1 Income Fluctuations Problem Part III: Huggett (1993)

We make an additional set of assumptions in addition to those in earlier classess in order to characterize the problem. As usual, the agent solves

$$\max_{c_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$
$$s.t.\ c_t + a_{t+1} \leq \left(1+r\right) a_t + y_t$$

We assume the endowment can take 2 possible values: $y_t \in [y_l, y_h]$ and a symmetric Markov transition probability $\pi\left(y_h|y_h\right) = \pi\left(y_l|y_l\right) = p \geq \frac{1}{2}$. Finally, we restrict $u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$.

1.1 Some preliminaries

We first write down the problem in recursive form:

$$v\left(a, y\right) = \max_{c, a'} u(c) + \beta \sum_{y'} v(a', y') \pi \left(y'|y\right)$$

Theorem 1 in Hugget (1993) applies standard dynamic program techniques to show that there exists a unique solution $v\left(a, y\right)$ to the above functional equation. Moreover, $v\left(a, y\right)$ is 1) bounded, 2) continuous, 3) strictly increasing, 4) strictly concave, 5) continuously differentiable. There exist optimal policy functions $c\left(a, y\right), a'(a, y)$ and $a'\left(a, y\right)$ is non-decreasing in a. We will use these result below.

Theorem 2 shows that there exist a set $S = [0, \bar{a}]$, where $\bar{a} < \infty$ such that if an agent enters the period with assets $a \in S$, then her next period's asset holdings $a'\left(a, y\right) \in S$. This is true for both realizations of the endowment. In other words, the state-space is bounded. The proof proceeds in 3 Lemmas and I reproduce it below (my notation differs slightly from the original paper).

Lemma 1: $a'\left(a, y_l\right) < a$ for $a > 0$

In other words, an agent who receives the lowest endowment decumulates assets. To prove this, we first show that $v'\left(a, y_h\right) \leq v'\left(a, y_l\right)$, that is, an agent with a higher endowment values an additional unit of assets less on the margin. To show this, proceed by induction and define v_n recursively as

$$\begin{aligned} v_0 &= 0 \\ v_{n+1} &= \max_{c, a'} u(c) + \beta \sum_{y'} v_n(a', y') \pi \left(y'|y\right) \end{aligned}$$

We first show $v_n'(a, y_h) \leq v_n'(a, y_l)$ and then invoke a lemma (3.7) in Stockey Lucas that proves pointwise convergence of v_n' to v' where v' is the derivative of the optimal value function.

To prove $v_n'(a, y_h) \leq v_n'(a, y_l)$, proceed by induction. First notice that this holds trivially for $n = 0$. Assume it holds for n. Then show it holds for v_{n+1}. To see the latter, notice that the Euler equations that characterize the optimal decision rules at iteration n are:

$$\begin{aligned} v_{n+1}'(a, y_l) &= u'\left((1+r)a + y_l - a'(a, y_l)\right) \geq \beta(1+r)\left[pv_n'\left(a'(a, y_l), y_l\right) + (1-p)v_n'\left(a'(a, y_l), y_h\right)\right] \\ v_{n+1}'(a, y_h) &= u'\left((1+r)a + y_h - a'(a, y_h)\right) \geq \beta(1+r)\left[(1-p)v_n'\left(a'(a, y_h), y_l\right) + pv_n'\left(a'(a, y_h), y_h\right)\right] \end{aligned}$$

The RHS of the y_l Euler equation evaluated at a' is greater or equal then the RHS of the y_h equation at a' from $v_n'(a, y_h) \leq v_n'(a, y_l)$ and $p \geq \frac{1}{2}$. Similarly, the LHS of the y_l equation is greater if evaluated at a'. It then follows that $u'((1+r)a + y_l - a'(a, y_l)) \geq u'((1+r)a + y_h - a'(a, y_h))$ as well, where the latter are evaluated at the optimal decision rules. You can convince yourself of this by analysing all the possible scenarios on whether the borrowing constraint binds or not, but the graphical argument we have used in class is the cleanest way to see this. We have thus shown that $v_{n+1}'(a, y_h) \leq v_{n+1}'(a, y_l)$ which completes the induction argument. Together with the lemma in Stockey Lucas this proves $v'(a, y_h) \leq v'(a, y_l)$.

Finally, to prove $a'(a, y_l) < a$, let us evaluate the Euler equation of the (a, y_l) agent at $a'(a, y_l) = a > 0$.

$$\begin{gathered} u'((1+r)a + y_l - a) - \beta(1+r)\left[pv'(a, y_l) + (1-p)v'(a, y_h)\right] \geq \\ u'((1+r)a + y_l - a) - \beta(1+r)v'(a, y_l) > \\ u'((1+r)a + y_l - a) - v'(a, y_l) = 0 \end{gathered}$$

where the first inequality is due to $v'(a, y_h) \leq v'(a, y_l)$ and the second due to $\beta(1+r) < 1$. Clearly, because $a > 0$ and so the constraint is not binding, we conclude that saving as much as a is suboptimal. Optimization requires driving down a so as to reduce the marginal utility of today's consumption relative to the marginal valuation of assets next period (follows from concavity of u and v).

Lemma 2 If $v'(a, y) > \beta(1+r)E[v'(a, y')|y]$ for $a \geq a^* > 0$ then $a'(a, y) < a$ for $a \geq a^*$

To show this, notices that the Euler equation at (a, y) is

$$u'((1+r)a + y_l - a) \geq \beta(1+r)\left[pv'(a, y_l) + (1-p)v'(a, y_h)\right]$$

Either the constraint binds and the $a' = 0 < a$ or the Euler equation must hold with equality. But at $a' = a$

$$v'(a, y) = u'((1+r)a + y_l - a) > \beta(1+r)E\left[v'\left(a, y'\right)|y\right]$$

and so optimality requires $a'(a,y) < a$

Lemma 3 There exists an a such that $a'(a,y_h) = a$.

Prove by contradition. Suppose not. Then $a'(a,y_h) > a$ for all a. From Lemma 1 $a'(a,y_l) < a$ so

$$a'(a,y_h) > a'(a,y_l)$$

Then simple algebra shows

$$\begin{aligned}
(1+r)a + y_l - a'(a,y_h) &\leq (1+r)a + y_l - a'(a,y_l) \\
c(a,y_h) - (y_h - y_l) &\leq c(a,y_l) \\
\frac{c(a,y_l)}{c(a,y_h)} &\geq 1 - \frac{(y_h - y_l)}{c(a,y_h)}
\end{aligned}$$

From v' and u' concave, totally differentiating we have $\frac{dc}{da} = \frac{v''}{u''} > 0$ so as $a \to \infty$, $c(a,y) \to \infty$ because v is bounded. Therefore

$$\frac{v'(a,y_h)}{v'(a,y_l)} = \frac{u'(c(a,y_h))}{u'(c(a,y_l))} = \left(\frac{c(a,y_l)}{c(a,y_h)}\right)^{\sigma} \geq \left[1 - \frac{(y_h - y_l)}{c(a,y_h)}\right]^{\sigma}$$

So if we make a arbitrarily large, $\frac{v'(a,y_h)}{v'(a,y_l)}$ is arbitrarily close to 1, so $v'(a,y_h) > \beta(1+r)v'(a,y_l)$. Also invoking $v'(a,y_l) \geq v'(a,y_h)$ which we have shown in lemma 1 we have

$$v'(a,y_h) > \beta(1+r)E\left[v'\left(a,y_h'\right)|y_h\right]$$

which by lemma 2 says $a'(a,y_h) < a$, a contradiction. Therefore $a'(a,y_h) = a$ does indeed have a fixed point and by the monotonicity of $a'(a,y_h)$ we have that the asset space is bounded.