\{c_{it},n_{it},K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[\int_0^1 \alpha_i u\left(c_{it}, 1-n_{it}\right) di\right] \tag{NP}$$

$$s.t.$$

$$\int_0^1 c_{it} di + K_{t+1} \leq Z_t f\left(K_t, H_t\right) + \left(1-\delta\right) K_t$$

$$H_t = \int_0^1 \varepsilon_{it} n_{it} di \tag{24}$$

$$K_0 \text{ given}$$

From the FOC of individual i in the Negishi problem:

$$\begin{aligned} \alpha_i\left(c_{it}\right)^{-\gamma} &= \theta_t \\ \alpha_i \psi\left(1-n_{it}\right)^{-\sigma} &= \theta_t Z_t f_2(K_t, H_t) \varepsilon_{it} \end{aligned} \tag{25}$$

Rearranging gives

$$\begin{aligned} c_{it} &= \left(\frac{\alpha_i}{\theta_t}\right)^{\frac{1}{\gamma}} \\ \left(1-n_{it}\right)\varepsilon_{it} &= \left(\frac{\psi \alpha_i}{\theta_t f_2(K_t, H_t)}\right)^{\frac{1}{\sigma}} \left(\varepsilon_{it}\right)^{1-1/\sigma} \end{aligned}$$

and note that consumption of individual i is proportional to its weight. Leisure is directly proportional to its weight and inversely proprtional to individual productivity: efficiency arguments induce the planner to make high-productivity individuals work harder.

Integrating the two FOCs across agents gives

$$\begin{aligned} C_t &= \int_0^1 c_{it} di = \int_0^1 \left(\frac{\alpha_i}{\theta_t}\right)^{\frac{1}{\gamma}} di \qquad (26) \\ 1 - H_t &= 1 - \int_0^1 \varepsilon_{it} n_{it} di = 1 - \int_0^1 \left(\frac{\psi\alpha_i}{\theta_t Z_t f_2(K_t, H_t)}\right)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di \end{aligned}$$

Now, note that

$$c_{it} = \frac{\left(\frac{\alpha_i}{\theta_t}\right)^{\frac{1}{\gamma}}}{\int_0^1 \left(\frac{\alpha_i}{\theta_t}\right)^{\frac{1}{\gamma}} di} C_t \Rightarrow c_{it} = \frac{(\alpha_i)^{\frac{1}{\gamma}}}{\int_0^1 (\alpha_i)^{\frac{1}{\gamma}} di} C_t \qquad (27)$$

$$\begin{aligned} (1 - n_{it})\,\varepsilon_{it} &= \left(\frac{\psi\alpha_i}{\theta_t Z_t f_2(K_t, H_t)}\right)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} \Rightarrow (1 - n_{it}) = \left(\frac{\psi\alpha_i}{\theta_t Z_t f_2(K_t, H_t)}\right)^{\frac{1}{\sigma}} (\varepsilon_{it})^{-1/\sigma} \\ &\Rightarrow (1 - n_{it}) = \frac{(\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{-1/\sigma}}{\int_0^1 (\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di} (1 - H_t) \qquad (28) \end{aligned}$$

Now, consider the social welfare function

$$\int_0^1 \alpha_i \frac{(c_{it})^{1-\gamma} - 1}{1-\gamma} + \psi\alpha_i \frac{(1 - n_{it})^{1-\sigma} - 1}{1-\sigma} d\mu_i$$

and substitute the two expressions in (27) into the social welfare function:

$$\begin{aligned} &\int_0^1 \alpha_i \frac{\left[\frac{(\alpha_i)^{\frac{1}{\gamma}}}{\int_0^1 (\alpha_i)^{\frac{1}{\gamma}} di} C_t\right]^{1-\gamma} - 1}{1-\gamma} + \psi\alpha_i \frac{\left[\frac{(\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{-1/\sigma}}{\int_0^1 (\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di} (1 - H_t)\right]^{1-\sigma} - 1}{1-\sigma} d\mu_i \\ &= \frac{\frac{\int_0^1 (\alpha_i)^{\frac{1}{\gamma}} di}{\left[\int_0^1 (\alpha_i)^{\frac{1}{\gamma}} di\right]^{1-\gamma}} C_t^{1-\gamma} - 1}{1-\gamma} + \psi \frac{\frac{\int_0^1 (\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di}{\left[\int_0^1 (\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di\right]^{1-\sigma}} (1 - H_t)^{1-\sigma} - 1}{1-\sigma} \end{aligned}$$

which yields the utility for the RA

$$\frac{C_t^{1-\gamma} - 1}{1-\gamma} + \psi X_t \frac{(1 - H_t)^{1-\sigma} - 1}{1-\sigma}$$

where

$$X_t = \frac{\left[\int_0^1 (\alpha_i)^{\frac{1}{\sigma}} (\varepsilon_{it})^{1-1/\sigma} di\right]^{\sigma}}{\left[\int_0^1 (\alpha_i)^{\frac{1}{\gamma}} di\right]^{\gamma}}$$

Therefore the RA problem which describes the aggregate allocations for this economy becomes:

$$\begin{aligned} & \max_{\{C_t, H_t, K_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} + \psi X_t \frac{(1-H_t)^{1-\sigma} - 1}{1-\sigma} \\ & s.t. \\ C_t + K_{t+1} &= (1-\delta) K_t + Z_t f(K_t, H_t) \\ & K_0 \text{ given} \end{aligned} \tag{29}$$

Some remarks are in order:

1. We have found a RA, but its preferences are not the ones of the individual agent. Note that we have H_t instead of N_t, and note that we have a new preference shifter X_t. First reason why this is not Gorman aggregation.

2. The preference shifter, in general, *depends on the distribution of shocks and endowments*, therefore aggregate dynamics do depend on the distribution. Second reason why this is not Gorman's aggregation. Note, however, that the distribution is exogenous: it's a simple problem.

3. Suppose $\gamma = \sigma$. Then utility is quasi-homothetic. If there are no skill shocks, but only differences in endowments, then $X_t = 1$ and $H_t = N_t$. We are back to Gorman's aggregation. If there are idiosyncratic shocks, then $X_t \neq 1$ and Gorman aggregation fails, which establishes that you need heterogeneity to be only in wealth for Gorman aggregation to hold.

4. Suppose $\gamma \neq \sigma$. Then utility is not quasi-homothetic. Even if there are no skill shocks, but only differences in endowments, then X_t depends on the distribution of endowments and Gorman's aggregation fails.

Finally, note that the assumption that agents can trade a full set of claims contingent on all possible realizations of idiosyncratic labor productivity shocks is not very realistic, as we will argue later in the course. It is mainly a useful theoretical benchmark.

2.4 Aggregation with indivisibilities and lotteries

This follows Rogerson (1988). We next study a static economy with no uncertainty in which agents are identical but their choice sets are non-convex because their labor supply is indivisible: agents can work 1 or 0 units of time and nothing in between. We show first that this economy admits an equilibrium in which agents receive different allocations even though there is no uncertainty and they are ex-ante identical. We then show that allowing agents to randomize their labor supply decision raises their welfare and derive an aggregation result in an economy with lotteries. We show that the aggregate implications of this economy are identical to those of a "representative agent" economy with no labor indivisibilities and with an infinite labor supply elasticity (a result independent of the labor supply elasticity of the individual agent).

The economy is static. A continuum $i \in [0,1]$ agents that own 1 unit of capital and 1 unit of time. They rent capital and labor to firms.

Technology

Firms solve

$$\max_{K,N} f(K,N) - wN - rK$$

where f is monotone and strictly concave in borth arguments, and N and K are the amount of capital and labor the firm hires.

Preferences

Households derive utility from consumption and disutility from work:

$$U(c,n) = u(c) - v(n)$$

where $n \in \{0,1\}$ is the supply of labor and is restricted to take only one of two values. Notice that $v(n)$ can have any form, but what is relevant is $v(0)$ and $v(1)$ which we will normalize to $v(0) = 0$ and $v(1) = m$, respectively.

Competitive equilibrium

A competitive equilibrium is a set of allocations c_i, n_i, k_i,N, K and prices w, r, such that

(i) for each agent i, c_i, n_i, k_i is a solution to

$$\max_{c,n,k} u(c) - mn$$

$$s.t.$$

$$c \leq wn + rk$$

$$n \in \{0, 1\}$$

$$0 \leq k \leq 1$$

(ii) N, K solve the firm's problem

$$\max_{N,K} f(K,N) - rK - wN$$

(iii) markets clear

$$K = \int_0^1 k_i di, \quad N = \int_0^1 n_i di, \quad f(K,N) = \int_0^1 c_i di$$

One can prove equilibrium existence in this economy, despite the non-convexity (Aumann (1966), Mas-Collel (1977)). What we want to show next is that there exist equilibria in this economy in which, even though agents are identical, they receive different allocations: some will work and consume more, some will not work and consume less. To do so, let's restrict $f(K,N) = K^{\alpha} N^{1-\alpha}$ and $u(c) = \ln(c)$.

Let's conjecture an equilibrium where some agents supply $n = 0$ and others supply $n = 1$. For this to be the case, it must be that agents are indifferent between the two alternatives. Notice from the budget constraint of the consumer that if she works, she receives $w + r$, while if she does not work, she receives r. Then indifference says:

$$u(w + r) - m = u(r)$$

which substituting further from the firm's first order conditions gives:

$$\ln(\alpha N^{1-\alpha} + (1-\alpha) N^{\alpha}) - m = \ln\left(\alpha N^{1-\alpha}\right)$$

(where we have used $K = 1$) which gives

$$N = \frac{1-\alpha}{\alpha} \frac{1}{e^m - 1}$$

So for example if $\alpha = 0.5$ and $m = \ln(3)$ we have $N = \frac{1}{2}$ which says that 1/2 of the households work.

Notice that in this economy agents would be better off if they would be able to randomize their labor supply decision. For example, consider an allocation that assigns agents consumption $\bar{c}$ to all agents, irrespective of whether they work, where $\bar{c} = Nc^1 + (1-N)\,c^0$ is the aggregate consumption in the economy characterized above and c^1, c^0 are the consumption allocations above. Then

$$u(c^1) - m = u(c^0) < N\,[u(\bar{c}) - m] + (1-N)\,[u(\bar{c})]$$

where the latter is the expected utility of the agent that can randomize. To see this, notice that the latter term can be rewritten as $u\,(Nc^1 + (1-N)\,c^0) - Nm > N\,[u(c^1) - m] + (1-N)\,u(c^0) = u(c^0) = u(c^1) - m$ where the first inequality follows from the concavity of u.

Let's consider next an economy which can implement the superior allocations above by means of lotteries. In particular, we allow agents to choose a probability ϕ that she works and consumption/capital supply decisions conditional on the outcome of the lotery c^1, c^0. Clearly $k^1 = k^0 = 1$ so let's focus on the consumption decisions. We also assume that agents can trade state-contingent claims so that agents trade consumption claims in both states in which the consumer can find herself. Let $p(\phi)$ the relative price of consumption in state 1 in terms of consumption in state 0. Then the consumer maximizes expected utility given by

$$\max_{\phi, c^1, c^0} \phi\left[u(c^1) - m\right] + (1-\phi)\,u(c^0)$$
$$s.t.\; p(\phi)c^1 + c^0 \leq p\,(\phi)\,[w + r] + r$$

Also notice that $p\,(\phi) = \frac{\phi}{1-\phi}$ in equilibrium. To see this, notice that one can think of agents as trading with a financial intermediary that exchanges claims on consumption in the two states of the world. The profits this intermediary makes are

$$\phi\,(-b) + (1-\phi)\,p\,(\phi)\,b$$

i.e, it sells b units of state 1 consumption and receives in exchange $p\,(\phi)\,b$ units of state 0 consumption, and thus pays $\phi\,(-b)$ to the ϕ agents that work and receives $(1-\phi)\,p\,(\phi)\,b$ from the agents that do not work. Substituting the expression for $p\,(\phi)$ and rearranging the budget constraint reads:

$$\phi c^1 + (1-\phi)\,c^0 \leq \phi\,[w + r] + (1-\phi)\,r$$

The Foc's w.r.t c^1 and c^0 are thus

$$\begin{aligned} \phi u'(c^1) &= \phi\lambda \\ (1-\phi)\, u'(c^0) &= (1-\phi)\,\lambda \end{aligned}$$

which says that $c^1 = c^0 = c$.

We can thus rewrite the agent's problem as one of choosing c to

$$\max_{c,\phi} u(c) - \phi m$$

$$s.t.$$

$$c \leq \phi w + r$$

and thus the budget set is convex and the allocations of this economy are identical to those of a "representative agent" economy in which agents choose consumption C and labor, N to

$$\max u(C) - mN$$

$$s.t.$$

$$C \leq wN + r$$

Interestingly, the "aggregate" Frish elasticity of labor supply is infinite, a result that is independent of the shape of $v()$ at the individual level.

2.5 Notes

Gorman (1953) developed the theory of aggregation of individual preferences. His main result is that when preferences are homothetic and households differ in initial wealth levels only, social preferences do not depend on the distribution of individual wealth. Stiglitz (1969) discusses the income and wealth distribution dynamics in the context of a Solow growth model, where individual savings are assumed to be linear in capital. Our discussion in sections 1.3 is based on Chatterjee (1994). Caselli and Ventura (2000) extend the Gorman result to economies where agents also differ in their endowments of efficiency units of labor. They work in continuous time and apply their results to the transitional dynamics of the neoclassical growth model and to an economy with Arrow-style externalities. Maliar and Maliar (2001, 2003) are examples of how the aggregation theorems apply to economies with both idiosyncratic and aggregate uncertainty when markets are complete. In particular, they show that even when preferences are non-homothetic, in certain cases one can obtain closed-form aggregation, although the aggregate preferences depend on the distribution of individual shocks. See also chapter 4.d in Mas Colell for a more abstract discussion of the existence of a representative consumer.

References

[1] Caselli, Francesco and Jaume Ventura (2000), "A representative Consumer Theory of Distribution", *American Economic Review.*

[2] Chatterjee, Satyajit (1994) "Transitional dynamics and the distribution of wealth in a neoclassical growth model", *Journal of Public Economics*

[3] Gorman, "Community preference field," *Econometrica*

[4] Maliar, Lilia and Serguei Maliar (2001) "Heterogeneity in capital and skills in a neoclassical stochastic growth model", *Journal of Economic Dynamics and Control*

[5] Maliar Lilia and Serguei Maliar (2003) "The representative consumer in the neoclassical growth model with idiosyncratic shocks*", Review of Economic Dynamics*

[6] Negishi Takashi (1960), "Welfare Economics and Existence of an Equilibrium for a Competitive Economy", *Metroeconomica*

[7] Samuelson, Paul (1956), "Social Indifference Curves," *Quarterly Journal of Economics*

[8] Stiglitz, Joseph (1969), "Distribution of Income and Wealth among Individuals", *Econometrica.*