1 The Income Fluctuation Problem: Consumption under Uncertainty (Part 2)

Last time we were after a problem of the form

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

$$s.t.$$

$$c_t + a_{t+1} = \left(1+r\right) a_t + y_t$$

$$a_{t+1} \geq 0$$

where we combine general preferences (i.e., we just impose $u' > 0, u'' < 0$) with a no-borrowing constraint.

In particular, we were interested in characterizing when the consumption and therefore the asset sequence (assuming bounded y_t) is bounded. We identify 6 possible cases, depending on the magnitude of $\beta\left(1+r\right)$ and whether y_t is deterministic or stochastic:

(c,a) bounded?	**no uncertainty**	**uncertainty**
$\beta\left(1+r\right) > 1$	no	no
$\beta\left(1+r\right) = 1$	yes	no
$\beta\left(1+r\right) < 1$	yes	?

1.1 $\beta\left(1+r\right) > 1$

Recall that with $\beta\left(1+r\right) > 1$ we have used Doob's supermartingale converge theorem to show that $M_t = \left[\left(1+r\right)\beta\right]^t u'\left(c_t\right)$ converges almost surely to a non-negative finite limit

$$\lim_{t\to\infty} M_t = \bar{M} < \infty$$

but since $\left[\left(1+r\right)\beta\right]^t \to \infty$ we must have $\lim_{t\to\infty} u'\left(c_t\right) = 0$ which means $c_t \to \infty$

1.2 $\beta\left(1+r\right) = 1$

With $\beta\left(1+r\right) = 1$ we were able to show that if y_t is deterministic then

$$\lim_{t\to\infty} c_t = \bar{c} = \sup_t \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j}$$

so c_t converges. For an example that illustrates this result, assume that $y_t = 0$ in even periods and 1 in odd periods. Then

$$x_t = \frac{r}{1+r}\left(1 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^4 + ...\right) = \frac{1+r}{2+r} \text{ if t odd}$$

and

$$x_t = \frac{r}{1+r}\left(0 + \left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^3 + ...\right) = \frac{1}{2+r} \text{ if t even}$$

Therefore c_t is equal to $\frac{1+r}{2+r}$ the first time the consumer reaches an odd period and stays there thereafter.

Interestingly, with uncertainty this result no longer goes through, that is, c_t will converge. We gave a simple proof for the case with $u''' > 0$, in which case the Euler equation together with Jensen's inequality says

$$u'(c_t) \geq E_t u'(c_{t+1}) > u'(E_t c_{t+1})$$

Alternatively, an alternative simple proof is available for the case y_t is iid when we no longer required $u''' > 0$.

To do so, let us write the problem recursively:

$$\begin{aligned} V(a,y) &= \max_{c,a\prime} u(c) + \beta EV(a', y') \\ a' &= (1+r)a + y - c \end{aligned}$$

Because y is iid we can use $w = (1+r)a + y$ (cash-in-hand) alone as a state variable (as the conditional expectation of $V(a', y')$ is independent of y). Then we can write

$$\begin{aligned} V(w) &= \max_{a'} u(w - a') + \beta EV(w') \\ w' &= (1+r)a' + y' \end{aligned}$$

The Euler equation is then

$$u'(c) \geq \beta(1+r) EV'(w')$$

and invoking the envelope condition

$$V'(w) = u'(c)$$

so

$$\begin{aligned} V'(w) &\geq \beta(1+r)EV'(w') \\ V'(w) &\geq EV'(w') \end{aligned}$$

Thus $V'(w)$ is a supermartingale and we have

$$\lim_{t\to\infty} EV'(w_t) = \bar{V}'$$

More so, we will show next that $\bar{V}' = 0$. To see this, notice that if $\bar{V}' > 0$, then the $\lim_{t\to\infty} w_t = \bar{w} < \infty$, that is w_t converges to a finite limit. But consumption $c = c(w)$ is a function of w alone because y is iid. Therefore next period's assets $a' = \bar{w} - \bar{c}$ converges to a finite limit. But then this is a contradiction because $w_t = (1+r)a_t + y_t$ and y_t is random.

Finally, Chamberlain and Wilson (2000) prove a more general result. They show that $\Pr(\lim_{t\to\infty} c_t = \infty) = 1$ as long as $\forall\alpha \in \mathbf{R}$ there is an ε such that

$$\Pr\left(\alpha \leq \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} \leq \alpha + \varepsilon\right) < 1 - \varepsilon$$

, i.e., provided that there is some uncertainty about the tail of the income process. The proof is involved and we do not discuss it in class.

1.3 $\beta(1+r) < 1$

1.3.1 No uncertainty

We said last time that if y_t is deterministic then we have that the asset and consumption sequence converge. Intuitively, given that they do so for $\beta(1+r) = 1$, it must be that asset accumulation is even slower when the consumer is impatient.

Let's consider next an example with $y_t = y$ constant to get used to this problem. Using the cash-in-hand notation above, we have

$$\begin{aligned} V(w) &= \max_{a'} u(w - a') + \beta V(w') \\ w' &= (1+r)a' + y \end{aligned}$$

The Euler equation is then

$$u'(c) \geq \beta(1+r)V'(w')$$

and invoking the envelope condition

$$V'(w) = u'(c)$$

Let's next invoke the concavity of V (from Stockey Lucas theorem 4.8 concavity of V follows as long as u is continuous, concave and bounded (or the state-space is bounded which we will show) and the choice set is convex) to show that $c'(w) > 0$. This follows by totally differentiating the envelope condition above:

$$u_{cc}\frac{dc}{dw} = V_{ww} \rightarrow \frac{dc}{dw} = \frac{V_{ww}}{u_{cc}} > 0$$

We next show that, as long as the borrowing constraint is not binding, cash-on-hand, w, decreases over time, i.e., if $a'(w) > 0$ then $w' < w$. To see this, notice that the Euler equation holds in this case with equality:

$$V'(w) = \beta(1+r)V'(w') < V'(w')$$

so $w' < w$ again from the concavity of the value function.

Moreover, we show that not only is w' decreasing, it will reach y in finite time (i.e., assets a will reach 0 at some point). The proof is by contradiction: assume instead that $a'(w_t) > 0$ for all t. Then the Euler equation holds with equality in all future periods:

$$0 \leq u'(c_t) = \beta(1+r)u'(c_{t+1}) = \lim_{j\to\infty}[\beta(1+r)]^j u'(c(w_{t+j})) \leq \lim_{j\to\infty}[\beta(1+r)]^j u'(c(y)) = 0$$

where the last inequality follows from the monotonicity of the consumption function and $w_{t+j} > y$.

We finally show that once w_t reaches y then $a'(y) = 0$ and $c(w) = y$ forever. Again, we prove this by contradiction: suppose $a'(y) > 0$. Then the Euler equation holds with equality and we have:

$$V'(y) = \beta(1+r)V'((1+r)a' + y) < V'((1+r)a' + y) < V'(y)$$

Thus, with a constant endowment stream and $\beta(1+r) < 1$ we have that the consumer eventually has 0 assets and consumes her endowment every period.

1.3.2 Uncertainty

This case is ambiguous. Under certain restrictions on the utility function and endowment process we can show that assets are bounded. We do so next for the case when y_t is *iid*. Recall in this case we can write $w_t = (1+r)\, a_t + y_t$ as a sufficient state variable.

We want to establish that there is an upper bound on w, $\bar{w} \geq y_{\max}$ such that if $w \in W \equiv [y_{\min}, \bar{w}]$ then $w'(w,y) = (1+r)\, a'(w) + y' \in W$.

For all $w \in W$ we have two cases: either $a'(w) = 0$ and $w' = y' \in W$ or $a'(w) > 0$ in which case the Euler equation holds with equality:

$$u'[c(w)] = \beta(1+r)\, Eu'[c(w')] = \beta(1+r)\frac{Eu'[c(w')]}{u'[c(w^{\max}(w))]}u'[c(w^{\max}(w))]$$

where $w^{\max}(w) = (1+r)(w - c(w)) + y_{\max}$ is the maximum realization of cash in hand next period, given that today's cash in hand is w. Suppose we could establish that

$$\lim_{w\to\infty}\frac{Eu'[c(w')]}{u'[c(w^{\max}(w))]} \leq 1$$

then we would have, for large enough w,

$$\begin{aligned} u'[c(w)] &= \beta(1+r)\, u'[c(w^{\max}(w))] < u'[c(w^{\max}(w))] \\ &\Longrightarrow w > w^{\max}(w) \end{aligned}$$

where we use $c'(w) > 0$. This says that we can find a $\bar{w}$ sufficiently large so that $\left|\frac{Eu'[c(w')]}{u'[c(w^{\max}(\bar{w}))]} - 1\right| < \varepsilon$ for each ε so that $\bar{w} > w^{\max}(\bar{w})$.

Thus we only need to establish conditions under which this condition holds. Compute a first-order Taylor approximation around $w' = w^{\max}$:

$$u_c(c(w')) \simeq u_c(c(w^{\max})) + u_{cc}(c(w^{\max}))\, c_w(w^{\max})(w' - w^{\max}).$$

Taking expectations of both sides

$$\begin{aligned} E[u_c(c(w'))] &\simeq u_c(c(w^{\max})) - u_{cc}(c(w^{\max}))\, c_w(w^{\max})\, E[w^{\max} - w'] \qquad (1) \\ &= u_c(c(w^{\max})) - u_{cc}(c(w^{\max}))\, c_w(w^{\max})\, E[y^{\max} - y'] \end{aligned}$$

where in the first line we use the fact that $w^{\max}$ is deterministic since it is implied by the specific income realization $y_{\max}$. And in the second line we use the fact that $w' \equiv (1+r)\, a' + y'$ which implies $w^{\max} - w' = y^{\max} - y'$.

Dividing equation (1) by $u_c\left(c\left(w^{\max}\right)\right)$ we obtain:

$$\frac{Eu'\left[c\left(w'\right)\right]}{u'\left[c\left(w^{\max}\left(\bar{w}\right)\right)\right]} \simeq 1+\alpha\left(c\left(w^{\max}\right)\right)c_w\left(w^{\max}\right)\left[y_{\max}-E\left(y'\right)\right],$$

where α is the coefficient of absolute risk aversion. Since both $\left[y_{\max}-E\left(y'\right)\right]$ and $c_w\left(w^{\max}\right)$ are positive and finite, the key condition that we need to satisfy for the limit in (**??**) to hold is

$$\lim_{w\to\infty}\alpha\left(c\left(w^{\max}\right)\right)=0. \tag{2}$$

In other words, we need *absolute risk aversion to be monotonically decreasing with asset holdings.* The faster it decreases, the smaller the upper bound on the asset space.

The intuition from these two propositions is clear: DARA means that the agent is less worried about income uncertainty as she gets rich because she becomes less risk averse, so she will consume more and accumulate less. This force limits precautionary accumulation as wealth increases. Remember that DARA is a sufficient condition and that this result holds for *iid* shocks. Huggett (1993) generalizes this result to a 2-state Markov chain for the income process (but only for the CRRA utility case).

We conclude by summarizing our findings in:

Result: In presence of borrowing constraints and uncertain income, the condition $\beta\left(1+r\right)<1$ *is necessary for the optimal consumption sequence and for the asset space to be bounded. Moreover, when* $\beta\left(1+r\right)<1$*, if income shocks are iid and absolute risk-aversion is decreasing (DARA utility), then the asset space is bounded. More in general, even with Markov shocks, as long as absolute risk aversion decreases fast enough with c, the state space will remain bounded.*