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Nonparametric Adaptive Learning with Feedback

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Recently macroeconomists and game theorists have dropped the mutual consistency assumption for rational expectations equilibrium (REE) or Nash equilibrium (NE), considering instead plausible adaptive agent learning behaviors yielding "sensible" REE or NE. But when agents do not have *detailed* knowledge of the relevant equilibrium relations, they can easily arrive at "incorrect belief equilibria" which are not rational and have no optimality properties [14]. REE and NE thus lose plausibility as outcomes of learning. Here we study *nonparametric* adaptive learning methods that enable agents to eventually learn the relevant equilibrium relations, leading to sensible REE and NE. *Journal of Economic Literature* Classification Numbers: C14, C73, D83. © 1998 Academic Press

1. INTRODUCTION

Recently macroeconomists (e.g., [17, 22]) and game theorists (e.g., [8, 11]) have considered the consequences of dropping the mutual consistency assumption in the concept of rational expectations equilibrium (REE) or Nash equilibrium (NE) and instead search for plausible agent learning behaviors that lead to a "sensible" REE or NE. These learning models are effectively parametric recursive method of moment estimation (or stochastic approximation) algorithms in which agents correctly specify the laws of motion and other relevant functional relationships up to some finite-dimensional vector of unknown parameters (say θ). At time zero, the agents start with an arbitrary estimate $\hat{\theta}_0$. When the next period arrives, they observe new information, update their estimates (beliefs), and then make current period decisions accordingly. The process repeats each period

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with agents having estimates $\hat{\theta}_n$, $n = 1, 2, ...$ Under reasonable conditions (see, e.g., [14, 17]), it has been shown that the parametric estimator sequence $\{\hat{\theta}_n\}$ converges (as $n \to \infty$) to the set of asymptotically stable solutions which characterize a set of "sensible" REE or NE solutions.

But what if agents do not know the functional relationships relevant to the learning environment? [14] show that agents can then easily arrive at "incorrect belief equilibria" in which they persist (even asymptotically) in basing their actions on incorrect beliefs about the underlying probability laws, with the consequence that the resulting equilibria are not rational and do not possess optimality properties for the agents. Indeed, correct specification is extraordinarily critical in these parametric learning models, since neglecting heteroskedasticity in otherwise correctly specified least-squares learning models of the sort considered by [17] leads to incorrect belief equilibria, in sharp contrast to the usual situation in econometric modeling where neglecting heteroskedasticity typically leads only to inefficiency, not inconsistency.

The present state of learning theory thus presents us with an uncomfortable dilemma: Either agents must be postulated to have amazing abilities to *intuit* (not learn) a correctly specified model of the *detailed* stochastic structure of equilibrium relations in their environment, or they face the unpleasant prospect of arriving at an incorrect belief equilibrium. This undermines the central position of REE and NE, as REE and NE are thus no longer necessarily plausible outcomes of learning.

The purpose of this paper is to provide a way past this dilemma by proposing and analyzing learning methods that enable agents to learn to correctly specify (in the limit) the relevant aspects of the economic environment. Development of such methods is not only of intellectual and aesthetic interest, as this permits re-establishment of sensible REE and NE as outcomes of learning, but it is also heuristically plausible that rational agents would be driven to arrive at such methods, as elimination of misspecification can often lead to (utility improving) optimality.

The key to our approach is to have agents learn nonparametrically, that is, to let θ belong to a suitable function space. For simplicity we work throughout with infinite-dimensional separable Hilbert spaces. This permits the object of learning to be a function, delivering sufficient flexibility that plausibly well-behaved economic relationships involving a given set of variables can be approximated in the limit as accurately as desired, avoiding misspecification. Effectively, our learning algorithms are nonparametric recursive method of moment (or Hilbert-valued stochastic approximation) procedures. Under sets of sufficient conditions analogous to those for the parametric learning models in [15] or in [14], we show that $\{\hat{\theta}_n\}$ converges (as $n \to \infty$) to the set of asymptotically stable solutions to a population moment condition $\bar{M}(\theta^*) = 0$, where $\bar{M}$ is a mapping from a function space to a function space. The "fixed" points (θ^*'s) are thus functions.

A further appealing feature of our approach is that the dynamic systems we treat have considerable utility in domains other than learning by economic agents. For example, they can be used to develop nonparametric analogs of the Kalman filter which have significant applications in forecasting, signal processing, and control, or to solve stochastic dynamic programming problems in discrete or continuous time, or to study the behavior of "recurrent neural networks," which have a range of interesting applications in cognitive science, computer science, and engineering. For succinctness, however, our focus here will be strictly restricted to learning by economic agents.

The plan of this paper is as follows. In Section 2 we introduce our nonparametric learning algorithms and establish their almost-sure convergence. We establish convergence under a set of conditions that are infinite-dimensional analogs of conditions of [15]; hence they are almost necessary and sufficient, and unavoidably abstract. In Section 3 we provide a set of simpler, more readily verifiable sufficient conditions. In particular, we introduce the concept of Hilbert-valued near epoch dependent (NED) functions of underlying mixing processes (observable or not). This is a widely applicable class of asymptotically ergodic processes that is well-suited to describing economic processes generated by agents' adaptive learning with feedback and hidden information. In Section 4 we illustrate our results with four examples. Example 4.1 is a modification of Bray and Savin's (1986) model where the equilibrium price is now a nonlinear function (of unknown form) of market signals (see [4]). If agents simply use Bray and Savin's method of recursive linear least squares to forecast price, they will end up with a fixed point which is no longer the REE price. Nevertheless, we show that agents can learn the REE price nonparametrically. Example 4.2 is a modification of the model of [9] where the exogenous shocks are no longer assumed to be identically and independently distributed. Hence there is no longer any obvious parametric learning rule for convergence without simultaneously learning about the law of motion of the serially correlated shocks. However, a nonparametric approach can achieve equilibrium. Example 4.3 is a generalization of fictitious play that involves "continuum strategies." Here, each player's chosen action is a function of the estimated density of the opponent's response and past actions. We provide for the first time conditions ensuring convergence to Nash equilibrium in such a game. Example 4.4 is a simple application to stochastic dynamic programming problems. Section 5 is a brief summary. Appendix A presents some definitions and technical conditions used in Section 3, while Appendix B contains all the mathematical proofs.

2. ALGORITHMS AND ALMOST-SURE CONVERGENCE

2.1. *The Basic Algorithm* (*Bounded Case*)

We consider learning algorithms that evolve according to

$$\hat{\theta}_{n+1} = \hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n),$$
$$\hat{\xi}_{n+1} = R_n(\hat{\xi}^n, \hat{\theta}^{n+1}, Z_{n+1}), \qquad n = 0, 1, 2, ..., \tag{2.1}$$

where $\hat{\xi}^n \equiv (\hat{\xi}_0, ..., \hat{\xi}_n)$, $\hat{\theta}^{n+1} = (\hat{\theta}_0, ..., \hat{\theta}_{n+1})$. The objects entering (2.1) are described by our formal assumptions. We begin by describing the underlying data generating process.

Assumption A.1. $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space on which is defined an exogenous stochastic process $\{Z_n: \Omega \to G; n = 0, 1, 2, ...\}$ (i.e., a sequence of $\mathscr{F}/\mathscr{B}(G)$-measurable mappings, generated by nature), where G is a real separable Hilbert space.

Typically $\{Z_n\}$ is a discrete-time weakly dependent process and G is a bounded subset of finite-dimensional Euclidean space. Nevertheless, A.1 allows for $\{Z_n\}$ to be a sequence of continuous-time or spatial random functions and G to be an infinite-dimensional Hilbert space.

The next assumption describes the decay of the learning rate a_n.

Assumption A.2. $\{a_n; n = 0, 1, 2, ...\}$ is a sequence of nonincreasing positive exogenous random numbers such that: $a_n \to 0$ as $n \to \infty$ *a.s.* $- \mathbb{P}$ and $\sum_{n=0}^{\infty} a_n = \infty$ *a.s.* $- \mathbb{P}$.

Throughout the paper, $\mathbb{H}$ denotes a real separable infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and the weak topology induced metric d. Ξ denotes a closed bounded convex subset of a real separable Hilbert space with norm $\|\cdot\|_{\Xi}$ and the weak topology induced metric d_{Ξ}. The next assumption (either A.3 or A.3′) describes the learning update (moment) mappings M_n.

Assumption A.3. The mapping $M_n: \Xi \times \mathbb{H} \to \mathbb{H}$ is measurable (for each n), and is (a) bounded uniformly in n; (b) weakly sequentially continuous for each n.

Assumption A.3′. (a) A.3(a) holds; (b) for each n, M_n is uniformly continuous on bounded sets.

Notice that either A.3(b) or A.3′(b) implies the boundedness of M_n for each n, ([1, p. 65]). We relax A.3(a) in Subsection 2.2 for a modified version of (2.1).

Next we describe the mapping R_n governing the underlying law of motion of the dynamic system.

Assumption A.4. For each n, R_n: $\Xi^n \times \mathbb{H}^{n+1} \times G \to \Xi$ is a known bounded Borel measurable mapping, where $\Xi^n \equiv \times_{t=0}^{n} \Xi$ and $\mathbb{H}^{n+1} \equiv \times_{t=0}^{n+1} \mathbb{H}$.

The next assumption specifies initial values $\hat{\xi}_0$ and $\hat{\theta}_0$ for the learning recursions.

Assumption A.5. $\hat{\xi}_0$: $\Omega \to \Xi$ and $\hat{\theta}_0$: $\Omega \to \mathbb{H}$ are arbitrary measurable mappings independent of $\{Z_n\}$, with $\|\hat{\theta}_0\| \leqslant B < \infty$ *a.s.* $- \mathbb{P}$.

With A.1–A.5 holding, the learning recursion (2.1) is a well-defined Hilbert space ($\mathbb{H}$) version of the Robbins–Monro ([19], RM) procedure with feedback (RMF) introduced in ([15], KC). Note that KC assume that the learning update function, $M_n \equiv M$: $\Xi \times \mathbb{R}^J \to \mathbb{R}^J$ (with fixed $J < \infty$), is independent of sample size or time, n. To allow nonparametric learning, we permit M_n: $\Xi \times \mathbb{H} \to \mathbb{H}$ to change with n so that the estimator $\hat{\theta}_n$ may belong to larger subsets of $\mathbb{H}$ as n gets larger. This is effectively a sieve recursive generalized method of moments [13] estimator. Subsection 2.2 contains an important alternative definition of M_n based on this point of view.

Denote $t_0 = 0$, $t_n = \sum_{0 \leqslant i \leqslant n-1} a_i$ and $m(t) \equiv \max[n \geqslant 0: t_n \leqslant t]$ if $t > 0$ and $m(t) \equiv 0$ if $t \leqslant 0$. The next assumption (A.6 or A.6′) relates M_n to its limiting (population moment) mapping $\bar{M}$.

Assumption A.6. There is a measurable mapping $\bar{M}$: $\mathbb{H} \to \mathbb{H}$ such that: (a) $\bar{M}$ is weakly sequentially continuous; (b) for each $0 < T$, $\bar{B} < \infty$, each $\theta \in \mathbb{H}$ with $\|\theta\| \leqslant \bar{B}$, and for any $\varepsilon > 0$

$$\lim_n \mathbb{P}\left[\sup_{j \geqslant n} \max_{t \leqslant T} d\left(\sum_{m(jT) \leqslant i \leqslant m(jT+t)-1} a_i[M_i(\xi_i(\theta), \theta) - \bar{M}(\theta)], 0\right) \geqslant \varepsilon\right] = 0,$$

where $\xi_n(\theta)$ is defined recursively as

$$\xi_{n+1}(\theta) = R_n(\xi^n(\theta), \theta^{n+1}, Z_{n+1}), \quad n = 0, 1, 2, ..., \quad \xi_0(\theta) = \xi^0(\theta) = \hat{\xi}_0,$$

$\xi^n(\theta) = (\xi_1(\theta), ..., \xi_n(\theta))$, and $\theta^{n+1} \equiv (\theta, ..., \theta)$ is the point in $\mathbb{H}^{n+1}$ with identical coordinate θ in each position.

Assumption A.6′. There is a measurable mapping $\bar{M}$: $\mathbb{H} \to \mathbb{H}$ such that: (a) $\bar{M}$ is uniformly continuous on bounded sets; (b) for each $0 < T$, $\bar{M} < \infty$, any $\theta \in \mathbb{H}$ with $\|\theta\| \leqslant \bar{B}$, and for any $\varepsilon > 0$

$$\lim_n \mathbb{P}\left[\sup_{j \geqslant n} \max_{t \leqslant T} \left\|\sum_{m(jT) \leqslant i \leqslant m(jT+t)-1} a_i[M_i(\xi_i(\theta), \theta) - \bar{M}(\theta)]\right\| \geqslant \varepsilon\right] = 0.$$

Notice that Assumption A.6′(b) implies A.6(b), and both are satisfied by most asymptotically ergodic processes (see Section 3 for one set of sufficient conditions).

Let $\theta^o(\cdot)$ be a piecewise linear interpolation of $\{\hat{\theta}\}$ with interpolation intervals $\{a_n\}$, i.e.,

$$\theta^o(t) \equiv \hat{\theta}_n \times (t_{n+1} - t)/a_n + \hat{\theta}_{n+1} \times (t - t_n)/a_n \qquad \text{for} \quad t \in [t_n, t_{n+1}).$$

Define $\theta^n(\cdot)$, the left shift of $\theta^o(\cdot)$, by $\theta^n(t) = \theta^o(t_n + t)$. We make one more assumption (A.7 or A.7′) to describe the effect of the feedback rule; however, because it is technical and because we will state a set of more interpretable sufficient conditions in Section 3, we place it in Appendix A in order to proceed directly to our first main result.

THEOREM 2.1. *Let* $\sup_n \|\hat{\theta}_n\| < \infty$ *a.s.* $-\,\mathbb{P}$. *Suppose the RMF*(2.1) *satisfies* A.1, A.2, A.4, A.5, *and one of the following two conditions*: (1) A.3, A.6 *and* A.7; *or* (2) A.3′, A.6′ *and* A.7′. *Then*:

(i) $\{\theta^n(\cdot)\}$ *is sequentially compact under the weak topology, and any weak limit* $t \to \theta(t)$ *satisfies the ODE* $\dot{\theta} = \bar{M}(\theta(\cdot))$.

Let Θ^* *be the set of* (*Lyapunov*) *locally asymptotically stable equilibria in* $\mathbb{H}$ *for this ODE with domain of attraction* $da(\Theta^*) \subset \mathbb{H}$. *Fix a subset* $C \subset da(\Theta^*)$, *compact under the weak topology.*

(ii) *If* $\hat{\theta}_n \in C$ *infinitely often, then* $\hat{\theta}_n \to \Theta^*$ *as* $n \to \infty$ *a.s.* $-\,\mathbb{P}$ *in the weak topology, i.e., there exists* Ω^* *with* $\mathbb{P}(\Omega^*) = 1$ *such that for any* $\omega \in \Omega^*$,

$$\lim_n \Big[\inf_{\theta \in \Theta^*} |\langle \hat{\theta}_n(\omega), h \rangle - \langle \theta, h \rangle| \Big] = 0, \qquad \text{for all} \quad h \in \mathbb{H}.$$

Remark 2.2. (a) In a finite-dimensional Euclidean space, all topologies generated by the different metrics are equivalent; hence Theorem 2.1 includes the parametric results of KC [15, p. 76, theorem 2.5.2] as a special case. Therefore, it also includes the results stated in [16, 17, 23, and 14 (KW)].

(b) If Θ^* has only finitely many isolated elements, the particular element to which $\{\hat{\theta}_n\}$ will converge depends on the initial conditions and the realization ω (i.e., realized noise). Therefore, we cannot rule out the possibility of sunspot equilibria. This also occurs in correctly specified parametric learning models, e.g., [22].

Remark 2.3. (a) Theorem 2.1 is established under mild conditions on M_n and $\bar{M}$. In particular, Assumptions A.3(b) and A.6(a) are satisfied by any linear bounded operators, completely continuous operators, and continuous monotone operators. Assumptions A.3′(b) and A.6′(a) are satisfied by any linear bounded operators, completely continuous operators, Hölder and Lipschitz operators, uniformly continuous operators, and certain continuous monotone operators, compact operators, and continuous operators with polynomial growth rates. (See Appendix A for the definitions, and [24, 25] for numerous examples.)

(b) Often $\bar{M}$ is a gradient operator (i.e., Gateaux derivative) of a real-valued function $F \in C^1(\mathbb{H}, \mathbb{R})$, which in turn is an objective function for some estimation problem. Now $\bar{M}(\theta) = 0$ for all $\theta \in \Theta^*$ gives all the critical points of F. By [25, Proposition 41.8 and Corollary 41.9, pp. 235–236], F is continuous with respect to weak convergence in $\theta \in \mathbb{H}$, provided that $\bar{M}$ is either (i) completely continuous, or (ii) monotone (F is convex or concave), or (iii) pseudomonotone and locally bounded, or (iv) compact, or (v) has negative semi-definite directional derivative. Therefore $\hat{\theta}_n \to \Theta^*$ weakly implies $F(\hat{\theta}_n) \to F(\Theta^*)$ strongly if $\bar{M}$ satisfies any one of (i)–(v). Consequently, in a utility space or a profit space, value functions evaluated at $\hat{\theta}_n$ will converge almost surely to the optimal values that would be achieved if learning were not necessary.

2.2. *A Modified Algorithm* (*Unbounded Case*)

In Theorem 2.1, we have assumed the uniform boundedness of $\{M_n\}$, $\{\hat{\xi}_n\}$, and $\{\hat{\theta}_n\}$. Because R_n embodies agents' actions, it is plausible to assume that agents always choose their actions $\hat{\xi}_n$ from a bounded set Ξ. To accommodate the possibility that $\|M_n\|$ may increase with n and $\hat{\theta}_n$ may become unbounded, we need to modify the basic algorithm (2.1).

First we redefine $M_n \equiv \pi_{k(n)} M$, where $M\colon \Xi \times \mathbb{H} \to \mathbb{H}$ is a measurable mapping, $\pi_k\colon \mathbb{H} \to \mathbb{H}_k$ is a projection operator (not necessarily orthogonal), $\{\mathbb{H}_k\colon k = 0, 1, 2, ...\}$ is a sequence of increasing closed, bounded, and convex subsets of $\mathbb{H}$ with $\bigcup_k \mathbb{H}_k = \mathbb{H}$, and $k(\cdot)\colon \mathbb{N} \to \mathbb{N}$ is a nondecreasing function of sample size n.

Assumption B.3. (a) $M\colon \Xi \times \mathbb{H} \to \mathbb{H}$ is bounded; (b) $\pi_k\colon \mathbb{H} \to \mathbb{H}_k$ is a continuous monotone projection operator such that $\sup_{\theta \in \mathbb{H}} \|\theta - \pi_k \theta\| \to 0$ as $k \to \infty$; (c) $k(\cdot)\colon \mathbb{N} \to \mathbb{N}$ is nondecreasing such that $k(n) \to \infty$ as $n \to \infty$ and $k(n) < n$.

This permits such nonparametric recursive estimators as kernel, orthogonal series, spline, wavelets, neural networks, etc. For example, neural network estimators are implemented by putting

$$\mathbb{H}_{k(n)} = \left\{ \theta(x) = \sum_{j=1}^{k(n)} \beta_j \phi(\gamma_0 + \gamma_j' x)\colon |\beta| + |\gamma| \leqslant b_n < \infty \right\}, \qquad \text{with}$$

$$b_n \to \infty, \qquad k(n) \to \infty \qquad \text{as} \quad n \to \infty,$$

and $\phi(\cdot)$ a bounded "activation" function (e.g., any cumulative distribution function). Another example is the finite-dimensional orthogonal projection RM algorithm without feedback (e.g., [23]):

$$\hat{\theta}_{n+1} = \hat{\theta}_n + a_n P_{k(n)}[\bar{M}(\hat{\theta}_n) + Z_n], \tag{2.2}$$

where $P_{k(n)}: \mathbb{H} \to \mathbb{H}_{k(n)}$ is an orthogonal projection, and $\{Z_n\}$ is a sequence of continuous-time-record martingale difference shocks.

Next we consider a "resetting" which differs from the traditional Ljung-type [16] "fixed resetting" methods for the parametric RMF algorithms. By definition, the "fixed resetting" procedure resets $\hat{\theta}_n$ back into a fixed bounded set that contains all asymptotic equilibrium points whenever $\hat{\theta}_n$ attempts to escape. The procedure introduced next instead allows the possibility that agents do not know the "correct" fixed bounded set; it is a modification of [23]'s approach for the orthogonal projection RM algorithm without feedback (2.2).

Let $\{B_n; n = 0, 1, 2, ...\}$ be a sequence of strictly increasing positive real numbers with $\lim_{n \to \infty} B_n = \infty$. Define a sequence of positive integer-valued random variables by

$$T(0) = 0, \qquad T(n+1) = T(n) + 1(J_n^c),$$

where 1(A) denotes the indicator of the set $A \in \mathscr{F}$ and $J_n \equiv \{\|\hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n)\| \leqslant B_{T(n)}\}$.

A randomly truncated RMF (TRMF) in a Hilbert space is

$$\begin{aligned} \hat{\theta}_{n+1} &= [\hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n)]\, 1(J_n) + \bar{\theta} 1(J_n^c), \\ \hat{\xi}_{n+1} &= R_n(\hat{\xi}^n, \hat{\theta}^{n+1}, Z_{n+1}), \qquad n = 0, 1, 2, ..., \end{aligned} \tag{2.3}$$

where $\hat{\xi}_0$ and $\hat{\theta}_0$ are as in Assumption A.5 and $J_n^c \equiv \{\|\hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n)\| > B_{(T(n))}\}$. We fix $0 < B < B_1$ and choose $\bar{\theta} \in \mathbb{H}$ to be an arbitrary fixed element with $\|\bar{\theta}\| < B$.

One problem with both the fixed resetting method and our TRMF is the possibility of an infinite number of resettings, affecting the limiting dynamics. To ensure that resetting happens only a finite number of times almost surely, we add two more assumptions to ensure that Θ^* is a set of asymptotically stable solutions.

Assumptions B.8. $\lim_{\|\theta\| \to \infty} \bar{M}(\theta) \neq 0$; $\Theta^* \equiv \{\theta \in \mathbb{H}: \bar{M}(\theta) = 0\}$ is compact in $(\mathbb{H}, d)$.

Assumption B.9. There is a bounded and twice continuously Frechet differentiable functional $V: \mathbb{H} \to \mathbb{R}$ such that: (a) $V(\theta) \geqslant 0$ for $\theta \in \mathbb{H}$, $\lim_{\|\theta\| \to \infty} V(\theta) = \infty$; (b) $V(\theta) \neq V(\theta^*)$, for $\theta \in \mathbb{H} - \Theta^*$ and $\theta^* \in \Theta^*$.

(c) $\langle V'(\theta), \bar{M}(\theta)\rangle < 0$, for all $\theta \in \mathbb{H} - \Theta^*$; and (d) $V(\bar{\theta}) < b$ and $[V(\bar{\theta}, b] \cap V(\Theta^*) \neq [V(\bar{\theta}, b]$, where $b \equiv \inf\{V(\theta): \|\theta\| = B\}$.

THEOREM 2.4. *Let the* TRMF(2.3) *satisfy* A.1, A.2, B.3, A.4, A.5, B.8, B.9, *and either* (1) A.3(b), A.6, A.7, *or* (2) A.3′(b), A.6′, A.7′. *Then all conclusions of Theorem* 2.1 *hold.*

2.3. *Almost-Sure Convergence in Norm*

Often in economic and econometric applications V is convex and $\bar{M}$ is monotone. For such cases, we can obtain stronger results on the structure of Θ^* and the convergence of $\hat{\theta}_n$.

Assumption B.9′. There is a convex and twice continuously Frechet differentiable functional $V: \mathbb{H} \to \mathbb{R}$ such that: (a) B.9(a), (c) and (d) hold; (b) $V(\theta) > V(\theta^*)$ for $\mathbb{H} - \Theta^*$ and $\theta^* \in \Theta^*$.

Assumption B.9″. There is a convex and twice continuously Frechet differentiable functional $V: \mathbb{H} \to \mathbb{R}$ such that: (a) B.9(c) and (d) hold; (b) $\langle V'(\theta) - V'(\tilde{\theta}), \theta - \tilde{\theta}\rangle \geqslant c\,\|\theta - \tilde{\theta}\|^p$ for some fixed $c > 0$, $p > 1$, and for all $\theta, \tilde{\theta} \in \mathbb{H}$.

COROLLARY 2.5. *Let the* TRMF(2.3) *satisfy* A.1, A.2, B.3, A.4, A.5 *and one of the following two conditions*: (1) A.3(b), A.6, A.7; *or* (2) A.3′(b), A.6′, A.7′.

(i) *If B.9′ holds, then $\hat{\theta}_n \to \Theta^*$ weakly, and $V(\hat{\theta}_n) \to V(\Theta^*)$ as $n \to \infty$ a.s. $-\ \mathbb{P}$, where Θ^* is closed, bounded, and convex.*

(ii) *If B.9″ holds, then $\Theta^* = \{\theta^*\}$ (a singleton), and $\|\hat{\theta}_n - \theta^*\| \to 0$ as $n \to \infty$ a.s. $-\ \mathbb{P}$.*

Assumption B.10. (a) $\langle \bar{M}(\theta), \theta\rangle\, \|\theta\|^{-1} \to -\infty$ as $\|\theta\| \to \infty$;

(b) $\langle \bar{M}(\theta) - \bar{M}(\tilde{\theta}), \theta - \tilde{\theta}\rangle \leqslant 0$ for all $\theta, \tilde{\theta} \in \mathbb{H}$.

Assumption B.11. $\langle \bar{M}(\theta) - \bar{M}(\tilde{\theta}), \theta - \tilde{\theta}\rangle \leqslant -g(\|\theta - \tilde{\theta}\|)\, \|\theta - \tilde{\theta}\|$ for all $\theta, \tilde{\theta} \in \mathbb{H}$, where g is strictly increasing and continuous on $[0, \infty)$ with $g(0) = 0$ and $g(y) \to \infty$ as $y \to \infty$.

COROLLARY 2.6. *Let the* TRMF(2.3) *satisfy* A.1, A.2, B.3, A.4, A.5 *and one of the following two conditions*: (1) A.3(b), A.6(b), A.7; *or* (2) A.3′(b), A.6′(b), A.7′.

(i) *If* B.10 *holds, then $\hat{\theta}_n \to \Theta^*$ weakly almost-surely, where Θ^* is closed, bounded, and convex.*

(ii) *If* B.11 *holds, then $\Theta^* = \{\theta^*\}$ (a singleton), and $\|\hat{\theta}_n - \theta^*\| \to 0$ as $n \to \infty$ a.s. $-\ \mathbb{P}$.*

3. SUFFICIENT CONDITIONS FOR ALMOST-SURE CONVERGENCE

In this section we provide less abstract sufficient conditions for A.6′(b) and A.7′. We first introduce some useful measures of stochastic dependence.

DEFINITION 3.1. (1) Let $\mathscr{A}$, $\mathscr{G}$ be two σ-subfields on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Define two measures of dependence as:

$$\alpha(\mathscr{A}, \mathscr{G}) \equiv \sup\left[|\mathbb{P}(A \cap C) - \mathbb{P}(A)\,\mathbb{P}(C)|: A \in \mathscr{A}, C \in \mathscr{G}\right];$$

$$\phi(\mathscr{A}, \mathscr{G}) \equiv \sup\left[|\mathbb{P}(C \mid A) - \mathbb{P}(C)|: A \in \mathscr{A}, \mathbb{P}(A) > 0, C \in \mathscr{G}\right].$$

(2) Let $\{D_n\}$ be a sequence of Banach-valued random elements ($\mathbb{B} - r.e.$'s) defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and denote $\mathscr{A}_a^b \equiv \sigma(D_j; a \leqslant j \leqslant b)$. Define

$$\alpha(m) \equiv \sup_n \left[\alpha(\mathscr{A}^n_{-\infty}, \mathscr{A}^{\infty}_{n+m})\right]; \qquad \phi(m) \equiv \sup_n \left[\phi(\mathscr{A}^n_{-\infty}, \mathscr{A}^{\infty}_{n+m})\right].$$

If $\lim_{m\to\infty} \alpha(m) = 0$, then $\{D_n\}$ is called an α-*mixing* sequence. If $\lim_{m\to\infty} \phi(m) = 0$, then $\{D_n\}$ is called a ϕ-*mixing* sequence.

(3) Let $\{D_n; -\infty < n < \infty\}$ be a $\mathbb{B} - r.e.$ sequence and $\{W_n; -\infty < n < \infty\}$ be an $\mathbb{H} - r.e.$ sequence. Then $\{W_n\}$ is called $L_p(\mathbb{H})$-*near epoch dependent* (NED) on $\{D_n\}$ if $\|W_n\|_p < \infty$, $1 \leqslant p < \infty$, and there exist constants $\{\mu_m \geqslant 0; m \geqslant 0\}$ with μ_m decreasing to zero as $m \to \infty$ and $\{d_n \geqslant 0; n \geqslant 1\}$ such that

$$\|W_n - E[W_n \mid \mathscr{A}^{n+m}_{n-m}]\|_p \leqslant \mu_m d_n, \qquad \text{where } \mathscr{A}^b_a \text{ is as in (2).}$$

We say that μ_m is of size $-a$ if $\sum_{m=0}^{\infty} [\mu_m]^{\delta} < \infty$ or $\mu_m = o(m^{-1/\delta})$ for some $a < (1/\delta)$ or $\mu_m = O(m^{\lambda})$ for some $\lambda < -a$.

[3, Section 21] first introduced the concept of real-valued NED functions of underlying mixing sequences. [12] illustrated how the NED concept nicely captures the characteristics of data generated by asymptotically stable dynamic processes relevant to econometric modeling. [14] shows that NED structures are well suited to analyzing parametric RM algorithms with feedback and correlated heterogeneous noise. Heuristically, the feedback present in learning systems introduces memory of the arbitrarily distant past into the system. A contraction property in the law of motion (see Assumption C.4 below) ensures that the effect of the distant past eventually becomes negligible. NED is the formal structure that enables us to exploit this asymptotic negligibility in establishing our convergence results. Recently [7] have introduced the notion of Hilbert-valued NED processes in order to study properties of non-parametric estimators for time series data, including those considered here.

Assumption C.1. (a) Assumption A.1 holds with G a norm-closed, bounded convex subset of a real separable Hilbert space with norm $\|\cdot\|_G$, containing the support of the distribution induced by Z_n, $n=0, 1, 2, ...$; (b) $\{Z_n; n=0, 1, 2, ...\}$ has finite L_r-norm with $r \geqslant 2$, and is a sequence of functions $L_2(G)$-NED on $\{D_n\}$ of size $-1/2$, where $\{D_n\}$ is a $\mathbb{B}$-valued mixing process on $(\Omega, \mathscr{F}, \mathbb{P})$ with ϕ_m of size $-1/2$ or α_m of size -1.

Assumption C.3. (a) B.3 and A.3(b) (or A.3′(b)) hold; (b) for each n, there exists $c_{3,n}$, independent of $\theta \in \mathbb{H}$, $\|\theta\| \leqslant \bar{B}$, with $0 \leqslant c_{3,n} < \infty$, such that for any $\xi_1, \xi_2 \in \Xi$,

$$\|M_n(\xi_1, \theta) - M_n(\xi_2, \theta)\| \leqslant c_{3,n} \|\xi_1 - \xi_2\|_\Xi.$$

Assumption C.2. A.2 holds, and $\sum_{n=0}^{\infty} (c_{3,n} a_n)^2 < \infty$ *a.s.* $- \mathbb{P}$.

Notice that C.3 and C.2 restrict the rate at which $k(n) \to \infty$ in $M_n \equiv \pi_{k(n)} M$.

Assumption C.4. $\rho_n: \Xi \times \mathbb{H} \times G \to \Xi$ is continuous and bounded uniformly in n, and

(a) For each $(\theta, z) \in \mathbb{H} \times G$, the mapping $\rho_n(\cdot, \theta, z)$: $(\Xi, \|\cdot\|_\Xi) \to (\Xi, \|\cdot\|_\Xi)$ is contractive uniformly in (θ, z, n); i.e., there exists a c_0 independent of (θ, z, n) with $0 \leqslant c_0 < 1$, such that for any $\xi_1, \xi_2 \in \Xi$,

$$\|\rho_n(\xi_1, \theta, z) - \rho_n(\xi_2, \theta, z)\|_\Xi \leqslant c_0 \|\xi_1 - \xi_2\|_\Xi.$$

(b) For each $(\xi, z) \in \Xi \times G$, the mapping $\rho_n(\xi, \cdot, z)$: $(\mathbb{H}, d) \to (\Xi, \|\cdot\|_\Xi)$ is Lipschitz continuous on $\{\theta: \|\theta\| \leqslant \bar{B} < \infty\}$ uniformly in (ξ, z, n).

(c) For each $(\xi, \theta) \in \Xi \in \mathbb{H}$, the mapping $\rho_n(\xi, \theta, \cdot)$: $(G, \|\cdot\|_G) \to (\Xi, \|\cdot\|_\Xi)$ is Lipschitz continuous uniformly in (ξ, θ, n).

Assumption C.6. For each $\theta \in \mathbb{H}$ with $\|\theta\| \leqslant \bar{B}$,

$$\lim_{n \to \infty} \|E[M_n(\xi_n(\theta), \theta)] - \bar{M}\| = 0.$$

THEOREM 3.2. *Suppose* $\hat{\xi}_{n+1} = \rho_n(\hat{\xi}_n, \hat{\theta}^{n+1}, Z_{n+1})$ *for all* $n \geqslant 0$, *and* C.1–C.4, A.5, C.6 *hold.*

(i) *If* A.6(*a*) (*or* A.6′(*a*)) *and* $\sup_n \|\theta_n\| < \infty$ *a.s.* $- \mathbb{P}$ *hold for the RMF*(2.1), *then all conclusions of Theorem* 2.1 *hold.*

(ii) *If* A.6(a) (*or* A.6′(a)), B.8 *and B.9 hold for the* TRMF(2.3), *then all conclusions of Theorem* 2.4 *hold.*

(iii) *If Assumption* B.11 *holds for the* TRMF(2.3), *then* $\|\hat{\theta}_n - \theta^*\| \to 0$ *as* $n \to \infty$ *a.s.* $- \mathbb{P}$.

4. APPLICATIONS

We first discuss two examples with fairly simple feedback structures. Despite this simplicity, our examples illustrate how our approach can be used to achieve substantive extensions of interesting learning models and to deliver useful insights.

EXAMPLE 4.1. A Modified Bray and Savin (1986) Model.

[4 (BS)] assume a continuum of firms on $[0, 1]$ producing homogeneous perishable goods. Each firm makes its production decision at t before the realization of an exogenous stochastic linear demand $d_t = a_1 - a_2 p_t + v_t$ with $a_2 > 0$, where $\{v_t\}$ is a sequence of zero-mean, *i.i.d.* unobservable demand shocks. Each firm $i \in [0, 1]$ has a quadratic cost function, $0.5q^2 - b'_i X_t q - w_{i,t} q$, where $b'_i \equiv (b_{i,1}, ..., b_{i,k})$ and $X'_t \equiv (X_{1,t}, ..., X_{k,t})$. Firm i observes production shocks X_t, $w_{i,t}$ before making its supply decision, where $\{X_t\}$ is a sequence of *i.i.d.* shocks observable by all firms, while $\{w_{i,t}\}$ is a sequence of zero-mean *i.i.d.* shocks observable only by firm i. Hence, firm i's optimal output at t is $q_i = p^e_{i,t} + b'_i X_t + w_{i,t}$, where $p^e_{i,t}$ is the firm's guess about p_t. The market average supply is $s_t = \int_{[0,1]} p^e_{i,t}\, di + b' X_t + w_t$, with $b = \int_{[0,1]} b_i\, di$, $w_t = \int_{[0,1]} w_{i,t}\, di$. The market clearing condition $d_t = s_t$ implies that

$$p_t = (a_2)^{-1} \left[a_1 - b' X_t - \int_{[0,1]} p^e_{i,t}\, di \right] + u_t, \qquad u_t = (v_t - w_t)/a_2.$$

Suppose that $\{X_t\}$ is independent of $\{v_{0,t}\}$ and $\{w_{i,t}, i \in [0, 1]\}$. Then there is a unique REE price $p^{REE}_t = [a_1 - X'_t b]/(a_2 + 1) + \mu_t$, and the REE price forecast is

$$p^e_{i,t} = E[p_t \mid p_{l-1}, X_l, w_{i,l}, 1 \leqslant l \leqslant t] = [a_1 - X'_t b]/(a_2 + 1) = \tilde{X}'_t \beta^*,$$

with $\tilde{X}'_t \equiv (1, X'_t)$, and $\beta^* \equiv (a_2 + 1)^{-1} (a_1, -b')'$. [4] consider several ways to form $p^e_{i,t}$. One method is to let firms guess $p^e_{i,t} = \theta(X_t) = \tilde{X}'_t \hat{\beta}_{t-1}$, and obtain $\hat{\beta}_{t-1}$ via OLS. To express learning recursively, we write $\hat{\beta}_0 = 0$, $Q_0 = 0$, and for $t \geqslant 1$,

$$\hat{\beta}_t = \hat{\beta}_{t-1} + t^{-1}(Q_t)^{-1}\, \tilde{X}_t(\hat{p}_t - \tilde{X}'_t\hat{\beta}_{t-1}),$$
$$Q_t = Q_{t-1} + t^{-1}(\tilde{X}_t, \tilde{X}'_t - Q_{t-1}),$$

where $\hat{p}_t = (a_2)^{-1}\,[a_1 - \tilde{X}'_t\hat{\beta}_{t-1} - X'_t b] + \mu_t$. Assuming that $E[X_t, X'_t]$ is finite and positive definite, it is easy to check that $\hat{\beta}_t \to \beta^*$ and $\hat{p}_t - [\tilde{X}'_t\beta^* + \mu_t] \to 0$ as $t \to \infty$ *a.s.* $-\,\mathbb{P}$. Hence the market clearing price generated by this parametric learning rule converges to the unique REE price $\tilde{X}'_t\beta^* + \mu_t$.

Now consider the following modification of the model of [4]. Suppose firms face the same stochastic linear demand as before, but now firm i's cost function is: $0.5q^2 - f_i(X_t)\,q - w_{i,t}q$, where X_t and $w_{i,t}$ are defined as before. Firm i knows its own nonlinear functional form $f_i(\cdot)$ but does not know the form of $f_l(\cdot)$ for $l \neq i$. Firm i's optimal output at t is $q_i = p^e_{i,t} + f_i(X_t) + w_{i,t}$, where $p^e_{i,t}$ is its guess about p_t. The market average supply is $s_t = \int_{[0,1]} p^e_{i,t}\, di + \bar{f}(X_t) + w_t$, where $\bar{f}(X_t) \equiv \int_{[0,1]} f_j(X_t)\, dj$, $w_t \equiv \int_{[0,1]} w_{i,t}\, di$. The market clearing condition implies:

$$p_t = (a_2)^{-1}\left[a_1 - \int_{[0,1]} p^e_{i,t}\, di - \bar{f}(X_t)\right] + u_t, \qquad u_t \equiv (a_2)^{-1}\,[v_t - w_t].$$

There is again a unique REE price $p^{REE}_t = [a_1 - \bar{f}(X_t)]/(a_2 + 1) + u_t$, and the REE price forecast is $p^e_{i,t} = [a_1 - \bar{f}(X_t)]/(a_2 + 1)$. (Notice here that our definition of a REE solution given information set $\mathscr{F}_t$ is the solution generated by the conditional expectation given $\mathscr{F}_t$. In contrast, [10] call a solution generated by the best linear prediction on $\mathscr{F}_t$ a REE solution.)

Case 1. Suppose that firms still use the linear guess $P^e_{i,t} = \tilde{X}'_t\hat{\beta}_{t-1}$, where $\hat{\beta}_{t-1}$ is again obtained via OLS, only now with $\hat{p}_t = (a_2)^{-1}\,[a_1 - \tilde{X}'_t\hat{\beta}_{t-1} - \bar{f}(X_t)] + u_t$. Under the positive definiteness of $E[X_t X'_t]$, it is easy to show that $\hat{\beta}_t \to \beta^{**}$, where

$$\beta^{**} \equiv (E[\tilde{X}_t, \tilde{X}'_t])^{-1}\, E[\tilde{X}_t(a_1 - \bar{f}(X_t))]/(a_2 + 1).$$

The market clearing price generated by learning converges to a steady state price $(a_2)^{-1}\,[a_1 - \bar{f}(X_t) - \tilde{X}'_t\beta^{**}] + u_t$, which in general is not the REE price unless $\bar{f}(x) \equiv b'x$ for some b and all $x \in$ support of $X_t \subset \mathbb{R}^k$. Note that this steady state price is the "REE" price according to the definition of [10], as it is the price generated by the best linear prediction given X_t. It is called an "approximate equilibrium" in [21] within the class of linear models. Nevertheless, this constitutes an "incorrect belief" equilibrium of the sort discussed by [14].

Case 2. Suppose instead that firms estimate $p^e_{i,t} = \theta_o(X_t)$ nonparametrically. As an example, suppose all firms use the same recursive kernel regression method to estimate θ_o. For this, let $\mathscr{K}: \mathbb{R}^k \to \mathbb{R}$ be some density function. Let $\{h_t\}$ be a sequence of positive numbers decreasing to zero as $t \to \infty$, the "bandwidth". Firms set $\hat{\theta}_0 \equiv 0$ and for $t \geqslant 0$,

$$\hat{\theta}_{t+1}(x) = \hat{\theta}_t(x) + (t+1)^{-1} \mathscr{K}((x - X_t)/h_t)[\hat{p}_t - \hat{\theta}_t(x)]/[h_t]^k, \qquad x \in \mathbb{R}^k,$$

where

$$\hat{p}_t = (a_2)^{-1} [a_1 - \hat{\theta}_t(X_t) - \bar{f}(X_t)] + u_t.$$

To match our general notation, we denote $\xi_{1,t} \equiv p_t$, $\xi_{2,t} = Z_{1,t} \equiv X_t$ and $\xi_{3,t} = Z_{2,t} \equiv u_t$, with

$$\xi_{1,t}(\theta) = p_t(\theta) = (a_2)^{-1} [a_1 - \theta(Z_{1,t}) - \bar{f}(Z_{1,t})] + Z_{2,t},$$

and $a_t = (t+1)^{-1}$ and

$$M_t(\xi_t(\theta), \theta(\cdot)) = (h_t)^{-k} \mathscr{K}((\cdot - \xi_{2,t})/h_t) \times [\xi_{1,t}(\theta) - \theta(\cdot)].$$

Put $\Xi \subset (0, \infty) \times \mathbb{R}^{k+1}$ (to be specified later), $\mathbb{H} = L_2(\mathbb{R}^k)$, and

$$\|\xi\|^2_{\Xi} = |\xi|^2; \qquad \|\theta\|^2 = \int_{\mathbb{R}^k} [\theta(x)]^2 \, dx.$$

Then firms have a Hilbert ($\mathbb{H}$)-valued RMF: $\hat{\theta}_{t+1} = \hat{\theta}_t + (t+1)^{-1} M_t(\hat{\xi}_t, \hat{\theta}_t)$. To allow $\{X_t\}$ to be temporally dependent and for simplicity, we restrict the supports of $\{X_t\}$ and $\{u_t\}$ to be compact.

PROPOSITION 4.1. *In Case* 2, *let* $\{u_t\}$ *be an i.i.d. sequence with compact support* $[l_1, l_2]$, *zero mean and finite variance, independent of* $\{X_t\}$. *Suppose the following conditions hold*:

(4.1.1) *each element of* $\{X_t; t \geqslant 0\}$ *has compact support* $[-1, 1]^k$ *and the same marginal unknown continuous density* f_X;

(4.1.2) $\{X_t\}$ *is* $L_2([-1, 1]^k)$-NED *with* $\sup_t d_t < \infty$ *and size* $-1/2$ *on a mixing sequence* $\{D_t\}$ *with* ϕ *of size* $-1/2$ *or* α *of size* -1;

(4.1.3) $\mathscr{K}(\cdot)$ *is Lipschitz continuous*, $\mathscr{K}(\cdot) \geqslant 0$, *symmetric about zero*, $\int_{\mathbb{R}^k} \mathscr{K}(x)\, dx = 1$, *and* $\int_{\mathbb{R}^k} |\mathscr{K}(x)|^2 \, dx \leqslant C < \infty$;

(4.1.4) $\{h_t\}$ *is a decreasing sequence of positive numbers satisfying* $h_t = O((t+1)^{-\delta})$ *for some fixed* $0 < \delta < 1/4k$;

(4.1.5) $\bar{f}(\cdot)$ *is Lipschitz continuous on* $[-1, 1]^k$;

(4.1.6) $\hat{\theta}_0$ *is an arbitrary random Lipschitz continuous function on* $[-1, 1]^k$, *independent of* $\{X_t, u_t\}$.

Then: $\int_{\mathbb{R}^k} (\hat{\theta}_t(x) - [a_1 - \bar{f}(x)](a_2+1)^{-1})^2 \, dx \to 0$ *as* $t \to \infty$ *a.s.* $-\mathbb{P}$.

Remarks. (a) One can relax (4.1.1) to allow $\{X_t\}$ to be nonstationary, but asymptotically stationary. More precisely, the marginal densities f_{X_t} can vary with t, as long as $\sup_x |f_{X_t}(x) - f_X(x)| \to 0$ as $t \to \infty$. We illustrate this point in the next examples. (b) This is an example of the RM algorithm [19] with simple feedback, as $\hat{p}_t$ is a function depending only on $(\hat{\beta}_t, \hat{\theta}_t)$ and (X_t, u_t), but not $\hat{p}_{t-1}$. We can rewrite this as a RM algorithm without feedback but with θ-dependent errors, and apply the results of [6] to obtain rate of convergence and asymptotic normality in addition to the almost-sure norm convergence. Nevertheless, this simple example illustrates the advantages of nonparametric adaptive learning in achieving convergence to REE in a wider context than has previously been available.

EXAMPLE 4.2. A Modified Evans and Honkapohja (1995) Model.

[9] (EH) consider stochastic nonlinear models where a variable of interest x_t (e.g., the choice variable in overlapping generation models) is generated according to $x_t = H(G(x_{t+1}, v_{t+1})^e, v_t)$, where v_t is an exogenous shock. The functional forms of H and G are assumed known and twice continuously differentiable. At time t, agents observe $G(x_t, v_t)$ and v_t and form a guess $G(x_{t+1}, v_{t+1})^e$ in order to generate $x_t = H(G(x_{t+1}, v_{t+1})^e, v_t)$. $\{v_t\}$ is assumed to be zero-mean and *i.i.d.* One case considered in [9] is as follows: If agents guess that the economy is in an equilibrium steady state with unknown parameter $G(x_{t+1}, v_{t+1})^e = b$ and they use a Ljung-type learning algorithm to recursively estimate b by $\hat{b}_t$ at time t, then the actual economy generated by $x_t = H(\hat{b}_t, v_t)$ will converge to the *rational steady state* solution $x_t = \bar{x}(v_t)$ with $\bar{x}(v) = H(\bar{b}, v) = H(E[G(\bar{x}(v_2), v_2) \mid v_1], v)$.

Although this setup can be used to analyze overlapping generations models with *i.i.d.* productivity or preference shocks, the *i.i.d.* assumption is quite restrictive. Instead it is natural to allow temporally dependent shocks. If, for example, $\{v_t\}$ is (known to be) a stationary nonlinear AR(1) process or an asymptotically stationary Markov process of order one, it does not make much sense for agents to form G^e as in EH. Instead, agents would more plausibly guess $G(x_{t+1}, v_{t+1})^e = \theta(v_t)$. There is no longer an obvious parametric model available, but with our procedures agents can in fact learn this function nonparametrically. Given that agents know the functional forms of H and G, we consider the situation in which at the beginning of period t, agents observe v_t, guess $G(x_{t+1}, v_{t+1})^e = \hat{\theta}_{t-1}(v_t)$, compute $x_t = H(\hat{\theta}_{t-1}(v_t), v_t)$ and $G(x_t, v_t)$, and update $\hat{\theta}_t$ via recursive kernel methods:

$$\hat{\theta}_t(v) = \hat{\theta}_{t-1}(v) + t^{-1}[G(x_t, v_t) - \hat{\theta}_{t-1}(v)]\, \mathscr{K}((v - v_{t-1})/h_t)/h_t,$$

where $\mathscr{K}(\cdot)$ is a kernel function and $\{h_t\}$ a bandwidth sequence.

To fix the notation, we let $\xi_{1,t} \equiv x_t$, $\xi_{2,t} = Z_{1,t} = v_t$ and $\xi_{3,t} = Z_{2,t} = v_{t-1}$ and put $M_t(\xi_t(\theta), \theta(\cdot)) = (h_t)^{-1}\, \mathscr{K}((\cdot - \xi_{3,t})/h_t)[G(\xi_{1,t}(\theta), \xi_{2,t} - \theta(\cdot)]$, $\xi_{1,t}(\theta) = H(\theta(Z_{1,t}), Z_{1,t})$.

PROPOSITION 4.2. *Suppose that G and H are twice continuously differentiable, and that*:

(4.2.1) *$\{v_t\}$ has compact support and is L_2-NED of size $-1/2$ on $\{D_t\}$, α-mixing of size -1;*

(4.2.2) *f_t, the density of v_t, is continuously differentiable, and satisfies $\sup_v |f_t(v) - f(v)| \to 0$ as $t \to \infty$, where f is a continuous density with $f(\cdot) > 0$ on the support of $\{v_t\}$;*

(4.2.3) *$q_{t|t-1}(\cdot \mid \cdot)$, the conditional density of $v_t \mid v_{t-1}$ is continuous in both arguments, continuously differentiable with respect to the second argument, and satisfies $\sup_{v, v'} |q_{t|t-1}(v \mid v') - q(v \mid v')| \to 0$ as $t \to \infty$, where q is a continuous conditional density*;

(4.2.4) *$\mathscr{K}(\cdot)$ is Lipschitz continuous, non-negative, symmetric about zero, $\int_{\mathbb{R}} \mathscr{K}(x)\, dx = 1$, and $\int_{\mathbb{R}} |\mathscr{K}(x)|^2\, dx \leqslant C < \infty$;*

(4.2.5) *$\{h_t\}$ is a decreasing sequence of positive numbers satisfying $h_t = O((t+1)^{-\delta})$ for some fixed $0 < \delta < 1/4$;*

(4.2.6) *$\hat{\theta}_0$ is Lipschitz continuous and independent of $\{v_t\}$;*

(4.2.7) *there exists a Lipschitz continuous function $\theta^*\colon \mathbb{R} \to \mathbb{R}$ satisfying: $\int G(H(\theta^*(v), v), v)\, q(dv \mid \cdot) - \theta^*(\cdot) = 0$;*

(4.2.8) *$\langle \bar{M}(\theta), \theta - \theta^* \rangle < 0$ for all $\theta \neq \theta^*$, where $\bar{M}(\theta(\cdot)) \equiv [\int G(H(\theta(v), v), v)\, q(dv \mid \cdot) - \theta(\cdot)]\, f(\cdot)$.*

Then: $\int (\hat{\theta}_t(y) - \theta^*(y))^2\, dy \to 0$ *as* $t \to \infty$ *a.s.* $-\, \mathbb{P}$.

Remarks. (a) The continuous differentiability assumptions in (4.2.2) and (4.2.3) can be replaced by Lipschitz continuity. One can make additional assumptions on G and H so that (4.2.7) and (4.2.8) hold.

(b) Suppose we replace (4.2.7) and (4.2.8) by the following conditions:

(4.2.7′) There exists a set (Θ^*) of Lipschitz continuous functions $\theta^*\colon \mathbb{R} \to \mathbb{R}$ satisfying (4.2.7);

(4.2.8′) Θ^* is an $L_2(\mathbb{R})$-norm bounded set and an asymptotically stable set for the ODE: $\dot{\theta}(\tau) = \bar{M}(\theta(\tau))$.

Then we have $\hat{\theta}_t \to \Theta^*$ in weak topology almost surely, and x_t will convergence to a set $\{H(\theta^*(v_t), v_t) : \theta^* \in \Theta^*\}$.

(c) As in the first example, we can rewrite the learning recursions as a RM with θ-dependent errors with no feedback so that the results of [6] apply to obtain rate of convergence and asymptotic normality.

While the foregoing examples used simple feedback rules $\hat{\xi}_t = f_t(\hat{\theta}_t, Z_t)$ to permit the key points of our discussion to stand out against a relatively simple background, the following examples require true feedback, albeit of the simplest sort, using laws of motion of the form $\hat{\xi}_t = \rho_t(\hat{\xi}_{t-1}, \hat{\theta}_t, Z_t)$.

Example 4.3. Generalization of Fictitious Play.

Fictitious play [5, 20] belongs to a class of famous models of learning and behavior in game theory. A typical fictitious play model assumes two players, finitely many strategies, simultaneous moves, and at the beginning of each period, each player uses the historical frequency of the other player's past actions to form a belief about the other player's possible strategy. The player then chooses his own strategy to maximize his current period's expected payoff given his belief. [11] have considered an augmented game of two-player, two-strategy fictitious play, consisting of the original game with both players' payoffs subject to *i.i.d.* shocks, and each player knowing only his own payoff shocks. [11] have applied convergence results of parametric stochastic approximation to show that learning players will converge to the unique Nash Equilibrium both in beliefs and in strategies.

In this example, we consider a generalization of fictitious play with continuum strategies. For concreteness, we assume that the game has the following structure: two players, simultaneous moves, infinite repetition, and each player's action space is [0, 1]. At the end of period t, player i observes both players' actions $\xi_{1,t}$ and $\xi_{2,t}$, and forms a belief $\theta_{ij,t+1}$ about player j's action $\xi_{j,t+1}$ in the next period where the belief $\theta_{ij,t+1}$ is a probability density over player j's action space [0, 1]. Player i then decides his action for the next period $\xi_{i,t+1}$ according to his response function $R_{i,t}$ which is derived from maximizing his current period's expected payoff given his beliefs. In particular, player i will choose $\xi_{i,t+1} \in [0, 1]$ to maximize

$$\int_{[0,1]} Q_{i,t+1}(\xi_{i,t+1}, \xi_{i,t}, Z_{i,t+1}, \xi_j)\, \theta_{ij,t+1}(\xi_j)\, d\xi_j,$$

where $Q_{i,t+1}$ is concave and differentiable in $\xi_{i,t+1}$, and $Z_{i,t+1}$ is a payoff shock to player i known at time $t+1$, which is unknown to the other player for all time. In contrast to [11], who take $\{Z_{i,t+1}\}$ to be an exogenous *i.i.d.* scalar sequence, we take $\{Z_{i,t+1}\}$ $(i=1,2)$ to be an exogenous weakly stationary dependent random process. Nevertheless, to keep things simple, we posit that player i assumes that $\xi_{j,t+1}$ is generated according to $\theta_{ij,t+1}$ as player i does not know that: (a) $\{Z_{j,t+1}\}$ may be time dependent; (b) $Q_{j,t+1}$ (hence $\xi_{j,t+1}$) may depend on $Z_{j,t+1}$. (If both players knew these facts they would plausibly estimate the conditional density of the other player's next period strategy, using the structure of $\{Z_t\}$. The resulting game will be totally different from the original fictitious play. We defer investigation of this type of game to future research.) Here we allow both players' payoff functions to be dependent on their own strategy last period, which can occur in OLG models in which parents care about children or in IO models where firms' profit functions depend on last period's investment, production, price, etc.

Given his belief $\hat{\theta}_{ij,t+1}$, player i's best action at time $t+1$, $\hat{\xi}_{i,t+1}=R_{i,t}(\xi_{i,t}, \hat{\theta}_{ij,t+1}, Z_{i,t+1})$, is a solution to

$$\int_{[0,1]} (d/d\xi_i)[Q_{i,t+1}(\hat{\xi}_{i,t+1}, \xi_{i,t}, Z_{i,t+1}, \xi_j)]\, \hat{\theta}_{ij,t+1}(\xi_j)\, d\xi_j = 0.$$

For example, if the payoff functions are quadratic,

$$\begin{aligned} &Q_{i,t+1}(\xi_{i,t+1}, \xi_{i,t}, Z_{i,t+1}, \xi_{j,t+1}) \\ &\qquad = -(\xi_{i,t+1} + a\xi_{i,t} + Z_{i,t+1} - \xi_{j,t+1})^2, \qquad |a|<1, \end{aligned}$$

then $\hat{\xi}_{i,t+1} = \int_{[0,1]} x\hat{\theta}_{ij,t+1}(x)\,dx - a\xi_{i,t} - Z_{i,t+1}$. Suppose player i forms his beliefs as

$$\begin{aligned} \hat{\theta}_{ij,t+1}(x) &= \hat{\theta}_{ij,t}(x) + (t+1)^{-1} \\ &\qquad \times [(h_{i,t})^{-1}\, \mathscr{K}_i((x-\hat{\xi}_{j,t})/h_{i,t}) - \hat{\theta}_{ij,t}(x)], \qquad x \in [0,1]. \end{aligned}$$

To fix notation, we let $\theta' \equiv (\theta_{12}, \theta_{21})$, $\xi'_t \equiv (\xi_{1,t}, \xi_{2,t})$, $M'_t \equiv (M_{1,t}, M_{2,t})$, and $Z'_t \equiv (Z_{1,t}, Z_{2,t})$. Then $\xi_{t+1}(\theta) = \int_{[0,1]} x\theta(x)\,dx - a\xi_t(\theta) - Z_{t+1}$ for the quadratic payoff function, and

$$\begin{aligned} M'_t(\xi_t, \theta_t(\cdot)) &= ([h_{1,t}]^{-1}\, \mathscr{K}_1((\cdot - (0,1)\,\xi_t)/h_{1,t}), \\ &\qquad [h_{2,t}]^{-1}\, \mathscr{K}_2((\cdot - (1,0)\,\xi_t)/h_{2,t})) - \theta_t(\cdot)'. \end{aligned}$$

Also let $G \subset \mathbb{R}^2$ (to be specified later), $\Xi = [0, 1] \times [0, 1] \in \mathbb{R}^2$, $\mathbb{H} = L_2([0, 1]) \times L_2([0, 1])$. For any $\xi' = (\xi_1, \xi_2) \in \Xi$ and any $\theta' = (\theta_{12}, \theta_{21}) \in \mathbb{H}$, put

$$\|\xi\|^2_{\Xi} \equiv |\xi_1|^2 + |\xi_2|^2; \qquad \|\theta\|^2 \equiv \int_{[0,1]} [\theta_{12}(x)]^2 \, dx + \int_{[0,1]} [\theta_{21}(x)]^2 \, dx.$$

PROPOSITION 4.3. *In the quadratic payoff function case, let* $\hat{\xi}_{1,0}$, $\hat{\xi}_{2,0}$ *be arbitrary* (0, 1)*-valued random variables, and let* $\hat{\theta}_{12,0}$, $\hat{\theta}_{21,0}$ *be arbitrary* $L_2([0, 1])$*-valued random continuous density functions. Suppose that* $\hat{\xi}_{1,0}$, $\hat{\xi}_{2,0}$, $\hat{\theta}_{12,0}$, $\hat{\theta}_{21,0}$, $\{Z_{1,t}\}$ *and* $\{Z_{2,t}\}$ *are mutually independent. Further, suppose for* $i = 1, 2$, *that the following hold*:

(4.3.1) *For all t, the elements of* $\{Z_{i,t}\}$ *have compact support* $[0, 1]$ *and unknown continuous density function* f^z_i;

(4.3.2) $\{Z_{i,t}\}$ *is* L_2*-NED of size* $-1/2$ *on an* α*-mixing sequence* $\{D_t\}$ *of size* 1;

(4.3.3) $\mathcal{K}_i(\cdot)$ *is Lipschitz continuous, symmetric about zero,* $\mathcal{K}_i(\cdot) \geqslant 0$, $\int_{\mathbb{R}} \mathcal{K}_i(x) \, dx = 1$, *and* $\int_{\mathbb{R}} |\mathcal{K}_i(x)|^2 \, dx \leqslant C_i < \infty$;

(4.3.4) $\{h_{i,t}\}$ *is a decreasing sequence of positive numbers satisfying* $h_{i,t} = O((t+1)^{-\delta_i})$ *for some fixed* $0 < \delta_i < 1/4$;

(4.3.5) *There exists a continuous density* θ^*_{ij} *such that* $\{\xi_{i,t}(\theta^*_{ij})\}$ *is stationary and has marginal continuous density* f_i.

Then: $\theta^*_{12} = f_2$, $\theta^*_{21} = f_1$, *and*

$$\int_{[0,1]} (\hat{\theta}_{12,t}(x) - f_2(x))^2 \, dx + \int_{[0,1]} (\hat{\theta}_{21,t}(y) - f_1(y))^2 \, dy \to 0$$

$$\text{as} \quad t \to \infty \ a.s. - \mathbb{P}.$$

Remarks. (a) Note that given the stationarity of $\{Z_t\}$ and $|a| < 1$, we have that $\{\xi_t(\theta)\}$ is stationary (recall $\xi_{i,t}(\theta_{ij}) = \int x\theta_{ij}(x) \, dx - a\xi_{i,t-1}(\theta_{ij}) - Z_{i,t}$), so Assumption (4.3.5) is reasonable given (4.3.1) and (4.3.2). Nevertheless, $\{\hat{\xi}_t \equiv \xi_t(\hat{\theta}_t)\}$ is only asymptotically stationary. (b) If we let f^z_i vary with t in (4.3.1), so that $f^z_{i,t}$ (say) is continuous and satisfies $(t+1) \sup_x |f^z_{i,t+1}(x) - f^z_{i,t}(x)| \to 0$ as $t \to \infty$, we can change (4.3.5) to assure that $\{\xi_{i,t}(\theta^*_{ij})\}$ has continuous marginal densities $\{f_{i,t}\}$ such that $(t+1) \sup_x |f_{i,t+1}(x) - f_{i,t}(x)| \to 0$ as $t \to \infty$, with $\sup_x |f_{i,t}(x) - f_i(x)| \to 0$ as $t \to \infty$. Then the above conclusion remains true. (c) By Remark 2.3(b) or Corollary 2.5, the value of the players' payoff functions converges strongly to the value that would obtain if learning were not necessary.

EXAMPLE 4.4. Optimal Savings/Stochastic Dynamic Programming.

An agent seeks to maximize $E[\sum_{t=0}^{\infty} \beta^t U(c_t)]$, where $0<\beta<1$, and U is a known increasing concave function. The budget constraint each period is: $c_t + s_t \leqslant r_t s_{t-1}$, $c_t \geqslant 0$, $s_t \geqslant 0$, where r' is the gross real return on savings s_t at period t with $r_t > 0$ *a.s.* $-\mathbb{P}$, and c_t is consumption at period t. The value of r_t is observed before c_t, s_t are chosen. Suppose that s_{-1} is a random variable with support $[0, \bar{s}]$, and that $\{r_t\}$ is independent of s_{-1} and is a Markovian process of order one with unknown transition probability density $q(r_{t+1} \mid r_t)$. A stationary optimal saving rule $s_t^* = S^*(s_{t-1}^*, r_t)$ for all $t \geqslant 0$ should satisfy the Euler equation:

$$U'(r_t s_{t-1}^* - s_t^*) = \beta \int U'(r_{t+1} s_t^* - S^*(s_t^*, r_{t+1}))\, r_{t+1} q(dr_{t+1} \mid r_t),$$

$$0 \leqslant s_t^* \leqslant r_t s_{t-1}^*.$$

To solve for S^*, the agent can choose $\hat{s}_t = \hat{\theta}_t(\hat{s}_{t-1}, r_t)$ at time t, observe r_{t+1} at the end of t (or beginning of $t+1$), and update her estimate $\hat{\theta}_t$ for S^* according to

$$\tilde{\theta}_{t+1}(s, r) = \hat{\theta}_t(s, r) + [(t+1)(h_t)^2]^{-1} \mathscr{K}((s - \hat{s}_{t-1})/h_t, (r - r_t)/h_t)\, Q_t,$$

$$Q_t \equiv \beta U'(r_{t+1} \hat{s}_t - \hat{\theta}_t(\hat{s}_t, r_{t+1}))\, r_{t+1} - U'(\hat{s}_{t-1} r_t - \hat{s}_t),$$

$$\hat{s}_t = \hat{\theta}_t(\hat{s}_{t-1}, r),$$

$$\hat{\theta}_{t+1}(s, r) = [\tilde{\theta}_{t+1}(s, r) - \hat{\theta}_t(s, r)] \times 1(0 \leqslant \tilde{\theta}_{t+1}(s, r) \leqslant \bar{s}) + \hat{\theta}_t(s, r),$$

$$\text{for all} \quad 0 \leqslant s \leqslant \bar{s},\ 0 \leqslant r \leqslant \bar{r},$$

where $\mathscr{K}$ is a kernel and h_t is a bandwidth as before.

PROPOSITION 4.4. *Suppose s_{-1} has support $[0, \bar{s}]$ and is independent of $\{r_t\}$. Suppose also that:*

(4.4.1) *$\{r_t\}$ has support $[0, \bar{r}]$, and is a Markovian process of order one with transition density q;*

(4.4.2) *$\mathscr{K}(\cdot)$ is Lipschitz continuous, non-negative, symmetric about zero, $\int_{\mathbb{R}^2} \mathscr{K}(x)\, dx = 1$, and $\int_{\mathbb{R}^2} |\mathscr{K}(x)|^2\, dx \leqslant C < \infty$;*

(4.4.3) *$\{h_t\}$ is a decreasing sequence of positive numbers satisfying $h_t = O((t+1)^{-\delta})$ for some fixed $0 < \delta < 1/8$.*

Suppose there exists a unique $S^*: [0, \bar{s}] \times [0, \bar{r}] \to [0, \bar{s}]$ such that the following hold:

(4.4.4) *For any* $r \in [0, \bar{r}]$, $s \in [0, \bar{s}]$,

$$U'(rs - S^*(s, r)) = \beta \int U'(r_{t+1} S^*(s, r) - S^*(S^*(s, r), r_{t+1})) \times r_{t+1} q(dr_{t+1} \mid r_t = r), \qquad 0 \leqslant S^*(s, r) \leqslant rs;$$

(4.4.5) *For any* $\theta \neq S^*$ *with* $0 \leqslant \theta(\theta(s, r), r') \leqslant r'\theta(s, r)$ *for* $r, r' \in [0, \bar{r}]$, $s \in [0, \bar{s}]$, *we have*

$$\iint [\theta(s, r) - S^*(s, r)] \times \left[\beta \int U'(r'\theta(s, r) - \theta(\theta(s, r), r'))\, r' q(dr' \mid r) - U'(rs - \theta(s, r)) \right] ds\, dr < 0;$$

(4.4.6) *The process* $\{s_t^*, r_{t+1}\}$ *generated according to* $s_t^* \equiv S^*(s_{t-1}^*, r_t)$, $s_{-1}^* = s_{-1}$ *is jointly stationary with continuous density* $f_{S^*} > 0$ *on* $[0, \bar{s}] \times [0, \bar{r}]$.

Then: $\iint [\hat{\theta}_t(s, r) - S^*(s, r)]^2\, ds\, dr \to 0$ *as* $t \to \infty$ *a.s.* $- \mathbb{P}$.

Remarks. (a) (4.4.5) and (4.4.6) are the appropriate identification and stability conditions for the present context. (b) By Remark 2.3(b) or Corollary 2.5, the value function converges in norm topology to the value that would obtain if learning were not necessary. (c) There are many other ways to solve stochastic dynamic programming problems similar to Example 4.4. For example, we can either (a) directly estimate the Markovian transition density $q(r_{t+1} \mid r_t)$ recursively, then solve the Euler equation with estimated q; or (b) directly estimate the value function recursively.

5. SUMMARY

The algorithms introduced in this paper provide tools for economists to study adaptive learning models for economic agents in useful and realistic contexts not previously available. The procedures are not only easy to compute, but also permit nonlinear updating functions, complicated feedback rules, and general correlated (exogenous) shocks. In particular, both economic agents and econometricians can almost surely learn the "truth," leading to potential utility improvements using these procedures regardless of their initial beliefs. We can thus avoid the pitfalls that arise using parametric learning models.

Here we obtain almost-sure convergence in a Hilbert space. We are currently considering sup-norm convergence in a functional space for RM procedures with feedbacks and Markovian (exogenous) shocks; such results are useful for studying models in which agents directly learn about value functions over time.

Important open questions include the speed of learning (i.e., the rate of convergence) and the asymptotic distribution of the estimator $\hat{\theta}_n$. [6] establish such results for the case of simple feedback rules $\hat{\xi}_n = f_n(\hat{\theta}_n, Z_n)$ with NED shocks Z_n. We plan to modify that approach to deal with the more general feedback rule $\hat{\xi}_n = \rho(\hat{\xi}_{n-1}, \hat{\theta}_n, Z_n)$.

APPENDIX A: DEFINITIONS AND CONDITIONS

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Let $M: X \to Y$ be an operator. Then M is (*norm*) *continuous* provided $x_n \to x$ in norm-$\|\cdot\|_X$ implies $M(x_n) \to M(x)$ in norm-$\|\cdot\|_Y$. M is *completely continuous* (or *strongly continuous*) provided $x_n \to x$ in the weak topology of X implies $M(x_n) \to M(x)$ in norm-$\|\cdot\|_Y$. M is *weakly sequentially continuous* provided $x_n \to x$ in the weak topology of X implies $M(x_n) \to M(x)$ in the weak topology of Y. M is *uniformly continuous* on $D \subseteq X$ provided for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $x, u \in D$ with $\|x - u\|_X < \delta(\varepsilon)$, we have $\|M(x) - M(u)\|_Y < \varepsilon$. M is *Lipschitz continuous* on $D \subseteq X$ provided there exists a fixed c such that for any $x, u \in D$, we have $\|M(x) - M(u)\|_Y \leqslant c\|x - u\|_X$. M is *contractive* provided M is Lipschitz continuous with $0 \leqslant c < 1$. M is *bounded* provided M maps bounded sets into bounded sets. M is *compact* provided M is continuous and maps bounded sets into relatively compact sets. The relations among the above various operators can be found in [24, pp. 596–597].

Let μ denote a sequence $\{\mu_n\}$ with $\mu_n \in \mathbb{H}$ for each n. Let $\mu^n \equiv (\mu_0, ..., \mu_n)$ and define the sequence $\{\xi_n(\mu)\}$ recursively as

$$\xi_{n+1}(\mu) = R_n(\xi^n(\mu), \mu^{n+1}, Z_{n+1}), \qquad n = 0, 1, 2, ...; \xi_0(\mu) = \xi^0(\mu) = \hat{\xi}_0.$$

Let $\bar{\mu}(\cdot)$ be the function with value μ_i on $[t_i, t_{i+1})$. Let $\hat{\xi}^o(\mu, \cdot)$ be the function with value $\xi_n(\mu)$ for $t \in [t_n, t_{n+1})$ (i.e., the piecewise right continuous constant interpolation of $\{\xi_n(\mu)\}$ on $[0, \infty)$ with interpolation intervals $\{a_n\}$).

Assumption A.7. There exists a $\mathbb{P}$-null set Ω_o such that for any $\omega \in \Omega - \Omega_o$, for each real $q_1 > 0$, $q_2 > 0$, we have: for each $\varepsilon > 0$, there exists $\delta > 0$ such that if for any infinite subsequence $\{n'\}$,

$$(1)\quad \limsup_{n'} \sup_{-q_2 \leqslant s,\, \tau \leqslant q_1} d(\bar{\mu}(t_{n'}+s), \bar{\mu}(t_{n'}+\tau)) \leqslant \delta \qquad \text{and}$$

$$(2)\quad \limsup_{n'} \sup_{-q_2 \leqslant s \leqslant q_1} d(\bar{v}(t_{n'}+s), \bar{\mu}(t_{n'}+s)) \leqslant \delta,$$

then for that subsequence $\{n'\}$,

$$(3)\quad \sup_{n'} \sup_{0 \leqslant s \leqslant q_1} d_{\Xi}(\bar{\xi}^o(\mu, t_{n'}+s), \bar{\xi}^o(v, t_{n'}+s)) \leqslant \varepsilon.$$

Here both $\mu \equiv \{\mu_n\}$ and $v \equiv \{v_n\}$ can be either sequences of constant values θ and θ' or one is constant and the other is the sequence of sample values $\{\hat{\theta}_n(\omega)\}$.

Assumption A.7′. A.7 holds when d_{Ξ} is replaced by $\|\cdot\|_{\Xi}$.

APPENDIX B: MATHEMATICAL PROOFS

Proof of Theorem 2.1. (i) Let Ω' denote the union of the set where $\{\hat{\theta}_n\}$ is not in $\mathbb{H}$ with the union of the exceptional sets in A.2, A.6(b) (or A.6′(b)) and A.7 (or A.7′). Fix $\omega \in \Omega - \Omega'$. Given $\sup_n \|\hat{\theta}_n\| \leqslant \bar{B} < \infty$ and the definition of $\{\theta^n(\cdot)\}$, we have that $\{\theta^n(\cdot)\}$ is uniformly bounded in norm. Let $i_0 \equiv m(t_n + t) = \max\{l > n: \sum_{j=n}^{l-1} a_j \leqslant t\}$ and $i_s \equiv m(t_n + t + s) - 1$. By the definition of $\theta^n(\cdot)$, the uniform boundedness of $\{\hat{\theta}_n\}$ and $\{\hat{\xi}_j\}$, and Assumption A.3(a), we have:

$$\|\theta^n(t+s) - \theta^n(t)\| \leqslant \sum_{i_0 \leqslant j \leqslant i_s} a_j \|M_j(\hat{\xi}_j, \hat{\theta}_j)\| \leqslant \sum_{i_0 \leqslant j \leqslant i_s} a_j \times \textit{constant}. \tag{b.0}$$

A.2 implies $\|\theta^n(t+s) - \theta^n(t)\| \to 0$ as $s \to 0$ for all n large enough. Hence $\{\theta^n(\cdot)\}$ is equicontinuous (for large n) under the norm topology. Since for each $h \in \mathbb{H}$ and any $0 < t < \infty$,

$$|\langle \theta^n(t), h \rangle| \leqslant \|\theta^n(t)\| \, \|h\|,$$

and

$$|\langle \theta^n(t+s) - \theta^n(t), h \rangle| \leqslant \|\theta^n(t+s) - \theta^n(t)\| \, \|h\| \to 0 \text{ uniformly in } n \text{ as } s \to 0,$$

$\{\theta^n(\cdot)\}$ is also uniformly bounded and equicontinuous under the weak topology. Since every set in $\mathbb{H}$ of the form $\{\theta: \|\theta\| \leqslant B < \infty\}$ is compact under the weak topology, we can apply Ascoli–Arzela's lemma to conclude that $\{\theta^n(\cdot)\}$ is sequentially compact. Thus, every sequence of $\{\theta^n(\cdot)\}$ has a convergent subsequence in the weak topology. For notational simplicity, we denote the convergent subsequence again by $\{\theta^n(\cdot)\}$ and the weak limit by $\theta(\cdot)$, i.e., $\langle \theta^n(t), h\rangle \to \langle \theta(t), h\rangle$ as $n \to \infty$, for each $h \in \mathbb{H}$. Moreover, the convergence is uniform on each compact t-set. Fix any $0 < T < \infty$ and let $t, t+s \in [0, T]$. For each $h \in \mathbb{H}$ and for this convergent subsequence $\{n\}$,

$$\langle \theta^n(t+s) - \theta^n(t), h\rangle = \left\langle \sum_{m(t_n+t) \leqslant j \leqslant m(t_n+t+s)-1} a_j M_j(\hat{\xi}_j, \hat{\theta}_j), h \right\rangle - o(1), \tag{b.1}$$

where $\sup_{0 \leqslant t, t+s \leqslant T} |\langle o(1), h\rangle| \to 0$ as $n \to \infty$. We chose a sequence $\{\delta_n > 0\}$ such that

$$\delta_n \downarrow 0 \qquad \text{and} \qquad \delta_n^{-1} \sup_{j \geqslant n} a_j \to 0 \qquad \text{as} \quad n \to \infty.$$

We select an increasing integer-valued sequence $\{r_l\}$ with $r_1 = n$ such that $\sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j = \delta_n$, modulo an end value. Hence $\delta_n^{-1}(t_{r_{l+1}} - t_{r_l}) \to 1$ as $n \to \infty$. Let $0 < \delta_n < 2\delta_n < \cdots \leqslant t+s$ be a partition of $[0, t+s]$ with equal length of subinterval δ_n. Then

$$\begin{aligned} &\sum_{m(t_n+t) \leqslant j \leqslant m(t_n+t+s)-1} a_j M_j(\hat{\xi}_j, \hat{\theta}_j) \\ &\quad = \sum_l \delta_n \times \left[\delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j M_j(\hat{\xi}_j, \hat{\theta}_j) \right] \\ &\quad = \sum_l \delta_n f_n(\tau) + \sum_l \delta_n A_n(\tau) = \int_{t \leqslant \tau < t+s} f_n(\tau)\, d\tau + \int_{t \leqslant \tau < t+s} A_n(\tau)\, d\tau, \end{aligned}$$

where $f_n(\cdot)$ and $A_n(\cdot)$ are the piecewise right continuous constant interpolations for $\tau \in [t_{r_l} - t_n, t_{r_{l+1}} - t_n)$,

$$f_n(\tau) \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j \bar{M}(\hat{\theta}_j),$$

and

$$A_n(\tau) \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j [M_j(\hat{\xi}_j, \hat{\theta}_j) - \bar{M}(\hat{\theta}_j)].$$

Due to the uniform boundedness of $\{\hat{\theta}_n\}$ and Assumption A.3(a),

$$\max_{r_l \leqslant j \leqslant r_{l+1}} \|\hat{\theta}_j - \hat{\theta}_{r_l}\| \leqslant \max_{r_l \leqslant j \leqslant r_{l+1}} \left\| \sum_{r_l \leqslant i \leqslant j} a_i M_i(\hat{\xi}_i, \hat{\theta}_i) \right\|$$
$$\leqslant \sum_{r_l \leqslant i \leqslant r_{l+1}} a_i \times constant \to 0 \qquad \text{as} \quad n \to \infty. \quad \text{(b.2)}$$

Now the equicontinuity (under the weak topology) of $\{\theta^n(\cdot)\}$ implies Assumption A.7(1); and the convergence of $\{\theta^n(\cdot)\}$ (under the weak topology) implies A.7(2). Thus Assumptions A.3 (or A.3′), A.6 (or A.6′) and A.7 (or A.7′) imply that for each $h \in \mathbb{H}$,

$$\sup_{0 \leqslant t,\, t+s \leqslant T} \left| \left\langle \int_{t \leqslant \tau < t+s} A_n(\tau)\, d\tau, h \right\rangle \right| \to 0 \qquad \text{as} \quad n \to \infty \ a.s. - \mathbb{P}.$$

Substituting into (b.1), we get that for each $h \in \mathbb{H}$,

$$\langle \theta^n(t+s) - \theta^n(t), h \rangle = \left\langle \int_{t \leqslant \tau < t+s} f_n(\tau)\, \tau, h \right\rangle + o(1), \qquad \text{(b.3)}$$

where $\sup_{0 \leqslant t,\, t+s \leqslant T} |o(1)| \to 0$ as $n \to \infty$. Due to the uniform convergence of $\{\theta^n(\cdot)\}$ in $[0, T]$ under the weak topology, the left hand side of (b.3) goes to $\langle \theta(t+s) - \theta(t), h \rangle$ as $n \to \infty$ for each $h \in \mathbb{H}$. To complete the proof, we need only show that the right hand side of (b.3) goes to $\langle \int_{t \leqslant \tau < t+s} \bar{M}(\theta(\tau))\, d\tau, h \rangle$ as $n \to \infty$ for each $h \in \mathbb{H}$. For each $\tau \in [t, t+s)$, we can fix $\tau \in [t_{r_l} - t_n, t_{r_{l+1}} - t_n)$, and write

$$f_n(\tau) \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j \bar{M}(\hat{\theta}_j) = b_{1n} + b_{2n} + b_{3n},$$

where

$$b_{1n} \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j \bar{M}(\theta(\tau)),$$

$$b_{2n} \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j [\bar{M}(\hat{\theta}_{r_l} - \bar{M}(\theta(\tau)))],$$

$$b_{3n} \equiv \delta_n^{-1} \sum_{r_l \leqslant j \leqslant r_{l+1}-1} a_j [\bar{M}(\hat{\theta}_j) - \bar{M}(\hat{\theta}_{r_l})].$$

By (b.2) and Assumption A.6′(a) and the triangle inequality, we get $\|f_n(\tau) - \bar{M}(\theta(\tau)\| \to 0$ as $n \to \infty$, thus (b.3) holds. Or by the definition of δ_n, the uniform boundedness of $\{\hat{\theta}_n\}$ and Assumption A.6(a), we have that $\|b_{i,n}\| \leqslant constant$ for $i = 1, 2, 3$. Clearly, $\langle b_{1n}, h \rangle \to \langle \int_{t \leqslant \tau < t+s} \bar{M}(\theta(\tau))\, d\tau, h \rangle$ as $n \to \infty$ for each $h \in \mathbb{H}$. In addition, the weak convergence of $\{\hat{\theta}_j\}$ and

Assumption A.6(a) imply that $\langle b_{2n}, h\rangle \to 0$ and $\langle b_{3n}, h\rangle \to 0$ as $n \to \infty$ for each $h \in \mathbb{H}$. This completes (i).

(ii) The proof is similar to [15, Theorem 2.5.2.].

LEMMA B.1. *Let the* TRMF(2.3) *satisfy* A.1, A.2, A.4, A.5, B.8, *B*.9 *and one of the following two conditions*: (1) A.3(b), A.6 *and* A.7; *or* (2) A.3′(b), A.6′ *and* A.7′. *Then* $\lim_n T(n) = T < \infty$ *a.s.* $- \mathbb{P}$.

Proof. Suppose the conclusion does not hold. Then $\{\hat{\theta}_n\}$ will cross the sphere $\{\theta: \|\theta\| = B\}$ infinitely often. Given B.8 and B.9, $V(\Theta^*)$ is a compact set. B.9(d) ensures that $[V(\bar{\theta}), b] - [V(\bar{\theta}), b] \cap V(\Theta^*)$ is a compact set with nonempty interior. Hence there exist positive reals δ_1 and δ_2 with $[\delta_1, \delta_2] \subset V(\bar{\theta}), b]$ and $\delta_1 \notin V(\Theta^*)$. Define

$$\mu \equiv \min[j \geqslant 1: V(\hat{\theta}_j) \geqslant \delta_1]; \qquad v \equiv \min[j \geqslant 1: V(\hat{\theta}_j) \geqslant \delta_2].$$

By definition, $\mu < v < \infty$ almost surely. Then $V(\hat{\theta}_j) \in [\delta_1, \delta_2]$ for $j \in [\mu, v]$. Hence the set $\{\hat{\theta}_j: \mu - 1 \leqslant j \leqslant v\}$ is bounded, in particular, for all $j \in [\mu - 1, v]$, $\|\hat{\theta}_j\| \leqslant B < B_1$. (Suppose there exists an $j \in [\mu - 1, v]$ with $\|\hat{\theta}_j\| > B$; then $V(\hat{\theta}_j) > b \geqslant \delta_2 > \delta_1$, a contradiction.) Choose a small $\eta > 0$ and a positive function ε such that $\varepsilon(\eta) \to 0$ as $\eta \to 0$. Define $t_0 = 0$, $t_n = \sum_{0 \leqslant i \leqslant n-1} a_i$ for $n > 0$, and $m(i, t) \equiv \max[n > i: t_n - t_i \leqslant t]$ if $t \geqslant 0$ and $m(i, t) \equiv 0$ if $t < 0$. By definition of μ and the choice of η,

$$\mu - 1 < m(\mu - 1, \eta) \leqslant v, \qquad \delta_1 \leqslant V(\hat{\theta}_{m(\mu - 1, \eta)}) \leqslant \delta_2.$$

Now we show that $V(\hat{\theta}_{m(\mu-1, \eta)}) < \delta_1$ to conclude the proof. By Taylor expansion, we have

$$V(\hat{\theta}_{m(\mu-1,\eta)}) - V(\hat{\theta}_{\mu-1}) = \langle V'(\hat{\theta}_{\mu-1}), [\hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}]\rangle + R_2(\hat{\theta}_{\mu-1}, \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}),$$

where

$$|R_2(\hat{\theta}_{\mu-1}, \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1})| \leqslant \mathit{constant} \times \|\hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}\|^2.$$

Since $\{\hat{\theta}_j: \mu - 1 \leqslant j \leqslant v\}$ is bounded by B, there is no truncation from $\hat{\theta}_{\mu-1}$ to $\hat{\theta}_{m(\mu-1,\eta)}$,

$$\|\hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}\| = \left\| \sum_{\mu-1 \leqslant j \leqslant m(\mu-1,\eta)-1} a_j M_j(\hat{\xi}_j, \hat{\theta}_j) \right\|,$$

and

$$\langle V'(\hat{\theta}_{\mu-1}), \hat{\theta}_{m(\mu-1,\eta)}\rangle = \sum_{\mu-1\leqslant j\leqslant m(\mu-1,\eta)-1} a_j\langle V'(\hat{\theta}_{\mu-1}), M_j(\hat{\xi}_j, \hat{\theta}_j)\rangle$$
$$= I_1 + I_2 + I_3,$$

where

$$I_1 \equiv \sum_{\mu-1\leqslant j\leqslant m(\mu-1,\eta)-1} a_j\langle V'(\hat{\theta}_{\mu-1}), \bar{M}(\hat{\theta}_{\mu-1})\rangle,$$

$$I_2 \equiv \left\langle V'(\hat{\theta}_{\mu-1}), \sum_{\mu-1\leqslant j\leqslant m(\mu-1,\eta)-1} a_j[M_j(\xi_j(\hat{\theta}_{\mu-1}), \hat{\theta}_{\mu-1}) - \bar{M}(\hat{\theta}_{\mu-1})]\right\rangle,$$

$$I_3 \equiv \left\langle V'(\hat{\theta}_{\mu-1}), \sum_{\mu-1\leqslant j\leqslant m(\mu-1,\eta)-1} a_j[M_j(\hat{\xi}_j, \hat{\theta}_j) - M_j(\xi_j(\hat{\theta}_{\mu-1}), \hat{\theta}_{\mu-1})]\right\rangle.$$

Since $\{\hat{\theta}_j\colon \mu-1\leqslant j\leqslant v\}$ is bounded by B and $\hat{\xi}_j\in\Xi$, a bounded set, A.3(b) (or A.3′(b)) and the fact that a finite union of bounded sets is a bounded set implies that $\sup_{\mu-1\leqslant j\leqslant m(\mu-1,\eta)-1}\|M_j(\hat{\xi}_j, \hat{\theta}_j)\|<\infty$, given the definition of $m(\mu-1,\eta)$, so that $\|\hat{\theta}_{m(\mu-1,\eta)}-\hat{\theta}_{\mu-1}\|=O(\eta)$, thus $|R_2|=O(\eta^2)$. Now $V'(\hat{\theta}_{\mu-1})$ is compact (hence bounded) (see, e.g., [1, p. 91, (2.4.6)], thus we have $|I_1|=O(\eta)$ by A.6(a) (or A.6′(a)), and $|I_2|=o(\eta)$ by A.6(b) (or A.6′(b)). Similar to the proof of $\|\hat{\theta}_{m(\mu-1,\eta)}-\hat{\theta}_{\mu-1}\|=O(\eta)$, we have $\|\hat{\theta}_j-\hat{\theta}_{\mu-1}\|=O(\eta)$ (hence $d(\hat{\theta}_j, \hat{\theta}_{\mu-1})=O(\eta)$), for any $j\in[\mu-1, m(\mu-1,\eta)-1]$. This together with A.3(b) (or A.3′(b)) and A.7 (or A.7′) implies that $|I_3|=o(\eta)$. Now B.9(c) implies $V(\hat{\theta}_{m(\mu-1,\eta)})-V(\hat{\theta}_{\mu-1})=-|O(\eta)|+o(\eta)+O(\eta^2)$. Since $V(\hat{\theta}_{\mu-1})<\delta_1$ by definition of μ, we get $V(\hat{\theta}_{m(\mu-1,\eta)})<\delta_1$, completing the proof.

Proof of Theorem 2.4. Lemma B.1 implies that $\sup_n\|\hat{\theta}_n\|<\infty$ *a.s.* $-\mathbb{P}$. By Assumption B.3, inequalities (b.0) and (b.2) are still valid. The result now follows by the same proof as in Theorem 2.1.

Proof of Corollary 2.5. By [25, Corollary 42.14, p. 252], B.9(a), B.9′(b), and with V being convex and twice continuously F-differentiable, we have that $V(\Theta^*)=0$ and Θ^* is closed, bounded, and convex. This together with B.9(c) implies B.8. Now Theorem 2.4 implies $\hat{\theta}_n\to\Theta^*$ weakly almost surely. By [25, Proposition 41.8, p. 235, H1], with convex F-differentiable V, we get $V(\hat{\theta}_n)\to V(\Theta^*)$ almost surely. This gives us the result (i). By [25, Corollary 42.7, p. 248], B.9″ implies B.9′; thus result (i) applies. [25, Theorem 42.A, p. 251] gives uniqueness of θ^*. By [25, Corollary 42.10, p. 249], $V'(\theta^*)=0$. Now by B.9″(b) and [25, Corollary 42.7, p. 248], $V(\hat{\theta}_n)-V(\theta^*)\geqslant cp^{-1}\|\hat{\theta}_n-\theta^*\|^p$ for all n. This gives the result (ii).

Proof of Corollary 2.6. By [24, p. 501], B.11 implies B.10. Clearly B.10 implies B.8, and B.9 is satisfied with $V(\theta) = \|\theta - \theta^*\|^2$ for $\theta^* \in \Theta^*$. Thus Theorem 2.4 implies $\hat{\theta}_n \to \Theta^*$ weakly. The results (i) and (ii) now follow directly from [24, Theorem 26.A, p. 557].

LEMMA B.2. *Given Assumptions* C.1, C.4(*a*) *and* (*c*), *recursively define* $\xi_{n+1}(\theta) = \rho_n(\xi_n(\theta), \theta, Z_{n+1})$ *for any constant sequence* $\{\theta\}$ *with* $\theta \in \mathbb{H}$ *and all* $n = 0, 1, 2, ...$ *Then for any* $\|\theta\| \leqslant \bar{B} < \infty$, $\{\xi_n(\theta); n = 0, 1, 2, ...\}$ *is a sequence of bounded functions* $L_2(\Xi)$-NED *on* $\{D_n\}$ *of size* $-1/2$.

Proof. This is a Hilbert space version of KW's (1994) Proposition 4.4. Their proof goes through here after proper translation.

Proof of Theorem 3.2. It suffices to show that Assumptions A.6′(b) and A.7′ are satisfied. For any fixed $\theta \in \mathbb{H}$ with $\|\theta\| \leqslant \bar{B}$, given Assumptions C.1, C.4(a) and (c), Lemma B.2 ensures that $\{\xi_n(\theta); n = 0, 1, 2, ...\}$ is a sequence of bounded functions $L_2(\Xi)$-NED on $\{D_n\}$ of size $-1/2$. By the minimum mean squared error property and Assumption C.3(c), we have

$$\begin{aligned}
&\|M_n(\xi_n(\theta), \theta) - E[M_n(\xi_n(\theta), \theta) \mid \mathscr{F}_{n-m}^{n+m}]\|_1 \\
&\quad \leqslant \|M_n(\xi_n(\theta), \theta) - M_n(E[\xi_n(\theta) \mid \mathscr{F}_{n-m}^{n+m}], \theta)\|_2 \\
&\quad \leqslant c_{3,n} \|\xi_n(\theta) - E[\xi_n(\theta) \mid \mathscr{F}_{n-m}^{n+m}]\|_2.
\end{aligned}$$

Hence for each fixed θ with $\|\theta\| \leqslant \bar{B}$, $\{M_n(\xi_n(\theta), \theta); n = 0, 1, 2, ...\}$ is a sequence of functions $L_2(\mathbb{H})$-NED on $\{D_n\}$ with $c_n = O(c_{3,n})$ and size $-1/2$. Now Lemma 4.3 of [7] implies that, for each $\theta \in \mathbb{H}$, $\{M_n(\xi_n(\theta), \theta) - E[M_n(\xi_n(\theta), \theta)]; n = 0, 1, ...\}$ is an $L_2(\mathbb{H})$-mixingale with $c_n = O(c_{3,n})$ and size $-1/2$, (see [7] for the definition of Hilbert-valued mixingales). Similar to the proof of Theorem A.1 in [14], but applying the strong law of large numbers of Corollary 3.8 in [7], we have that Assumptions A.1, A.2, A.4, and C.6 imply Assumption A.6′(b). Similar to the proof of lemma A.2 in [14], but using different metrics, we have that Assumptions C.1(a), C.4(a) and (b) imply A.7′.

Proof of Proposition 4.1. It is obvious that Assumption C.1(a) is satisfied by $G = [-1, 1] \times [l_1, l_2]$, and C.1(b) is satisfied given (4.1.2) and the conditions on $\{\mu_t\}$. Given the conditions, we have that Ξ is a compact subset of $\mathbb{R}$. For each t and any (ξ, θ), $(\bar{\xi}, \bar{\theta}) \in \Xi \times \mathbb{H}$ with $\|\theta\| \leqslant B$, $\|\bar{\theta}\| \leqslant B$, the triangle inequality gives $\|M_t(\xi, \theta) - M_t(\bar{\xi}, \bar{\theta})\| \leqslant I_1 + I_2$, where

$$\begin{aligned}
I_1 &\equiv \|[h_t]^{-k} \mathscr{K}((\cdot - \xi_2)/h_t)[(\xi_1 - \theta) - (\bar{\xi}_1 - \bar{\theta})]\|, \\
I_2 &\equiv \|[h_t]^{-k} [\mathscr{K}((\cdot - \xi_2)/h_t) - \mathscr{K}((\cdot - \bar{\xi}_2)/h_t)](\bar{\xi}_1 - \bar{\theta})\|.
\end{aligned}$$

Now

$$I_1^2 \leqslant \|[h_t]^{-k} \mathscr{K}((\cdot - \xi_2)/h_t)\|^2 \|(\xi_1 - \bar{\xi}_1) - (\theta - \bar{\theta})\|^2$$
$$= [h_t]^{-k} \int_{[-h_t^{-1} h_t^{-1}]^k} [\mathscr{K}(y - \xi_2 [h_t]^{-1})]^2 \, dy \times \|(\xi_1 - \bar{\xi}_1) - (\theta - \bar{\theta})\|^2$$
$$\leqslant [h_t]^{-k} C \|(\xi_1 - \bar{\xi}_1) - (\theta - \bar{\theta})\|^2,$$

where the last inequality is due to (4.1.3). By the triangle inequality, we get $I_1 \leqslant [h_t]^{-k/2} C_1 [|\xi_1 - \hat{\xi}_1| + \|\theta - \bar{\theta}\|]$. Next

$$I_2^2 \leqslant \|[h_t]^{-k} [\mathscr{K}((\cdot - \xi_2)/h_t) - \mathscr{K}((\cdot - \bar{\xi}_2)/h_t)]\|^2 \times \|\bar{\xi}_1 - \bar{\theta}\|^2$$
$$\leqslant [h_t]^{-4k} [C_2]^2 |\xi_2 - \bar{\xi}_2|^2 \times [c_3 + B]^2.$$

Hence $I_2 \leqslant [h_t]^{-2k} C_2 |\xi_2 - \hat{\xi}_2| [C_3 + B]$. We get

$$\|M_t(\xi, \theta) - M_t(\bar{\xi}, \bar{\theta})\|$$
$$\leqslant \max([h_t]^{-k/2} C_1, [h_t]^{-2k} C_2 [C_3 + B], 1) \times [\|\xi - \bar{\xi}\|_{\Xi} + \|\theta - \bar{\theta}\|].$$

So for each t, M_t is Lipschitz continuous on $\Xi \times \{\theta: \|\theta\| \leqslant B\}$; thus it is uniformly continuous on $\Xi \times \{\theta: \|\theta\| \leqslant B\}$, and Assumption C.3(a) is satisfied.

From the above proof, we also have: for any $\xi, \hat{\xi} \in \Xi$, and $\|\theta\| \leqslant B$,

$$\|M_t(\xi, \theta) - M_t(\bar{\xi}, \theta)\| \leqslant O([h_t]^{-2k}) \times \|\xi - \bar{\xi}\|_{\Xi}.$$

Given (4.1.4), we can set $c_{3,t} = O([h_t]^{-2k})$, and Assumption C.3(b) is satisfied. Assumption C.2 is satisfied given (4.1.4) and $a_t = (t+1)^{-1}$. Since $\rho_t(\xi, \theta, z) = (a_2)^{-1} [a_1 - \bar{f}(x) - \theta(x)] + u_t$, Assumptions C.4(a) and (b) and (c) are satisfied with $c_0 = 0$, $c_1 = -1/a_2$, and $c_3 = \max(-c_f/a_2, c_\theta/a_2, 1)$ respectively, where c_f is the Lipschitz constant for the function $\bar{f}$ and c_θ the Lipschitz constant for the Lipschitz continuous function $\theta(\cdot)$. Assumption A.5 is directly assumed by (4.1.6). Since $\{X_t\}$ and $\{u_t\}$ are independent and $E[u_t] = 0$, we have

$$E[M_t(\xi_t(\theta), \theta(\cdot))]$$
$$= E[(h_t)^{-k} \mathscr{K}((\cdot - \xi_{2,t})/h_t)(\xi_{1,t} - \theta(\cdot))]$$
$$= E[(h_t)^{-k} \mathscr{K}((\cdot - X_t)/h_t)][(a_2)^{-1} (a_1 - \bar{f}(X_t)) - \theta(\cdot)]$$
$$= \int_{[-1,1]^k} (h_t)^{-k} \mathscr{K}((\cdot - y)/h_t)$$
$$\times [(a_2)^{-1} (a_1 - \bar{f}(y) - \theta(y)) - \theta(\cdot)] f_X(y) \, dy$$
$$= \int \mathscr{K}(\eta)[(a_2)^{-1} (a_1 - \bar{f}(\cdot + \eta h_t) - \theta(\cdot + \eta h_t)) - \theta(\cdot)] f_X(\cdot + \eta h_t) \, d\eta.$$

Given (4.1.1), (4.1.3)–(4.1.6), and the construction of $\{\hat{\theta}_t\}$, Theorem 2.1.1 in [18] gives

$$\lim_{t\to\infty} \sup_{x\in[-1,1]^k} |E[M_t(\xi_t(\theta), \theta(x))] - [(a_2)^{-1}(a_1 - \bar{f}(x) - \theta(x)) - \theta(x)] f_X(x)| = 0.$$

Hence

$$\lim_{t\to\infty} \|E[M_t(\xi_t(\theta), \theta(\cdot))] - [(a_2)^{-1}(a_1 - \bar{f}(\cdot) - \theta(\cdot)) - \theta(\cdot)] f_X(\cdot)\|^2 = 0.$$

Set

$$\bar{M}(\theta(\cdot)) \equiv [(a_2)^{-1}(a_1 - \bar{f}(\cdot) - \theta(\cdot)) - \theta(\cdot)] f_X(\cdot).$$

Then Assumptions C.6 and A.6′(a) are satisfied. Moreover,

$$\Theta^* = \{\theta \in L_2(\mathbb{R}^k): \bar{M}(\theta) = 0\} = \{\theta_o = [a_1 - \bar{f}(\cdot)]/(a_2 + 1)\}.$$

Define a functional $V: L_2(\mathbb{R}) \to \mathbb{R}$ as $V(\theta) = \|\theta - \theta_o\|^2$. Then

$$\langle V'(\theta), \bar{M}(\theta)\rangle = \langle 2[\theta - \theta_o], f_X[1 + (a_2)^{-1}][\theta_o - \theta]\rangle < 0, \quad \text{for all} \quad \theta \neq \theta_o.$$

Hence B.8 and B.9, and even B.11 are satisfied. The result now follows from Theorem 3.2.

Proof of Proposition 4.2. The proof is similar to that for Proposition 4.1; we only sketch the main steps here. For fixed θ,

$$\begin{aligned} &E[M_t(\xi_t(\theta), \theta(\cdot))] \\ &\quad = E\left[(h_t)^{-1} \mathscr{K}((\cdot - v_{t-1})/h_t) \times \left[\int G(H(\theta(v), v), v)\, q_{t\mid t-1}(dv \mid v_{t-1}) - \theta(\cdot)\right]\right] \\ &\quad = \int \mathscr{K}(\eta) \left[\int G(H(\theta(v), v), v)\, q_{t\mid t-1}(dv \mid \cdot + h_t\eta) - \theta(\cdot)\right] \times f_{t-1}(\cdot + h_t\eta)\, d\eta. \end{aligned}$$

Now we only need to set $\bar{M}(\theta(\cdot)) = [\int G(H(\theta(v), v), v)\, q(dv \mid \cdot) - \theta(\cdot)] f(\cdot)$ and specify the Lyapunov functional $V(\theta) = \|\theta - \theta^*\|^2$.

Proof of Proposition 4.3. Since Z_t is uniformly bounded in t, there exists G, a norm-bounded convex subset of $\mathbb{R}^2$, containing the support of Z_t for all t, so that Assumption C.1(a) is satisfied. Assumption C.1(b) is given by (4.3.2). For any $(\xi, \theta), (\hat{\xi}, \bar{\theta}) \in \Xi \times \mathbb{H}$, we have

$$\|M_t(\xi, \theta) - M_t(\bar{\xi}, \bar{\theta})\|^2$$
$$= \|M_{1,t}(\xi, \theta) - M_{1,t}(\bar{\xi}, \bar{\theta})\|^2 + \|M_{2,t}(\xi, \theta) - M_{2,t}(\bar{\xi}, \bar{\theta})\|^2 \equiv I_1 + I_2.$$

By the triangle inequality and then the Lipschitz continuity of $\mathscr{K}_1$ assumed in (4.3.3), we get

$$I_1 \leqslant [h_{1,t}]^{-2} \|\mathscr{K}_1((\cdot - \xi_2)/h_{1,t}) - \mathscr{K}_1((\cdot - \bar{\xi}_2)/h_{1,t})\|^2 + \|\theta_1 - \bar{\theta}_1\|^2$$
$$= [h_{1,t}]^{-1} \int_{[0,1]} [\mathscr{K}_1((x - \xi_2)/h_{1,t}) - \mathscr{K}_1((x - \bar{\xi}_2)/h_{1,t})]^2 \, dx + \|\theta_1 - \bar{\theta}_1\|^2$$
$$\leqslant [h_{1,t}]^{-4} C_1^2 |\xi_2 - \bar{\xi}_2|^2 + \|\theta_1 - \bar{\theta}_1\|^2.$$

Similarly we get

$$I_2 \leqslant [h_{2,t}]^{-4} C_2^2 |\xi_1 - \bar{\xi}_1|^2 + \|\theta_2 - \bar{\theta}_2\|^2.$$

Denote $c_{3,t} = \max(C_1[h_{1,t}]^{-2}, C_2[h_{2,t}]^{-2})$; then

$$\|M_t(\xi, \theta) - M_t(\bar{\xi}, \bar{\theta})\|^2 \leqslant c_{2,t}^2 \|\xi - \bar{\xi}\|_{\Xi}^2 + \|\theta - \bar{\theta}\|^2.$$

Hence

$$\|M_t(\xi, \theta) - M_t(\bar{\xi}, \bar{\theta})\| \leqslant \max(c_{3,t}, 1)[\|\xi - \bar{\xi}\|_{\Xi}^2 + \|\theta - \bar{\theta}\|^2]^{1/2}.$$

Hence for each t, the mapping $M_t \colon \Xi \times \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous and thus is uniformly continuous. Thus Assumption C.3(a) is satisfied. Now for each fixed $\theta \in \mathbb{H}$, take any $\xi, \bar{\xi} \in \Xi$; we also get from the above that $\|M_t(\xi, \theta) - M_t(\bar{\xi}, \theta)\| \leqslant c_{3,t} \|\xi - \bar{\xi}\|_{\Xi}$, and we have that Assumption C.3(c) is satisfied. Since $a_t = (t+1)^{-1}$, given (4.3.4), $c_{3,t} a_t = O((t+1)^{2\delta - 1})$ for some $0 < \delta < 1/4$; thus Assumption C.2 is satisfied. Since $\rho_t(\xi, \theta, z) = \int_{[0,1]} x\theta(x) \, dx - a\xi - z$, Assumptions C.4(a, b, c) are satisfied with $c_0 = |a|$, $c_1 = 1$ and $c_2 = 1$ respectively. Assumption A.5 is directly assumed. For each fixed $\theta \in L_2([0, 1]) \times L_2([0, 1])$,

$$E[M_{1,t}(\xi_t(\theta), \theta)] = E[(h_{1,t})^{-1} \mathscr{K}_1((\cdot - \xi_{2,t}(\theta))/h_{1,t})] - \theta_{12},$$

and

$$E[M_{2,t}(\xi_t(\theta), \theta)] = E[(h_{2,t})^{-1} \mathscr{K}_2((\cdot - \xi_{1,t}(\theta))/h_{2,t})] - \theta_{21}.$$

Under conditions (4.3.1), (4.3.3), and (4.3.4), Theorem 2.1.1 in [18] implies

$$\lim_{t \to \infty} \sup_{x \in [0,1]} |E[(h_{1,t})^{-1} \mathscr{K}_1((x - \xi_{2,t}(\theta))/h_{1,t})] - f_2(x)| = 0.$$

Hence

$$\lim_{t \to \infty} \|E[M_{1,t}(\xi_t(\theta), \theta)] - (f_2 - \theta_{12})\|$$

$$= \left(\int_{[0,1]} [\lim_{t \to \infty} E[(h_{1,t})^{-1} \mathscr{K}_1((x - \xi_{2,t}(\theta))/h_{1,t})] - f_2(x)]^2 \, dx \right)^{1/2} = 0.$$

Hence $\lim_{t \to \infty} E[M_{1,t}(\xi_t(\theta), \theta)] = f_2 - \theta_{12}$ in the norm topology. Similarly we have $\lim_{t \to \infty} E[M_{2,t}(\xi_t(\theta), \theta)] = f_1 - \theta_{21}$ in norm. Thus Assumptions A.6′(a) and C.6 are satisfied with

$$\bar{M}(\theta)' = (\bar{M}_1(\theta), \bar{M}_2(\theta)) = (f_2 - \theta_{12}, f_1 - \theta_{21}) = (f_2, f_1) - \theta'.$$

Again,

$$\begin{aligned} \Theta^* &= \{\theta \in L_2([0,1]) \times L_2((0,1]) \colon \bar{M}(\theta) = 0\} \\ &= \{\theta^{*\prime} = (f_2, f_1)\}, \end{aligned}$$

and we can define a functional $V\colon L_2([0,1]) \times L_2([0,1]) \to \mathbb{R}$ as $V(\theta) = \|\theta - \theta^*\|^2$. Thus B.8, B.9, and even B.11 are satisfied. The result now follows from Theorem 3.2.

Proof of Proposition 4.4. It is easy to show that

$$\begin{aligned} \bar{M}(\theta(s,r)) = f_{s_{t-1}(\theta), r_t}(s,r) \Big[\beta \int U'(r_{t+1}\theta(s,r) - \theta(\theta(s,r), r_{t+1})) \\ \times r_{t+1} q(dr_{t+1} \mid r) - U'(sr - \theta(s,r)) \Big]. \end{aligned}$$

Set $V(\theta) = \|\theta - S^*\|^2$, and the result follows.

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