1 Consumption Insurance

Throughout the chapter we will use the following notation to represent uncertainty. Let $s_t \in S_t$ be the current state of the economy and let $s^t = \{s_0, s_1, ..., s_t\}$ be the history up to time t, with $s^t \in S^t \equiv S_0 \times S_1 \times ... \times S_t$. Let $\pi(s^t)$ the probability of this history occurring.

1.1 Incomplete Markets

We briefly discuss several ways in which one can depart from the assumption of complete markets, and contrast these to the complete markets benchmark. In all economies we consider below, we assume the following preferences:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c_{it}\left(s^t\right)\right)$$

We assume that agent i is endowed with $y_{it}(s^t)$ units of consumption at date t at history s^t, with $\sum_{i \in I} y_{it}(s^t) = Y_t(s^t)$ denoting the aggregate endowment/income.

1.1.1 Complete Markets

The household faces the following sequence of budget constraints at date t and history s^t:

$$c_{it}\left(s^t\right) + \sum_{s_{t+1} \in S_{t+1}} q_t\left(s_{t+1}, s^t\right) a_{it+1}\left(s_{t+1}, s^t\right) = y_{it}\left(s^t\right) + a_{it}\left(s^t\right), \tag{1}$$

where $q(s_{t+1}, s^t)$ is the price at date t and state s^t of an Arrow security that pays one unit of consumption if state s_{t+1} is realized next period and $a_{it+1}(s_{t+1}, s^t)$ is the quantity of such a security that the agent i buys at date t[1]. Obviously, this sequential formulation of the complete markets model gives rise to the same conclusion, i.e. full risk sharing.[2]

Notice that the first order condition for the consumption choice is

$$u'\left(c_i\left(s^t\right)\right) = \lambda_{it}\left(s^t\right)$$

[1] Notice $s^t = \left(s_t, s^{t-1}\right)$.

[2] It can be proved that the sequence of constraints here together with the no Ponzi-scheme condition

$$\lim_{t\to\infty} \sum_{s^t \in S^t} q_t\left(s_t, s^{t-1}\right) a_t^i\left(s_t, s^{t-1}\right) \geq 0$$

is equivalent to the Arrow-Debreu constraint we studied earlier. See chapter 8 in LS.

and for the security choice is:

$$\pi\left(s^t\right) q_t\left(s_{t+1}, s^t\right) \lambda_{it}\left(s^t\right) = \beta\pi\left(s^{t+1}\right) \lambda_{it+1}\left(s^{t+1}\right).$$

Notice next that the second expression reduces to

$$q_t\left(s_{t+1}, s^t\right) \lambda_{it}\left(s^t\right) = \pi\left(s^{t+1}|s^t\right) \lambda_{it+1}\left(s^{t+1}\right)$$

or

$$\frac{\lambda_{it+1}\left(s^{t+1}\right)}{\lambda_{jt+1}\left(s^t\right)} = \frac{\lambda_{it}\left(s^t\right)}{\lambda_{jt}\left(s^t\right)} = ... = \frac{\lambda_{i0}}{\lambda_{j0}}$$

Which implies, from the first FOC, that

$$\frac{u'\left(c_i\left(s^t\right)\right)}{u'\left(c_j\left(s^t\right)\right)} = \frac{\lambda_{i0}}{\lambda_{j0}} = \frac{\lambda_i}{\lambda_j}$$

Assume CRRA preferences $u\left(c_t^i\left(s_t\right)\right) = \frac{c_t^i(s_t)^{1-\sigma}}{1-\sigma}$

$$\frac{c_t^i\left(s^t\right)}{c_t^j\left(s^t\right)} = \left(\frac{\lambda_{i0}}{\lambda_{j0}}\right)^{\frac{1}{\sigma}},$$

Hence the ratio of consumption allocations is constant. Summing over $j = 1, ..., I$, this implies clearly that

$$c_{it}\left(s^t\right) = \left[\frac{\left(\lambda_i\right)^{\frac{1}{\sigma}}}{\sum_{j=1}^{I}\left(\lambda_j\right)^{\frac{1}{\sigma}}}\right] C_t\left(s^t\right), \quad (2)$$

so, individual consumption tracks perfectly aggregate consumption for every household.[3]

1.1.2 Autarky

The most stringent departure from complete markets is autarky: we assume here that insurance markets to trade across states s_t at a given point in time t are completely absent, and there is no storage technology to transfer resources across periods (e.g., the consumption good is perishable). In this economy, an individual i who receives a random stream of income shocks $\left\{\left\{y_{it}\left(s^t\right)\right\}_{s^t \in S^t}\right\}_{t=0}^{\infty}$ has no other choice than consuming her income every period, i.e.

$$c_{it}\left(s^t\right) = y_{it}\left(s^t\right). \quad (3)$$

[3] In general, full risk sharing does not mean constant consumption. Individual consumption is constant over time and across states only if aggregate consumption is constant. Even if the aggregate endowment is constant, with flexible labor supply and non-separability between consumption and leisure in preferences, consumption may not be constant over time.

1.1.3 Bond economy

We next allow agents to trade one-period bonds that are not contingent on the realization of uncertainty: the bond pays the same amount in all states of the world. The sequence of budget constraints the household faces is now more restrictive than under complete markets:

$$c_{it}\left(s^t\right) + q_t\left(s^t\right) a_{it+1}\left(s^t\right) = y_{it}\left(s^t\right) + a_{it}\left(s^{t-1}\right),$$

where $q_t\left(s^t\right)$ is the price at date t and state s^t of an asset that pays one unit of consumption next period, *independently* of the realization of the state s_{t+1}. In other words, the agent is cut-off from every state-contingent insurance market and has only access to a simple financial instrument to transfer resources over time. The absence of insurance opportunities induces the consumer to hold a certain amount of the bond in order to smooth consumption.

Notice that the first order conditions here reduce to

$$u'\left(c_i\left(s^t\right)\right) = \lambda_{it}\left(s^t\right)$$

which is identical to the one above, but that now the Euler equation says:

$$q_t\left(s^t\right)\pi\left(s^t\right)\lambda_{it}\left(s^t\right) = \beta\sum_{s^{t+1}}\pi\left(s^{t+1}\right)\lambda_{it+1}\left(s^{t+1}\right)$$

or

$$q_t\left(s^t\right)\lambda_{it}\left(s^t\right) = \beta\sum_{s^{t+1}}\pi\left(s^{t+1}|s^t\right)\lambda_{it+1}\left(s^{t+1}\right)$$

Substituting the expression for $\lambda_{it}\left(s^t\right)$ we have that

$$q_t\left(s^t\right)u'\left(c_i\left(s^t\right)\right) = \beta\sum_{s^{t+1}}\pi\left(s^{t+1}|s^t\right)u'\left(c_i\left(s^{t+1}\right)\right)$$

so that the ratio of the growth rate of the marginal utility of consumption across two households is no longer constant as with complete markets, but rather constant on average:

$$\beta\sum_{s^{t+1}}\pi\left(s^{t+1}|s^t\right)\frac{u'\left(c_i\left(s^{t+1}\right)\right)}{u'\left(c_i\left(s^t\right)\right)} = \frac{q_t\left(s^t\right)}{\beta} = \beta\sum_{s^{t+1}}\pi\left(s^{t+1}|s^t\right)\frac{u'\left(c_j\left(s^{t+1}\right)\right)}{u'\left(c_j\left(s^t\right)\right)}$$

1.1.4 A note on when markets are complete

Whether markets are complete or not clearly depends not only on the set of financial assets we allow agents to trade, but also on the nature of uncertainty. In general, let $|S^t|$ denote the cardinality of (number of elements in) S^t : then markets are complete whenever agents have access to K securities with an $|S^t| \times K$ return matrix R (so that the typical element R_{ij} is the return of the security j in state i) that has rank (maximal number of linearly independent columns) equal to $|S^t|$. So if $|S^t| = 3$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the Arrow securities we have considered above, but allowing agents to trade securities with returns $R' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ would also complete markets (*e.g.* purchasing 1 unit of an Arrow security that only pays in state 2 is equivalent to purchasing 1 unit of security 2 and selling 1 unit of security 3 simultaneously).

1.1.5 Empirical Implications

The full risk sharing hypothesis can be tested empirically. Under CRRA preferences, for example, equation (2) implies that the log-change in individual consumption should equal the log-change in aggregate consumption, for every individual, in every period. If we estimate from microdata the relationship

$$\Delta \log c_t^i = \beta_1 \Delta \log C_t + \beta_2 \Delta \log y_t^i + \varepsilon_t^i,$$

where y_t^i is current individual income, then the full risk-sharing hypothesis implies $(\beta_1 = 1, \beta_2 = 0)$. Contrast this prediction with the "autarky hypothesis" which implies $(\beta_1 = 0, \beta_2 = 1)$, i.e. consumption tracks perfectly current income. In general, they are both rejected, albeit the data seem to be much closer to full risk-sharing in many contexts.[4]

Remark: A good model (empirically, at least) for consumption lies between autarky and full risk-sharing, i.e. it must be a model where agents have access to "partial" consumption insurance.

We next study in more detail the bond economy described above in which a riskless one-period bond is the only security agents trade. To simplify the notation for the next

[4] See Mace (1991), Cochrane (1991).

sections, we assume away fluctuations in the aggregate endowment $Y_t(s^t)$, hence

$$q_t\left(s^t\right) = q \equiv \frac{1}{1+r},$$

where r is the interest rate on a risk-free bond, and reformulate the budget constraint with lighter notation as

$$a_{t+1} = (1+r)\left(y_t + a_t - c_t\right), \tag{4}$$

i.e. we will omit the explicit dependence on histories.

1.2 Permanent Income Hypothesis (PIH)

The PIH is a special case of the bond economy. The strict version of the PIH makes three key assumptions: 1) households have quadratic utility

$$u(c) = b_1 c_t - \frac{1}{2} b_2 {c_t}^2,$$

2) the interest rate on the one-period bond equals the inverse of the discount rate, or $\beta(1+r) = 1$, and 3) loose borrowing limits, i.e. we only impose a No-Ponzi scheme condition stating that in the limit assets cannot be negative, i.e.

$$E_t\left[\lim_{\tau\to\infty}\left(\frac{1}{1+r}\right)^{\tau} a_{t+\tau}\right] \geq 0$$

and de-facto we ignore borrowing constraints in our calculations.

Consumption as a random walk– From the consumption Euler equation:

$$b_1 - b_2 c_t = \beta(1+r) E_t\left(b_1 - b_2 c_{t+1}\right) \Rightarrow E_t c_{t+1} = c_t. \tag{5}$$

from which we recover the well known result that consumption is a martingale.

It is useful to note that from the law of iterated expectations and the martingale property:

$$E_t c_{t+2} = E_t\left[E_{t+1} c_{t+2}\right] = E_t c_{t+1} = c_t$$

and, more in general:

$$E_t c_{t+j} = c_t, \text{ for any } j \geq 0. \tag{6}$$

Iterating forward one period on the budget constraint $a_{t+1} = (1+r)(y_t + a_t - c_t)$, we obtain

$$c_t = y_t + a_t - \frac{1}{1+r} a_{t+1} = y_t + a_t - \frac{1}{1+r}\left[c_{t+1} - y_{t+1} + \frac{a_{t+2}}{1+r}\right]$$

and rearranging

$$c_t + \frac{1}{1+r} c_{t+1} = a_t + y_t + \frac{1}{1+r} y_{t+1} - \frac{1}{1+r} a_{t+2}$$

If we keep iterating and then apply the conditional expectations operator and the law of iterated expectations, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t c_{t+j} &= a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t y_{t+j} \\ c_t &= \frac{r}{1+r} \left[a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t y_{t+j} \right] = \frac{r}{1+r} (a_t + H_t) \end{aligned} \quad (7)$$

where the LHS of the second row uses property established in (6), and in the last equality we denoted human wealth, i.e. the expected discounted value of future earnings, with H_t. Recall that financial wealth is a_t. Define permanent income as the annuity value (i.e. $\frac{r}{1+r}$) of total (human and financial) wealth $W_t \equiv (a_t + H_t)$. Therefore, we have the following result:

Result 2.1: If preferences are quadratic, and $\beta(1+r) = 1$, then consumption follows a martingale process and equals permanent income, i.e. the annuity value of human and financial wealth.

Certainty equivalence– Notice that, if one solves the non-stochastic version of the PIH problem stated earlier, from the FOC (5) one obtains $c_{t+1} = c_t$ and, by iterating forward on the budget constraint,

$$c_t = \frac{r}{1+r} \left[a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} \right].$$

Compared to equation (7), the above equation suggests that the consumption satisfies *certainty equivalence* in the sense that to obtain the solution of the stochastic problem, one can 1) solve the deterministic problem and 2) substitute conditional expectations of the forcing variables (y_{t+j}) in place of the variables themselves. Put differently, *the variance and higher moments of the income process do not matter for the determination of consumption.* This property descends directly from the linear-quadratic objective function.

An example

Suppose the income process follows

$$y_t = \rho y_{t-1} + \varepsilon_t$$

Then $E_{t+j} y_{t+j} = \rho^j y_t$ and the consumption rule is

$$c_t = \frac{r}{1+r} a_t + \frac{r}{1+r-\rho} y_t$$

So if $\rho = 1$ and endowment shocks are permanent, then the consumer consumes her entire endowment. If $\rho = 0$ and the shock is purely transitory, then she consumers only a fraction $\frac{r}{1+r}$ of it.

Consumption dynamics– From (7) , the change in consumption at time t equals

$$\Delta c_t = c_t - c_{t-1} = c_t - E_{t-1} c_t = \frac{r}{1+r} \left[W_t - E_{t-1} W_t \right],$$

where we have used the random walk property. Now, use the definition of total wealth W_t to define the innovation (i.e. the unexpected change) in permanent income, at time t as

$$\begin{aligned} W_t - E_{t-1} W_t &= a_t - E_{t-1} a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left[E_t y_{t+j} - E_{t-1} \left(E_t y_{t+j} \right) \right], \qquad (8) \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(E_t - E_{t-1} \right) y_{t+j}, \end{aligned}$$

where we have used the law of iterated expectations $E_{t-1} \left(E_t y_{t+j} \right) = E_{t-1} y_{t+j}$, and the fact that $a_t = E_{t-1} a_t$, since there is no uncertainty at time t about the evolution of wealth next period: just look at the budget constraint (**??**). Putting together (8) and the expression above for the change in consumption we arrive at

$$\Delta c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(E_t - E_{t-1} \right) y_{t+j}. \qquad (9)$$

This equation states another useful result:

Result 2.2: under the PIH, the change in consumption between $t-1$ and t *is proportional to the revision in expected earnings due to the new information ("news") accruing in that same time interval.*

Wealth Dynamics. Borrowing constraints

First, note that, from the budget constraint (**??**)

$$a_{t+1} = (1+r)\left(y_t + a_t - c_t\right),$$

rearranging, we obtain an expression for the change in wealth

$$\Delta a_{t+1} = (1+r)\, y_t + r a_t - (1+r)\, c_t. \tag{10}$$

Substituting into the above equation the optimal consumption choice from (7) reproduced below

$$c_t = \frac{r}{1+r}\left[a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \tag{11}$$

we obtain:

$$\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} E_t \Delta y_{t+j}$$

Now, suppose that the income process follows a random walk, $y_t = y_{t-1} + \varepsilon_t$. Then it is easy to see that $\Delta y_{t+j} = \varepsilon_{t+j}$ and $\Delta a_{t+1} = 0$. Therefore, the initial wealth endowment perpetuates itself (i.e., it is constant) so if the individual starts above the borrowing constraint, it will never be binding. The reason for this result is that wealth changes only if the individual is consuming just a part of its income (and saving the remaining part) in order to smooth consumption. With permanent shocks, all the income shock is consumed in every period. Note, indeed, that when income is a unit root, from (11)

$$c_t = \frac{r}{1+r} a_t + y_t$$

which substituted into equation (10) yields $a_{t+1} = a_t$.

However, if the income process is *iid*, we have that $\Delta y_{t+1} = \varepsilon_{t+1} - \varepsilon_t$, $\Delta y_{t+2} = \varepsilon_{t+2} - \varepsilon_{t+1}$, therefore

$$\begin{aligned}
\Delta a_{t+1} &= -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} E_t \Delta y_{t+j} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} E_t\left[\varepsilon_{t+j} - \varepsilon_{t+j-1}\right] \\
&= -E_t\left[\varepsilon_{t+1} - \varepsilon_t\right] - \frac{1}{1+r} E_t\left[\varepsilon_{t+2} - \varepsilon_{t+1}\right] - \ldots \\
&= \varepsilon_t
\end{aligned}$$

since all other terms are zero. This means that wealth follows a random walk and, as a result, any constraint on asset holdings will be binding with probability one sooner or later.

To conclude, whether ignoring borrowing constraint is troublesome or not depends on the specific income process. However, in general this result highlights the fact that borrowing constraints cannot be ignored.

2 Precautionary Savings: Prudence and Borrowing Constraints

In this section we study conditions under which savings react to changes in income uncertainty. Recall that in the PIH, when you abstract from borrowing constraints, certainty equivalence implies that "mean preserving spreads" of the income distribution do not impact on saving.

2.1 Prudence: A two-period Model

Consider the simple two-period consumption-saving problem

$$\max_{\{c_0,c_1,a_1\}} u(c_0) + \beta E[u(c_1)] \tag{12}$$

$$s.t.$$

$$c_0 + a_1 = y_0$$

$$c_1 = (1+r)a_1 + \tilde{y}_1$$

where y_0 is given, and income next period $\tilde{y}_1$ is also exogenous but stochastic.[5] If we retain the assumption $\beta(1+r) = 1$ to simplify the algebra, the Euler equation gives

$$u'(y_0 - a_1) = E[u'((1+r)a_1 + \tilde{y}_1)],$$

which is one equation in one unknown, a_1. Note that current consumption c_0 is determined by the period-zero budget constraint

$$c_0^* = y_0 - a_1^*,$$

hence a rise in savings leads to a fall in current consumption.

[5] Note that the timing of this problem is slightly different from the one adopted in the description of the PIH. There, we assumed that individuals receive income and consume at the beginning of the period and the payments of interests occurs at the end of the period. Here, we assume the payment of interests occurs at the beginning of the period, income is paid at the end of the period and individuals consume at the end of the period. In general, results are robust to this timing, it is a mater of convenience which one to choose.

Mean-preserving spread– What happens to optimal consumption at $t = 0$ if the uncertainty over income next period $\tilde{y}_1$ rises, i.e. as future income becomes more risky? Consider a mean-preserving spread of $\tilde{y}_1$. Define

$$\tilde{y}_1 = \bar{y}_1 + \varepsilon_1,$$

where ε_1 is the stochastic component and $\bar{y}_1$ is the mean. Assume that $E(\varepsilon_1) = 0$ and $var(\varepsilon_t) = \sigma_\varepsilon$. The Euler equation becomes

$$u'(y_0 - a_1) = E\left[u'\left((1+r)\,a_1 + \bar{y}_1 + \varepsilon_1\right)\right],$$

which shows that *if u' is convex*, then by Jensen's inequality, a mean-preserving spread of ε_1 will increase the value of the RHS which shifts upward, inducing a rise in a_1^* and a fall in c_0^*. This is an application of the famous result by Rotschild and Stiglitz (1970).

Prudence– The convexity of the marginal utility (or $u''' > 0$) is called "prudence" and is a property of preferences, like risk aversion: risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function.[6]

Result 2.3: If the marginal utility is convex ($u''' > 0$), then the individual is "prudent" and a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.

It can be easily seen that any utility function with decreasing absolute risk aversion, i.e. in the DARA class (which includes CRRA) displays positive third derivative. Let $\alpha(c)$ be the coefficient of absolute risk aversion. Then:

$$\alpha(c) = \frac{-u''(c)}{u'(c)} \Rightarrow \alpha'(c) = \frac{-u'''(c)\,u'(c) + [u''(c)]^2}{[u'(c)]^2}.$$

Since with DARA $\alpha'(c) < 0$, then we have that

$$-u'''(c)\,u'(c) + [u''(c)]^2 < 0 \Rightarrow u'''(c) > \frac{[u''(c)]^2}{u'(c)} > 0.$$

Intuitively, a rise in uncertainty reduces the certainty-equivalent income next period and with DARA effectively increases the degree of risk-aversion of the agent, inducing him to save more.

[6] Precisely, Kimball (1990) defines the index of absolute prudence as the ratio $-u'''(c)/u''(c)$, so in a similar vein to the Arrow-Pratt index of absolute risk-aversion $-u''(c)/u'(c)$.

Prudence is a motive for additional savings in order to take precaution against possible negative realizations of the income shock next period. In this sense, savings induced by prudence are called "precautionary savings" or "self-insurance". In this simple, two-period partial equilibrium model one can define precautionary wealth due to income uncertainty σ_ε as the difference between the optimal asset choice under uncertainty $a_1^*(\sigma_\varepsilon)$ and the optimal asset choice under certainty over next period income, i.e. $a_1^*(0)$.

Saving motives– This is a good time to make a short remark about "saving motives". The saving motive associated to $\beta R > 1$ which pushes the individual to postpone consumption because of patience and/or returns to savings is called *intertemporal motive*. The saving motive of the pure PIH where utility is quadratic (hence uncertainty has no role) and $\beta R = 1$ (hence intertemporal motives are inactive) is called *smoothing motive*. The individual wants to smooth consumption through income shocks. Finally, as explained above, the saving motive associated to future income uncertainty is called *precautionary or self-insurance motive*. We add that in a life-cycle model where the individual faces a retirement period, during the working stage of the life-cycle the individual would have a *life-cycle motive* for saving associated to the desire of smoothing consumption throughout her life.

2.2 Borrowing Constraints:

To isolate the role of borrowing constraints, we abstract from prudence altogether and focus on the quadratic utility case. To account for the possibility that the borrowing constraint is binding, the Euler equation needs to be modified. Suppose households face a *no-borrowing constraint* $a_{t+1} \geq 0$. Then, (5) becomes

$$c_t = \begin{cases} E_t c_{t+1} & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{cases}$$

where the first line is just the FOC of the agent when the constraint is not binding, while the second line descends directly from the budget constraint $a_{t+1} = R(y_t + a_t - c_t)$ when the constraint is binding ($a_{t+1} = 0$). The households would like to borrow to finance consumption, but she is not allowed, so she consumes all her resources.

In which scenarios is the borrowing constraint binding? We can rewrite the borrowing constraint as

$$c_t \leq y_t + a_t$$

Consider the AR(1) process for income $y_t = \rho y_{t-1} + \varepsilon_t$. Recall that in the absence of borrowing constraints the optimal consumption rule is

$$c_t = \frac{r}{1+r}a_t + \frac{r}{1+r-\rho}y_t$$

Clearly this consumption rule would violate the date t budget constraint as long as

$$\frac{r}{1+r}a_t + \frac{r}{1+r-\rho}y_t > y_t + a_t$$

or

$$\frac{1-\rho}{1+r-\rho}y_t < \frac{1}{1+r}a_t$$

Thus if $\rho = 1$ the borrowing constraint does not bind, while if $\rho = 0$ the consumer may receive a very low income shock which would not allow her to smooth consumption.

So, if the constraint is binding $c_t = y_t + a_t$, whereas if it is not binding, the agent will save some income and $c_t < y_t + a_t$. The above pair of conditions can be written in compound form as

$$c_t = \min\{y_t + a_t, E_t c_{t+1}\} = \min\{y_t + a_t, E_t[\min\{y_{t+1} + a_{t+1}, E_{t+1}c_{t+2}\}]\}.$$

Now, suppose that the uncertainty about income y_{t+1} increases. Very low realizations of income y_{t+1} become more likely, which makes the borrowing constraint more likely to bind in the future and reduces the value of $E_t[\min\{y_{t+1} + a_{t+1}, E_{t+1}c_{t+2}\}]$. This, in turn, reduces the value of $E_t c_{t+1}$. Thus, if the borrowing constraint is not already binding at time t but it may be binding in the future, then agents consume less today.

Intuitively, when agents face borrowing constraints, they fear getting several consecutive bad income realizations which would push them towards the constraint and force them to consume their income *without the ability of smoothing consumption.* To prevent this situation, they save for self-insurance (precautionary motive). Thus, we have an important result: prudence is not strictly necessary for precautionary saving behavior, or:

Result 2.4: Even in absence of prudence (e.g. with quadratic preferences), in presence of borrowing constraints a rise in future income uncertainty leads to a rise in current savings for precautionary reasons and to a decline in current consumption.

Even though we showed this result for quadratic utility, it is a general results that holds for concave utility.

2.2.1 A Natural Debt Limit

This is a good place to discuss how to model debt limits. We started by imposing an exogenous borrowing constraint like $a_{t+1} \geq -\phi$, where ϕ is a parameter (in our previous case $\phi = 0$). However, one may wonder if there is a "natural" borrowing limit that the household faces.

Suppose the income process $\{y_t\}_{t=0}^{\infty}$ is deterministic. Impose non-negativity of consumption throughout the life of the household, i.e. $c_t \geq 0$ for all t and iterate forward on the budget constraint

$$\begin{aligned} c_t &= a_t + y_t - \frac{a_{t+1}}{1+r} \geq 0 \quad \Rightarrow \quad a_t \geq -y_t + \frac{a_{t+1}}{1+r} \\ a_t &\geq -y_t + \frac{a_{t+1}}{1+r} \geq -y_t + \frac{1}{1+r}\left[-y_{t+1} + \frac{a_{t+2}}{1+r}\right] \geq ... \\ a_t &\geq -\sum_{j=0}\left(\frac{1}{1+r}\right)^j y_{t+j}. \end{aligned}$$

In other words, by imposing this constraint, the household is not allowed to accumulate more debt than what she will ever be able to repay by consuming just zero every period.

If the income process is stochastic, then how can we be sure that whatever the household borrows she will repay *almost surely (i.e. with probability 1)*? Then, we need to substitute y_t at each t with the lowest possible realization of the income shock, call it $y_{\min}$, and we have the *natural debt limit*

$$a_t \geq -\left(\frac{1+r}{r}\right) y_{\min}. \tag{13}$$

No exogenous borrowing constraint can ever be looser than the natural debt limit.

Inada conditions and natural borrowing limit– Keep in mind an important property: if the utility function satisfies the Inada condition $u(0) = -\infty$, then the consumer will never want to borrow up to the natural debt limit.[7] Suppose she does borrow up to $a_t = -\left(\frac{1+r}{r}\right) y_{\min}$ and suppose the income realization is precisely $y_{\min}$ which has

[7] For example, CRRA utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ with $\gamma \geq 1$ satisfies the Inada condition $u(0) = -\infty$.

positive probability. From the budget constraint:

$$\begin{aligned} c_t &= a_t + y_{\min} - \frac{a_{t+1}}{1+r} = -\left(\frac{1+r}{r}\right) y_{\min} + y_{\min} - \frac{a_{t+1}}{1+r} \\ &= -\frac{1}{r} y_{\min} - \frac{a_{t+1}}{1+r} \leq -\frac{1}{r} y_{\min} - \frac{1}{1+r} \underbrace{\left[-\left(\frac{1+r}{r}\right) y_{\min}\right]}_{\text{max that can be borrowed}} \\ &\leq 0 \end{aligned}$$

which shows that, with positive probabiility next period the consumer will have to consume zero. However that would lead to an infinitely negative utility, and so the consumer will never to reach that state.

The preferences alone will insure that the *natural* borrowing limit will never bind. In other words, in solving for the optimal consumption you can safely assume interior solutions for the Euler equation. This is not true for ad-hoc debt limits!