

1 Firm Dynamics

We next extend the tools we have learned to an environment in which heterogeneity is on the firm side. In particular, we extend the equilibrium concept we used above to allow for firm entry and exit. The model we study below is a variant of that in Hopenhayn (1992) and Hopenhayn and Rogerson (1993) and is a workhorse model in macroeconomics, industrial organization and international trade.

This class of models is motivated by the observation that in the data there is substantial heterogeneity at the firm level and large reallocation of resources from unproductive firms (contracting and exiting) to productive firms (that expand and enter new markets). One third of jobs and 40% of firms are destroyed over a 5-year period and replaced by new ones. A question that this framework seeks to address is: what are the aggregate consequences of policies that tamper with this process (subsidies on large firms, firing/hiring restrictions, financing frictions etc.)

1.1 Environment

The key new ingredients of this model are 1) idiosyncratic firm-level productivity shocks, 2) a fixed cost of operating each period (which makes exit optimal for unproductive firms), 3) a sunk cost of entering.

Consumers

Consumers (we normalize their mass to 1) have preferences over consumption, c_t and work, n_t . Labor supply is restricted to be either 0 and 1: $n_t \in \{0, 1\}$. Consumers face no uncertainty and are ex-ante and ex-post identical (they own equal shares of the value of each firm) and maximize the discounted sum of future utility:

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - v(n_t)]$$

We have shown (Rogerson 1988) that in the presence of employment lotteries the economy behaves as if there were a representative agent with preferences

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - \psi N_t]$$

where N_t is the fraction of agents that works. These assumptions ensure that the representative agent's problem collapses to the static problem of choosing consumption and labor supply to

$$\begin{aligned} & \max u(c) - aN \\ & s.t. pc \leq N + \Pi + T \end{aligned}$$

where we normalize the wage to 1, Π are profits from the firms, and T are lump-sum transfers from the government.

Firms

Firms have a stochastic production function which uses labor as the only input. If a firm employs n workers in a given period, it receives profits equal to

$$\pi_t = p_t f(a_t, n_t) - n_t - p_t c^f - g(n_t, n_{t-1})$$

where p_t is the output price, n_t its labor input (wage is normalize to 1), c^f is a fixed operating cost (expressed in terms of units of output the firm must expend to continue operating) and $g(n_t, n_{t-1})$ represents costs of changing the labor input (e.g. hiring and firing taxes levied by the government). For example

$$g(n_t, n_{t-1}) = \tau \min(0, n_t - n_{t-1})$$

represents a fixed payment τ for each job destroyed. Productivity follows a first-order markov process $F(a, a')$, where $F(a, \cdot)$ is the distribution of next period's value for the shock.

We will be characterizing a stationary equilibrium in which the price of ouput will be constant, p . The Bellman equation characterizing the firm's problem is the following.

$$V(a, n) = \max_{n'} pf(a, n') - n' - pc^f - g(n', n) + \beta \max \left[\int V(a', n') dF(a, a'), -g(0, n') \right]$$

The firm's continuation value is the maximum of the value of 2 options it has: 1) continuing operating and receive $\int V(a', n') dF(a, a')$ or exit. Exiting requires firing all workers and thus paying firing taxes $g(0, n')$.

No hiring/firing costs

Let us assume $g(\cdot) = 0$ for now and simply focus on the exit-decision (Hopenhayn 1992). We will characterize first firm decision rules and then define an equilibrium. We will then return to the more general case with firing costs.

Notice that the Bellman equation reduces to

$$V(a) = \max_n pf(a, n) - n - pc^f + \beta \max \left[\int V(a') dF(a, a'), 0 \right]$$

The firm receives 0 in the event of exit and n is no longer a state-variable as we have assumed away the adjustment costs. Let $\pi(a, p)$ denote the profits a firm with productivity a makes when the price is p : $\pi(a, p) = \max_n pf(a, n) - n - pc^f$. Let $q(a, p)$ denote the associated output supplied by the firm, and $n(a, p)$ the optimal labor demand.

Let us assume:

A1. $f(a, n)$ is such that a) q and n are single valued, increasing in a and continuous, b) π is continuous, strictly increasing in a and bounded.

A2. F is continuous in a, a' and $F_a(a, a') < 0$, $a \in A \equiv [0, 1]$

A3. F is recurrent: for any $\varepsilon > 0$, there exists an integer n such that $F^n(a, \varepsilon) > 0$ where F^n is the distribution of a_{t+n} given $a_t = a$.

Under A1. one can show that $V(a)$ is unique, bounded, continuous and strictly increasing in a . Notice that this rules out production functions with constant returns to scale: $f = an$ as in this case profits are unbounded as long as $pa > 1$ (or the firm exits otherwise).

A2. says that the a process is serially correlated. This, together with A1, implies that the continuation value $\int V(a') dF(a, a')$ increases in a . This implies that the cutoff productivity level x at which the firm finds optimal to continue, $x = \inf \{x \in A : \int V(a') dF(x, a') \geq 0\}$ ($x = 1$ if this set is empty) is unique.

We have seen an assumption similar to A3 (we referred to it as the monotone mixing condition) when discussing the existence of a stationary distribution in the uninsured labor income economy.

New entrants

To start operating, a firm must pay a fixed setup cost c^e . After payment of this cost, the firm draws a productivity a from distribution G and starts producing. The value of

the entrant is therefore:

$$V^e = -pc^e + \int v(a) dG(a)$$

Measure of firms

Let M denote the mass of entrants. Let μ denote the measure of firms (over productivity) in the economy: $\mu(\mathcal{A})$ is the mass of firms with productivity in \mathcal{A} , where $\mathcal{A} \subset A$ is a Borel set. This measure evolves over time according to

$$\mu'([0, a']) = \int_x^1 F(a, a') d\mu(a) + M'G(a')$$

where the integral sums over all firms that decide to continue operating (whose productivity a is above the exit cutoff x).

Stationary Equilibrium

A stationary equilibrium is a vector $(p, c, N, x, M, \mu, \Pi, V)$ (taxes T are zero for now so that labor income and profits from ownership of firms are the sole source of income for consumers) such that

(i) $p = \frac{u'(c)}{\psi}$ (consumer optimizes: this is what the first-order conditions for the choice of c and N reduce to

(ii) $c = \int_0^1 [q(a, p) - c^f] d\mu(a) - Mc^e$ (goods market clears)

(iii) $N = \int_0^1 n(a, p) d\mu(a)$ (labor market clears)

(iv) $\int_0^1 V(a') dF(x, a') = 0$ (cutoff for exit is optimally chosen)

(v) $V^e \leq 0$ (with equality if $M > 0$)

(vi) $\mu([0, a']) = \int_x^1 F(a, a') d\mu(a) + MG(a')$

(vii) $\Pi = \int_0^1 \pi(a, p) d\mu(a) - Mf^e$

(viii) V is the unique solution to the Bellman equation.

We next outline a proof for the existence/uniqueness of an equilibrium with positive entry ($M > 0$). First, notice that p, c, N, Π, V can be written as functions of μ alone from i, ii, iii, iv and viii. One can also show that, given μ , these are unique. We can thus restrict attention to the following 3 equations:

1. $\mu(a') = \int_x^1 F(a, a') d\mu(a) + MG(a')$

2. $\int_0^1 V(a', \mu) dF(x, a') = 0$

3. $-c^e + \int v(a, \mu) dG(a) = 0$

and a stationary equilibrium is then given by (x, M, μ) that satisfy these.

Suppose that $m(x, M)$ is an invariant measure of firms given exit rule x and mass of entrants M (the solution to 1). Hopenhayn shows that m is continuous, decreasing in x and strictly increasing in M (the recurrence assumption A3 above guarantees that m is well-defined and unique).

For a fixed exit cutoff $x \in [0, 1]$, define $M_1(x)$ by $\int V(a', m(x, M_1(x))) dF(x, a') = 0$. In other words, $M_1(x)$ is the mass of entrants with the property that for the invariant measure $m(x, M_1(x))$ the exit rule x is optimal (the mass of entrants needed to rationalize an exit rule x). Clearly, $M_1(x)$ is the solution to 2. given $m(x, M)$ and a given x . Hopenhayn shows that M_1 is continuous and strictly increasing.

Finally, define $M_2(x)$ for $x \in [0, 1]$ by $\int V(a, m(x, M_2(x))) dG(a) - pc^e = 0$, i.e. the mass of entrants that is needed so that the value of entry is 0 (the mass of entrants necessary to rationalize an entrant's value of 0). Hopenhayn shows that M_2 is continuous and non-decreasing. Again, this is the solution to 3. given a particular x and $m(x, M)$.

Showing uniqueness/existence then boils down to showing that M_1 and M_2 intersect, and only once. Hopenhayn shows that this is so if the fixed cost of entry is sufficiently low and the profit function satisfies $\pi(a, p) = h(a)\phi(p)$ (which is the case, for example, for production functions $f(a, n) = ay(n)$ and y is homogenous of degree $k \in (0, 1)$.)

Hopenhayn also conducts a number of comparative statics exercises. One can show that if c^e increases, the M_2 curve shifts downward, so that x and M decrease in equilibrium. This results in higher prices (less competition) and higher profits for larger firms, as well as less turnover. An increase in c^f lowers both M_1 and M_2 , but under the assumption on the production function above the net effect is to increase equilibrium x and thus turnover. Also notice that all firms eventually die: otherwise with $M > 0$ the number of firms will grow over time. This is again ensured by the recurrence assumption above.

Calibration

Hopenhayn and Rogerson (1993) calibrate this economy to US data and then conduct policy experiments. The benchmark model is the one with $g(n', n) = 0$. Firm productivity is assumed to evolve according to $\log(a_t) = \bar{a} + \rho \log(a_{t-1}) + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma^2)$ (so no longer restricted to $[0, 1]$), $u(c) = \ln(c)$, $f(a, n) = an^\theta$.

They set $\beta = 0.8$ (4% interest rate per year (recall in this model $\beta(1+r) = 1$)) and the period is 5 years., $\theta = 0.64$ in line with RBC literature, and the rest of the parameters are chosen to match moments from US size distribution of firms, as well as serial correlation of log employment, variance of changes in log employment, mean employment, as well as the exit rate of 37%. In this model c^e is not identified separately from p (pf^e is what affects the entry decision), so one can normalize any of these to 1.

The authors then compute stationary equilibria in this economy, as well as in the economy with firing taxes, $\tau > 0$ (the definition of equilibrium with firing costs is similar, but now the measure μ is over (a, n)), and compute how aggregate prices, quantities and most importantly productivity, are affected by these taxes. Clearly, higher τ prevent unproductive firms from firing workers and thus distort the allocation of labor across firms and reduce aggregate productivity. The authors find aggregate productivity losses as large as 2% for a firing tax equal to 1 year's worth of wages.

This framework has been subsequently extended to allow for

- 1) aggregate uncertainty (entry and exit comove strongly with the cycle, esp. for small firms)
- 2) capital accumulation
- 3) borrowing frictions
- 4) occupational choice: agents choose whether to work or become entrepreneurs and create new firms.
- 5) international trade