1 Numerical Methods to Solve the Income Fluctuations Problem

Consider the following problem

$$V_0 = \max E_0 \sum \beta^t u(x_t) \tag{1}$$
$$s.t$$
$$x_t + a_{t+1} = (1+r)\,a_t + \exp(y_t)$$
$$a_t \geqslant -\phi \tag{2}$$

where x_t now denotes consumption, ϕ is the borrowing constraint, i.e., the maximum amount of debt we allow the agent to hold and we assume $\beta(1+r) < 1$. We assume

$$y_t = \rho y_{t-1} + \varepsilon_t$$

and ε_t is $N(0, \sigma^2)$, with ρ determining the persistence of income shocks. Also $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

The first order conditions for this problem are:

$$u'(x_t) \geq \beta E_t u'(x_{t+1}), \quad (= \ \ if \ \ a_{t+1} = -\phi)$$

We want to solve for the decision rules $x(a,y)$ and $a'(a,y)$ that solve this problem for each a and y in the state-space. To do so, it suffices to solve for $x(a,y)$ and $a'(a,y) = (1+r)\,a + y - x(a,y)$ follows from the budget constraint.

One popular line of attack is the so-called 'collocation' method in which we approximate the unknown $x(a,y)$ with a linear combination of low-order polynomials and solve for the coefficients on this polynomials by requiring that the approximant satisfies the Euler equation at a number of (suitably chosen) nodes in the state-space. In addition to approximating $x(a,y)$, we also need to be able to compute the integral on the RHS of the Euler equation, as well as solve the complementarity problem which in this univariate case can be solved by a bisection method. See the slides that accompany this lecture for a description of each of the methods. I discuss an algorithm for solving the problem below. Also see the files start.m, euler.m and solve.m. start.m is the main file which does all the work. euler.m evaluates the difference between the LHS and RHS of the Euler equation. solve.m solves for the c that solve the Euler equation. Use help to see what the Compecon routines are doing to read this code.

1. Discretize distribution of shocks ε_t. Use Gaussian quadrature with K nodes e_s and weights w_s. Recall these $2K$ unknowns are chosen to ensure the first $2K-1$ moments of this distribution are equal to those of the original (Gaussian in this case).

2. Declare bounds for state space (a, y). For y this is easy. Given $e_{\max}$ and $e_{\min}$ are the lower and upper bound in the discretized distribution, we have $y_{\min} = \frac{e_{\min}}{1-\rho}$ and $y_{\max} = \frac{e_{\max}}{1-\rho}$. For a we know $-\phi$ is a lower bound, as for the upper bound, guess a large number, say $10\exp(y_{\max})$ and then verify that we never exceed the bounds. It is important that the optimal decision rules induce laws of motion for the endogenous state variable a are such that we will not extrapolate (evaluate the approximants outside the bounds of the state space).

3. Declare the family of approximants you will use to approximate $x(a, y)$ and the order of the approximation in each dimension. We want more nodes in the endogenous state variable, a. Associated with each function approximation family are "optimal" nodes at which you want to solve the problem.

4. Guess an optimal consumption rule $\tilde{x}^0(a, y)$. E.g., guess that $\tilde{x}^0(a, y) = y$ and the consumer consumes her endowment each period. We will update this guess at each iteration i. Call $\tilde{x}^i(a, y)$ the guess at each iteration. Recall that this guess corresponds to a guess on the coefficients on the basis functions: $\tilde{x}^i(a, y) = \sum_{l=1}^{M_a} \sum_{j=1}^{M_y} c_{lj}^i \phi_l(a) \phi_j(y)$ where e.g., $\phi_l(a)$ is an $l-th$ order basis function (e.g. Chebyshev polynomial) and c_{lj}^i is the unknown coefficients.

We will update these coefficients by solving for the optimal consumption rule at each node in the state space and then solving $\Phi c^{i+1} = x(c^i)$ where $x(c^i)$ is the x that solves the Euler equation given an initial guess c^i. Thus, at each node a_j, y_j, we solve

$$u'(x_j) - \beta \sum_{k=1}^{K} w_k u'\left(\tilde{x}^i\left((1+r)\, a_j + y_j - x_j, \rho y_j + \varepsilon_k\right)\right) \geq 0, \;\; (= \;\; if \;\; a' > -\phi)$$

where recall ε_k is the shock in state k and w_k the probability of that event in the discretized distribution. We can write this also as

$$u'(x_j) - \beta \sum_{k=1}^{K} w_k u' \left(\sum_{l=1}^{M_a} \sum_{j=1}^{M_y} c_{lj}^i \phi_l \left((1+r) a_j + y_j - x_j \right) \phi_j \left(\rho y_j + \varepsilon_k \right) \right) \geq 0, \quad (= \; if \; a' > -\phi)$$

in order to emphasize that x_j is the consumption that solves the Euler equation at state j and $\tilde{x}^i$ is the guess for what consumption is next period at the corresponding states. This is clearly a non-linear complementarity problem in $x_{j.}$

5. File euler.m computes the expression above at each nodes in the state space for a vector of conjectured consumption at each node.

6. File solve.m solves for the optimal x_j that solve the Euler equation. We first declare the upper and lower bounds on x_j (the lower bound is an arbitrarily small number, the upper bound is $(1+r)\,a_j + y_j$ and then use the bisection method, by checking the sign of the above expression and halving the range accordingly. Notice how at the end we set $x_j = (1+r)\,a_j + y_j$ if the expression above is positive at the upper bound on x_j.

7. Construct a finer partitioning of the state-space, say 2 times more nodes in each dimension than earlier and evaluate the Euler equations at these nodes. Accurary requires that the errors at these nodes are small.

8. Plot the consumption function in the a and y space.