1 The Income Fluctuation Problem: Consumption under Uncertainty

We are now ready to examine the problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.$$

$$c_t + a_{t+1} = (1+r)\, a_t + y_t$$

$$a_{t+1} \geq 0$$

where we combine general preferences (i.e., we just impose $u' > 0, u'' < 0$) with a no-borrowing constraint. The value for the interest rate is always *exogenously given* and constant. Note the slight difference in timing of the budget constraint with respect to the problem stated when we studied the PIH. We explained that it only depends on the timing of the consumption choice. Finally, assume the income sequence $\{y_t\}$ is bounded.

The first order necessary condition for optimality is

$$u'(c_t) = \beta(1+r)\, E_t[u'(c_{t+1})] + \lambda_t, \tag{1}$$

where $\lambda_t > 0$ is the multiplier on the no-borrowing constraint. Condition (1) implies the Euler equation

$$u'(c_t) \geq \beta(1+r)\, E_t[u'(c_{t+1})]. \tag{2}$$

Thus, either we have

$$c_t = c_{t+1} \text{ and } a_{t+1} > 0$$

or

$$c_t < c_{t+1} \text{ and } a_{t+1} = 0$$

. The consumption sequence is thus non-decreasing. Intuitively, the consumer would like to smooth consumption and will do so whenever she has sufficient assets. A decreasing consumption sequence can be improved upon by saving an additional unit of consumption today and consuming it next period.

We want to understand whether the optimal consumption sequence $\{c_t\}$ is bounded above or whether it is unbounded and it will be diverging as $t \to \infty$. This characterization is important because, if the consumption sequence is bounded, then the endogenous state space for assets $[0, \bar{a}]$ is compact, i.e. there exists an upper bound $\bar{a}$ which is finite. This is a crucial requirement for proving the existence of an equilibrium in economies populated by many agents who choose their optimal consumption by solving income fluctuations problems. If the consumption sequence diverges, then $\bar{a} \to \infty$. This means that, in such an economy, there will be an infinite supply of assets and no equilibrium can be found at that given *interest rate*.

The convergence properties of the consumption sequence will depend on the value of the term $\beta(1+r)$. We always examine three separate cases: $\beta(1+r)$ above, equal to or below one.

We start from a problem where income fluctuations are deterministic, i.e., perfectly foreseen. Next we move to stochastic income fluctuations.

1.1 Deterministic Income Fluctuations

Case $\beta(1+r) > 1$: Without uncertainty, since u is strictly concave, the Euler equation (2) implies

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1}) > u'(c_{t+1}) \Rightarrow c_{t+1} > c_t,$$

thus consumption grows indefinitely. Since borrowing is limited at zero, assets must grow to finance consumption, so also assets diverge. The individual is "too patient" or/and the rate of return on savings is too high. Both forces push her to accumulate too much wealth.

Case $\beta(1+r) = 1$: We will prove that consumption and assets converge to a finite value. In particular, suppose we know that the last date at which the borrowing constraint never binds is $t-1$ (so that $a_t = 0$). Then the Euler equation says $c_{t+j} = c$ for all $j \geq 0$. From the lifetime budget constraint $\sum_{j=0}\left(\frac{1}{1+r}\right)^j c_{t+j} = a_t + \sum_{j=0}\left(\frac{1}{1+r}\right)^j y_{t+j}$

$$c_{t+j} = c_t = x_t = \frac{r}{1+r}\sum_{j=0}\left(\frac{1}{1+r}\right)^j y_{t+j} \;\forall j \geq 0$$

where x_t be the annuity value of the income process which starts from date t. In other words, if the borrowing constraint binds only finitely often, the consumption process converges.

More generally, a proposition in L.S (chapter 16.3.1) states a stronger result:

Proposition 1 ***Proof.***

Proposition 2 *Given a borrowing constraint and a non-stochastic endowment stream, the limit of the non-decreasing optimal consumption path is*

$$\bar{c} \equiv \lim_{t \to \infty} c_t = \sup_t x_t \equiv \bar{x}$$

■

Proof. The proof is by contradiction. We first rule out $\bar{c} > \bar{x}$. Notice that if $\bar{c} > \bar{x}$, then there is a date t such that 1) $a_t = 0$ and 2) $c_{t+j} \geq x_t \ \forall j > 0$. 2) follows from $\lim_{t\to\infty} c_t = \sup_t x_t$ while 1) follows from the fact that $c_{t+1} = c_t$ unless $a_t = 0$. (in other words, once we find a date t such that $c_{t+j} > x_t \ \forall j \geq 0$, we know $c_t = c_{t-1}$ unless $a_t = 0$, so we have $c_{t+j-1} > x_{t-1t} \ \forall j > t$ and we can go backward until we find the first date at which $a_t = 0$).At that date t we have $\sum_{j=0} \left(\frac{1}{1+r}\right)^j c_{t+j} > \sum_{j=0} \left(\frac{1}{1+r}\right)^j x_t = \sum_{j=0} \left(\frac{1}{1+r}\right)^j y_{t+j}$ which violates the life-time budget constraint and contradicts the assertion that $\bar{c} > \bar{x}$. ■

To rule out $\bar{c} < \bar{x}$, notice that if $\bar{c} < \bar{x}$ than there is a date t such that $c_{t+j} < x_t$ $\forall j \geq 0$. Thus we have

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j} < \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j x_t = \sum_{j=0} \left(\frac{1}{1+r}\right)^j y_{t+j} \leq a_t + \sum_{j=0} \left(\frac{1}{1+r}\right)^j y_{t+j}$$

where the latter inequality follows from $a_t \geq 0$. But then the consumer is not fully exhausting his life-time budget constraint, contradicting the optimality of the conjecture decision rule. QED.

Case $\beta(1+r) < 1$**:** A fortiori, we have convergence.

1.2 Stochastic Income Fluctuations

We now turn to the stochastic case. Here there is an additional motive for saving, the *precautionary motive*, due to 1) prudence (for some utility functions) and 2) the interaction between risk-aversion (i.e., aversion to consumption fluctuations) and the borrowing constraint. Given the extra saving motive that pushes consumption upward over time, we should expect that the condition under which $\{c_t\}$ converges will be more stringent than in the deterministic case. It turns out that we will need $\beta(1+r) < 1$.

A useful supermartingale– Multiply both sides of

$$u'(c_t) \geq \beta(1+r) E_t [u'(c_{t+1})]$$

by $\beta^t (1+r)^t$ and define $M_t \equiv [\beta (1+r)]^t u' (c_t) > 0$. Then equation (2) can be written as

$$M_t \geq E_t M_{t+1}$$

which asserts that M_t follows a *supermartingale.* Since M_t is non-negative, by the supermartingale convergence theorem (Doob, 1995), this stochastic process converges almost surely to a non-negative finite limit $\bar{M}$, i.e.,

$$\lim_{t \to \infty} M_t = \bar{M} < \infty. \tag{3}$$

Case $\beta (1+r) > 1$**:** According to the convergence theorem above, $[\beta (1+r)]^t u' (c_t)$ has a finite limit. Since $[\beta (1+r)]^t \to \infty$, then marginal utility $u' (c_t)$ can only converge to $u' (\bar{c}) = 0$. Given the Inada condition, $c_t \to \infty$. Since debt is limited by the liquidity constraint, divergence of consumption means $a_t \to \infty$, hence there is no upper bound in the asset space. This is the same result we found for the certainty case.

Case $\beta (1+r) = 1$**:** The convergence result with $\beta (1+r) = 1$ of the deterministic case does not hold in the stochastic case. See LS (16.5) for a complete formal proof of this result. We can give a simple proof of this result for general income process if we assume that $u''' > 0$. From the Euler Equation

$$u' (c_t) \geq E_t [u' (c_{t+1})] .$$

From convexity of the marginal utility, by Jensen's inequality

$$u' (c_t) \geq E_t [u' (c_{t+1})] > u' (E_t (c_{t+1})) .$$

By concavity, we have that $E_t (c_{t+1}) > c_t$, so consumption will always tend to ratchet upward over time which prevents the consumption sequence from being bounded almost surely. To see this, notice that at every date $t$there is a positive mass on realizations greater than c_t which contradicts the almost-sure boundedness.

Case $\beta (1+r) < 1$**:** Consider the case of *iid* income shocks. Let x be cash in hand, i.e $x = Ra + y$. From the Euler Equation:

$$u_c (c (x)) = \beta R E [u_c (c (x'))] = \beta R \frac{E [u_c (c (x'))]}{u_c (c (x'_{\max}))} u_c (c (x'_{\max})) , \tag{4}$$

where $x'_{\max} = Ra'(x) + y_{\max}$ is the consumption associated to the maximum realization of income (and therefore cash in hand) next period, given that todays' cash in hand is x. Suppose that the limit

$$\lim_{x\to\infty} \frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} \leq 1. \tag{5}$$

Then, for x large enough, since $\beta R < 1$, the Euler equation (4) yields

$$u_c\left(c\left(x\right)\right) = \beta R u_c\left(c\left(x'_{\max}\right)\right) < u_c\left(c\left(x'_{\max}\right)\right).$$

Concavity of u and monotonicity of c wrt x (proved in the deterministic case earlier) implies that

$$c\left(x'_{\max}\right) < c\left(x\right) \Rightarrow x'_{\max}\left(x\right) < x$$

thanks to the fact that $c_x\left(x\right) > 0$. And we would be done, because we have demonstrated that cash in hand does not increase forever: for x large enough $x' < x$ for sure.

Therefore, we only need to establish conditions under which the limit in (5) holds. We give two alternative proofs that DARA implies (5).

Proof 1 (based on Schechtman-Escudero, 1977). Note that:

$$\frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} \leq \frac{u_c\left(c\left(x'_{\min}\right)\right)}{u_c\left(c\left(x'_{\max}\right)\right)}.$$

Suppose that the following condition about the marginal utility of consumption is true: if $c_1 > c_0$, we have that, for some finite $\alpha > 0$, and for c_0 large enough

$$\frac{u'\left(c_1\right)}{u'\left(c_0\right)} \leq \left(\frac{c_0}{c_1}\right)^{\alpha}. \tag{6}$$

Then, putting together the last two equations, for x large enough:

$$\frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} \leq \frac{u_c\left(c\left(x'_{\min}\right)\right)}{u_c\left(c\left(x'_{\max}\right)\right)} = \left[\frac{c\left(x'_{\max}\right)}{c\left(x'_{\min}\right)}\right]^{\alpha}$$

Now, note that $c\left(x'_{\max}\right) = c\left(Ra'\left(x\right) + y_{\max}\right) = c\left(Ra'\left(x\right) + y_{\min}\right) + h\left(x\right)\left(y_{\max} - y_{\min}\right)$ where $0 \leq h\left(x\right) \leq 1$. Hence:

$$\begin{aligned}
\frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} &\leq \left[\frac{c\left(x'_{\max}\right)}{c\left(x'_{\min}\right)}\right]^{\alpha} = \left[\frac{c\left(x'_{\min}\right) + h\left(x\right)\left(y_{\max} - y_{\min}\right)}{c\left(x'_{\min}\right)}\right]^{\alpha} \\
&= \left[1 + \frac{h\left(x\right)\left(y_{\max} - y_{\min}\right)}{c\left(Ra'\left(x\right) + y_{\min}\right)}\right]^{\alpha}
\end{aligned}$$

and taking the limit as $x \to \infty$ we have:

$$\lim_{x\to\infty} \frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} \leq 1$$

since $a'(x)$ is strictly increasing in x.[1]

The last step is in understanding what type of restriction (6) imposes on the utility function. Arrow shows a useful equivalence between the relative risk aversion property of u and condition (6). If for some c_0 we have that for all $c \geq c_0$, $-\left[u''(c)\,c\right]/u'(c) \leq \alpha$ then for all $c \geq c_0$ we also have that $u'(c)/u'(c_0) \leq \left(\frac{c_0}{c}\right)^{\alpha}$.

In particular, if u is CRRA, then the condition (6) is satisfied and an upper bound for the asset space exists. Note that CRRA utility belongs to the DARA class, which is useful for the next proposition.

Proof 2 Consider $u_c\left(c\left(x'\right)\right)$ and compute a first-order Taylor approximation around $x' = x'_{\max}$:

$$u_c\left(c\left(x'\right)\right) \simeq u_c\left(c\left(x'_{\max}\right)\right) + u_{cc}\left(c\left(x'_{\max}\right)\right) c_x\left(x'_{\max}\right)\left(x' - x'_{\max}\right).$$

Taking expectations of both sides

$$\begin{aligned} E\left[u_c\left(c\left(x'\right)\right)\right] &\simeq u_c\left(c\left(x'_{\max}\right)\right) - u_{cc}\left(c\left(x'_{\max}\right)\right) E\left[x'_{\max} - x'\right] c_x\left(x'_{\max}\right) \qquad (7) \\ &= u_c\left(c\left(x'_{\max}\right)\right) - u_{cc}\left(c\left(x'_{\max}\right)\right) E\left[y_{\max} - y'\right] c_x\left(x'_{\max}\right) \end{aligned}$$

where in the first line we use the fact that $x'_{\max}$ is deterministic since it is implied by the specific income realization $y_{\max}$. And in the second line we use the fact that $x' \equiv (1+r)\,a' + y'$ which implies $x'_{\max} - x' \equiv y_{\max} - y'$.

Dividing equation (7) by $u_c\left(c\left(x'_{\max}\right)\right)$ we obtain:

$$\frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}\right)\right)} \simeq 1 + \alpha\left(c\left(x'_{\max}\right)\right)\left[y_{\max} - E\left(y'\right)\right] c_x\left(x'_{\max}\right),$$

where α is the coefficient of absolute risk aversion. Since both $\left[y_{\max} - E\left(y'\right)\right]$ and $c_x\left(x'_{\max}\right)$ are positive and finite, the key condition that we need to satisfy for the limit in (5) to hold is

$$\lim_{x\to\infty} \alpha\left(c\left(x\right)\right) = 0. \qquad (8)$$

[1]Note that there is no contradiction in taking the limit as $x \to \infty$ in order to show that x has an endogenous upper bound. The problem is defined also for x very large. The point is that assets endogenously will never go beyond a limit, so one can restrict attention to a smaller set of the state space.

In other words, we need *absolute risk aversion to be monotonically decreasing with asset holdings*. The faster it decreases, the smaller the upper bound on the asset space.

The intuition from these two propositions is clear: DARA means that the agent is less worried about income uncertainty as she gets rich because she becomes less risk averse, so she will consume more and accumulate less. This force limits precautionary accumulation as wealth increases. Remember that DARA is a sufficient condition and that this result holds for *iid* shocks. Huggett (1993) generalizes this result to a 2-state Markov chain for the income process (but only for the CRRA utility case).

We conclude by summarizing our findings in:

Result 2.5: In presence of borrowing constraints and uncertain income, the condition $\beta(1+r) < 1$ *is necessary for the optimal consumption sequence and for the asset space to be bounded. Moreover, when* $\beta(1+r) < 1$*, if income shocks are iid and absolute risk-aversion is decreasing (DARA utility), then the asset space is bounded. More in general, even with Markov shocks, as long as absolute risk aversion decreases fast enough with* c*, the state space will remain bounded.*